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A fixed point theorem and the Hyers-Ulam stability in Riesz spaces

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Abstract

We prove a fixed point theorem and show its applications in investigations of the Hyers-Ulam type stability of some functional equations (in single and many variables) in Riesz spaces.

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1 Introduction

The Hyers-Ulam stability for functional, but also for difference, differential and integral equations, is a very quickly growing area of investigations (for more details and further references, see, *e.g.*, [1–7]; examples of some recent results can be found in [8–20]). It is related to the notions of shadowing (see, *e.g.*, [21–23]) as well as to the theories of perturbation (see, *e.g.*, [24, 25]) and optimization. The first known result on such stability is due to Pólya and Szegö [26] and reads as follows.

> For every real sequence $(a_n)_{n \in \mathbb{N}}$ with $\sup_{n,m \in \mathbb{N}} |a_{n+m} - a_n - a_m| \leq 1$, there is a real number ω such that $\sup_{n \in \mathbb{N}} |a_n - \omega n| \leq 1$. Moreover, $\omega = \lim_{n \to \infty} a_n/n$.

But it was Ulam (*cf.* [27, 28]) who in 1940 gave the main motivation for the study of that subject, at the University of Wisconsin, where he presented some unsolved problems and in particular the following one.

> Let G_1 be a group and (G_2, d) a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$?

In 1941 Hyers [27] published an answer to it, which is presented below in a bit generalized form (see, *e.g.*, [4]).

> Let $(X, +)$ be a commutative semigroup, Y be a Banach space and $\varepsilon > 0$. Then for every $g : X \to Y$ with $\sup_{x,y \in X} \|g(x + y) - g(x) - g(y)\| \leq \varepsilon$ there is a unique $f : X \to Y$ such that $\sup_{x \in X} \|g(x) - f(x)\| \leq \varepsilon$ and

$$f(x + y) = f(x) + f(y), \quad x, y \in X. \tag{1}$$

By now we express that result simply saying that *the Cauchy functional equation* (1) *is Hyers-Ulam stable (or has the Hyers-Ulam stability) in the class of functions* Y^X.

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The Hyers-Ulam stability in Riesz spaces have already been studied in [29] (with a direct method) and in [11] (with an application of the spectral representation theorem). The main motivation for this kind of investigations follows from the pretty natural concept to pose the stability problem for a given functional equation in the settings of an ordered structure as an alternative for the topological or metric ones. To preserve the spirit of the Hyers-Ulam stability, we do that in a way which allows to sustain the full coincidence with the classical notion in the simplest case of real functions.

The fixed point approach has been already applied in the investigation of the Hyers-Ulam stability, *e.g.*, in [8, 12, 13, 30–37] and it seems that Baker (see [30]) has used this tool for the first time in this field; for a survey on this subject, see [38]. In this paper we continue this direction following the approach presented in [12, 13, 31]. We start our paper with a natural fixed point theorem and next derive some stability results from it.

2 Preliminaries

In this paper, as usual, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of all positive integers, nonnegative integers, real numbers and nonnegative real numbers, respectively.

For the readers' convenience, we quote basic definitions and properties concerning Riesz spaces (see [39]).

Definition 1 (*cf.* [39], Definition 11.1 and Definition 22.1) We say that a real linear space L, endowed with a partial order $\leq \subset L^2$, is a *Riesz space* if $\sup\{x, y\}$ exists for all $x, y \in L$ and

$$ax + y \leq az + y, \quad x, y, z \in X, x \leq z, a \in \mathbb{R}_+;$$

we define the absolute value of $x \in L$ by the formula $|x| := \sup\{x, -x\} \geq 0$.

A Riesz space L is called *Archimedean* if, for each $x \in L$, the inequality $x \leq 0$ holds whenever the set $\{nx : n \in \mathbb{N}\}$ is bounded from above.

It is easily seen that in a Riesz space L we have $|v| \geq 0$ and $\alpha u \leq \beta u$ for every $v, u \in L$, $u \geq 0$, and $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$.

There are several types of convergence that may be defined according to the order structure. One of them is the *relatively uniform convergence* defined as follows.

Definition 2 (*cf.* [39], Definition 39.1) Let L be a Riesz space and $e \in L$. A sequence $\{f_n\}_{n \in \mathbb{N}}$ in L is said to *converge e-uniformly* to an element $f \in L$ whenever, for every $\varepsilon > 0$, there exists a positive integer n_0 such that $|f - f_n| \leq \varepsilon e$ holds for all $n \geq n_0$. A sequence $\{f_n\}_{n \in \mathbb{N}}$ in L is called *e-uniform Cauchy sequence* whenever, for every $\varepsilon > 0$, there exists a positive integer n_1 such that $|f_m - f_n| \leq \varepsilon e$ holds for all $m, n \geq n_1$.

Let us point out (see [39, p.252]) that in a Riesz space L that is Archimedean, the e-uniform limit of a sequence in it, if exists, is unique and the fact that $\{f_n\}_{n \in \mathbb{N}}$ converges e-uniformly to f will be denoted by $\lim_{n \to \infty}^e f_n = f$ (in particular, if $v \in L$ and $f_n \geq v$ for $n \in \mathbb{N}$, then $f \geq v$).

Definition 3 The series $\sum_{i=0}^{\infty} a_i$ in a Riesz space L is said to *converge e-uniformly* (for a given $e \in L$) to an element $S \in L$ whenever the sequence $S_n := \sum_{i=0}^{n} a_i$ converges e-uniformly to S. If L is Archimedean, then we write $\sum_{i=0}^{\infty} a_i \overset{e}{=} S$.

Definition 4 (*cf.* [39], Definition 39.3) A Riesz space L is called *e-uniformly complete* (with a given $e \in L$) whenever every e-uniform Cauchy sequence has an e-uniform limit.

There is a large class of spaces satisfying the above conditions. In particular, every Dedekind σ-complete space (that is such that any non-empty at most countable subset which is bounded from above has a supremum) is an Archimedean e-uniformly complete space for every $e \geq 0$ (see [39, pp.125, 252, 253]).

3 Main result

In what follows X is a nonempty set and Y is an Archimedean u-uniformly complete Riesz space for some given $u \in Y$.

We start that part with a lemma that is quite elementary. However, since we use it several times, we present it with a proof for the convenience of readers.

Lemma 1 *Let L be an Archimedean Riesz space and $e \in L$. The sequence of rests $r_n \overset{e}{:=} \sum_{i=n+1}^{\infty} a_i$ of an e-uniformly convergent series $\sum_{i=0}^{\infty} a_i$ in L is e-uniformly convergent to 0.*

Proof Let us observe that $|\sum_{i=n+1}^{m} a_i - S + S_n| = |S_m - S_n - S + S_n| = |S_m - S|$. Consequently, $\sum_{i=n+1}^{\infty} a_i \overset{e}{=} S - S_n \in L$ as S_m converges e-uniformly to S and, therefore, r_n is well defined. Moreover, we have $\lim_{n \to \infty}^{e} r_n = \lim_{n \to \infty}^{e} (S - S_n) = 0 \in L$. □

The following two hypotheses will be useful.

(H1) Mappings $f_1, \ldots, f_k : X \to X$ and $L_1, \ldots, L_k : X \to \mathbb{R}_+$ are given and $\mathcal{J} : Y^X \to Y^X$ is an operator satisfying the inequality

$$\left| (\mathcal{J}a)(t) - (\mathcal{J}b)(t) \right| \leq \sum_{i=1}^{k} L_i(t) \left| a(f_i(t)) - b(f_i(t)) \right|, \quad a, b \in Y^X, t \in X. \tag{2}$$

(H2) $\Lambda : Y^X \to Y^X$ is a linear operator given by

$$(\Lambda \delta)(t) := \sum_{i=1}^{k} L_i(t) \delta(f_i(t)), \quad \delta \in Y^X, t \in X. \tag{3}$$

Observe that Λ is monotone with respect to the pointwise ordering (because L_i is non-negative). Our main theorem reads as follows.

Theorem 1 *Let us assume that hypotheses* (H1) *and* (H2) *are fulfilled and that functions $\varphi, \varepsilon, S : X \to Y$ satisfy the conditions*

$$\left| (\mathcal{J}\varphi)(t) - \varphi(t) \right| \leq \varepsilon(t), \quad t \in X, \tag{4}$$

$$\sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(t) \overset{u}{=} S(t), \quad t \in X. \tag{5}$$

Then $\mathcal{J}$ has a unique fixed point $\psi \in Y^X$ with

$$\left| \varphi(t) - \psi(t) \right| \leq S(t), \quad t \in X. \tag{6}$$

Moreover,

$$\psi(t) = \lim_{n \to \infty}{}^{u}\big(\mathcal{J}^{n}\varphi\big)(t), \quad t \in X. \tag{7}$$

Proof The proof is somewhat similar to the proof of [12, Theorem 1], but since there are also significant differences between them, we present it here.

First we show by induction that

$$\big|\big(\mathcal{J}^{n}\varphi\big)(t) - \big(\mathcal{J}^{n+1}\varphi\big)(t)\big| \le \big(\Lambda^{n}\varepsilon\big)(t), \quad t \in X. \tag{8}$$

For $n = 0$, (8) and (4) coincide. So, let us suppose that (8) holds for a fixed $n \in \mathbb{N}$. Using (2) and the definition of Λ (*i.e.*, (3)), we have

$$\big|\big(\mathcal{J}^{n+1}\varphi\big)(t) - \big(\mathcal{J}^{n+2}\varphi\big)(t)\big|$$

$$\le \sum_{i=1}^{k} L_i(t)\big|\big(\mathcal{J}^{n}\varphi\big)\big(f_i(t)\big) - \big(\mathcal{J}^{n+1}\varphi\big)\big(f_i(t)\big)\big|$$

$$\le \sum_{i=1}^{k} L_i(t)\big(\Lambda^{n}\varepsilon\big)\big(f_i(t)\big)$$

$$= \big(\Lambda^{n+1}\varepsilon\big)(t), \quad t \in X,$$

which completes the proof of (8). Therefore, for $n, k \in \mathbb{N}$, we get

$$\big|\big(\mathcal{J}^{n}\varphi\big)(t) - \big(\mathcal{J}^{n+k}\varphi\big)(t)\big| \le \sum_{i=n}^{n+k-1}\big(\Lambda^{i}\varepsilon\big)(t), \quad t \in X. \tag{9}$$

This, according to (5) and Lemma 1, means that $((\mathcal{J}\varphi)^{n}(t))_{n\in\mathbb{N}}$ is a u-uniform Cauchy sequence for any $t \in X$, and therefore u-uniformly convergent, as Y is u-uniformly complete. Let us denote by $\psi(t)$ its relative uniform limit. Using (9) with $n = 0$, we have

$$\big|\varphi(t) - \big(\mathcal{J}^{k}\varphi\big)(t)\big| \le \sum_{i=0}^{k-1}\big(\Lambda^{i}\varepsilon\big)(t), \quad t \in X. \tag{10}$$

Write $S_{k-1}(t) := \sum_{i=0}^{k-1}(\Lambda^{i}\varepsilon)(t)$ for $k \in \mathbb{N}$ and $t \in X$. By the triangle inequality and (10), for any $k \in \mathbb{N}$ and $t \in X$, we obtain

$$\big|\varphi(t) - \psi(t)\big| \le \big|\varphi(t) - \big(\mathcal{J}^{k}\varphi\big)(t)\big| + \big|\big(\mathcal{J}^{k}\varphi\big)(t) - \psi(t)\big|$$

$$\le S_{k-1}(t) + \big|\big(\mathcal{J}^{k}\varphi\big)(t) - \psi(t)\big|$$

$$\le \big|S_{k-1}(t) - S(t)\big| + \big|\big(\mathcal{J}^{k}\varphi\big)(t) - \psi(t)\big| + S(t).$$

From the u-uniform convergence of $((\mathcal{J}\varphi)^{k}(t))_{k\in\mathbb{N}}$ to $\psi(t)$ and $(S_{k-1}(t))_{k\in\mathbb{N}}$ to $S(t)$, for $t \in X$, we derive

$$\big|\varphi(t) - \psi(t)\big| \le S(t) + \frac{1}{n}u, \quad n \in \mathbb{N}, t \in X, \tag{11}$$

which results in (6) as Y is Archimedean.

We will show that ψ is a fixed point of $\mathcal{J}$. Having applied (2) with $a = \psi$ and $b = \mathcal{J}^{n-1}\varphi$, for any $t \in X$, one has

$$\big|(\mathcal{J}\psi)(t) - \mathcal{J}(\mathcal{J}^{n-1}\varphi)(t)\big| \le \sum_{i=1}^{k} L_i(t)\big|\psi(f_i(t)) - \mathcal{J}^{n-1}\varphi(f_i(t))\big|, \quad n \in \mathbb{N}. \tag{12}$$

Next, using the triangle inequality $|(\mathcal{J}\psi)(t) - \psi(t)| \le |(\mathcal{J}\psi)(t) - (\mathcal{J}^n\varphi)(t)| + |(\mathcal{J}^n\varphi)(t) - \psi(t)|$ (for $n \in \mathbb{N}$, $t \in X$), we infer that

$$\big|(\mathcal{J}\psi)(t) - \psi(t)\big| \le \frac{1}{m}u\left(\sum_{i=1}^{k} L_i(t) + 1\right), \quad t \in X, m \in \mathbb{N}, \tag{13}$$

which means that $\mathcal{J}\psi = \psi$ as Y is Archimedean.

For the proof of the uniqueness of ψ, assume that $\psi_1, \psi_2 \in Y^X$ are two fixed points of $\mathcal{J}$ satisfying

$$\big|\varphi(t) - \psi_i(t)\big| \le S(t), \quad t \in X, i = 1, 2. \tag{14}$$

At first let us observe that for every $n \in \mathbb{N}$, we get

$$\big|(\mathcal{J}^n\psi_1)(t) - (\mathcal{J}^n\psi_2)(t)\big| \le 2\sum_{i=0}^{\infty}(\Lambda^i\varepsilon)(t), \quad t \in X. \tag{15}$$

In fact, for $n = 0$ inequality (15) follows directly from (14). Now assume that (15) holds for a given $n \in \mathbb{N}$. Then, by (12), for every $t \in X$, we have

$$
\begin{aligned}
\big|(\mathcal{J}^{n+1}\psi_1)(t) &- (\mathcal{J}^{n+1}\psi_2)(t)\big| \\
&= \big|(\mathcal{J}(\mathcal{J}^n\psi_1))(t) - (\mathcal{J}(\mathcal{J}^n\psi_2))(t)\big| \\
&\le \sum_{i=1}^{k} L_i(t)\big|(\mathcal{J}^n\psi_1)(f_i(t)) - (\mathcal{J}^n\psi_2)(f_i(t))\big| \\
&\le 2\sum_{i=1}^{k} L_i(t) \sum_{j=n}^{\infty}(\Lambda^j\varepsilon)(f_i(t)) \\
&= 2\sum_{j=n}^{\infty}\sum_{i=1}^{k} L_i(t)(\Lambda^j\varepsilon)(f_i(t)) \\
&= 2\sum_{j=n+1}^{\infty}(\Lambda^j\varepsilon)(t).
\end{aligned}
$$

Thus we have proved (15). Letting $n \to \infty$ and taking into account Lemma 1, we infer that $\psi_1 = \psi_2$. $\qquad\square$

Remark 1 If in (2) we take $k = 1$, $f_1(x) = x$ and $L_1(x) = \lambda \in [0, 1)$ for $x \in X$, then Theorem 1 becomes a result corresponding to the classical Banach contraction principle.

Remark 2 We can associate to a given $u \in Y$ an extended (*i.e.*, admitting the infinite value) norm on Y denoted by $\| \cdot \|_u$ and defined by

$$\|v\|_u := \inf\{\lambda \in \mathbb{R}_+ : |v| \le \lambda u\}, \quad v \in Y.$$

From Theorem 1 we can easily deduce the following corollary.

Corollary 1 *Let hypothesis* (H1) *be fulfilled,* $\Lambda_0 : \mathbb{R}_+^X \to \mathbb{R}_+^X$ *be a linear operator given by*

$$(\Lambda_0\delta)(t) := \sum_{i=1}^{k} L_i(t)\delta\big(f_i(t)\big), \quad \delta \in \mathbb{R}_+^X, t \in X, \tag{16}$$

and functions $\varphi : X \to Y$ *and* $\varepsilon_0 : X \to \mathbb{R}_+$ *satisfy the conditions*

$$\big\|(\mathcal{J}\varphi)(t) - \varphi(t)\big\|_u \le \varepsilon_0(t), \quad t \in X, \tag{17}$$

$$S_0(t) := \sum_{n=0}^{\infty} \big(\Lambda_0^n\varepsilon_0\big)(t) < \infty, \quad t \in X. \tag{18}$$

Then $\mathcal{J}$ *has a unique fixed point* $\psi \in Y^X$ *with*

$$\big\|\varphi(t) - \psi(t)\big\|_u \le S_0(t), \quad t \in X. \tag{19}$$

Moreover, ψ *is given by* (7).

Proof Define $S, \varepsilon : X \to Y$ by $S(t) := S_0(t)u$ and $\varepsilon(t) := \varepsilon_0(t)u$ for $t \in X$. Next let $\Lambda : Y^X \to Y^X$ be given by (3). Then it is easily seen that (4) and (5) are valid. Consequently, in view of Theorem 1, $\mathcal{J}$ has a unique fixed point $\psi \in Y^X$ such that (6) holds; moreover, ψ is given by (7). Clearly, (6) and (19) are equivalent, whence we derive the statement concerning the uniqueness of ψ. □

4 The Hyers-Ulam stability

In the simplest case $k = 1$, Theorem 1 yields at once the following corollary concerning the Hyers-Ulam stability of a quite general functional equation

$$\psi(t) = F\big(t, \psi\big(\xi(t)\big)\big) \tag{20}$$

in the class of functions Y^X, with given $F : X \times Y \to Y$ and $\xi : X \to X$.

Corollary 2 *Let* $F : X \times Y \to Y$, $\xi : X \to X$, $\lambda : X \to \mathbb{R}_+$, $\varphi, \varepsilon, S : X \to Y$,

$$\big|F(t, u) - F(t, v)\big| \le \lambda(t)|u - v|, \quad t \in X, u, v \in Y, \tag{21}$$

$$S(t) \stackrel{u}{=} \varepsilon(t) + \sum_{n=0}^{\infty}\left(\prod_{i=0}^{n} \lambda\big(\xi^i(t)\big)\right)\varepsilon\big(\xi^{n+1}(t)\big), \quad t \in X, \tag{22}$$

and

$$\big|\varphi(t) - F\big(t, \varphi\big(\xi(t)\big)\big)\big| \le \varepsilon(t), \quad t \in X. \tag{23}$$

Then there is a unique solution $\psi \in Y^X$ of functional equation (20) *such that*

$$\left|\psi(t) - \varphi(t)\right| \leq S(t), \quad t \in X. \tag{24}$$

Moreover, ψ is given by (7) *with* $(\mathcal{T}g)(t) = F(t, g(\xi(t)))$ *for* $g \in Y^X$, $t \in X$.

Proof Use Theorem 1 with $k = 1$, $f_1 = \xi$, $L_1 = \lambda$, and $(\mathcal{J}g)(t) = F(t, g(\xi(t)))$ for $g \in Y^X$, $t \in X$. □

Example 1 If there is $\alpha \in \mathbb{R}_+$ such that (in Corollary 2) $\lambda(t) \leq \alpha$ for $t \in X$, then clearly we get $S(t) \leq \sum_{n=0}^{\infty} \alpha^n \varepsilon(\xi^n(t))$ for $t \in X$, provided $\sum_{n=0}^{\infty} \alpha^n \varepsilon(\xi^n(t))$ is *u*-uniformly convergent in Y. Moreover, in the case where $\varepsilon(f(t)) \leq \beta \varepsilon(t)$ for all $t \in X$ and some $\beta \in \mathbb{R}_+$ with $\alpha\beta < 1$ (*e.g.*, when $\alpha < 1$ and ε is a constant function), $S(t) \leq \frac{1}{1-\alpha\beta}\varepsilon(t)$ for $t \in S$. This means that Corollary 2 implies some analogues of the results in [30] for the Riesz spaces.

Analogously, from Corollary 1 we obtain the following.

Corollary 3 *Let* $F : X \times Y \to Y$, $\xi : X \to X$, $\varepsilon_0, \lambda : X \to \mathbb{R}_+$, $\varphi : X \to Y$, (21) *holds,*

$$S_0(t) := \varepsilon_0(t) + \sum_{n=0}^{\infty}\left(\prod_{i=0}^{n}\lambda\left(\xi^i(t)\right)\right)\varepsilon_0\left(\xi^{n+1}(t)\right) < \infty, \quad t \in X, \tag{25}$$

$$\left\|\varphi(t) - F\left(t, \varphi\left(\xi(t)\right)\right)\right\|_u \leq \varepsilon_0(t), \quad t \in X. \tag{26}$$

Then there is a unique solution $\psi \in Y^X$ of equation (20) *such that*

$$\left\|\psi(t) - \varphi(t)\right\|_u \leq S_0(t), \quad t \in X. \tag{27}$$

Moreover, ψ is given by (7).

Example 2 If (in Corollaries 2 and 3) $F(t, v) = b(t)v + H(t)$ for $t \in X$, $v \in Y$ with some $b : X \to \mathbb{R}$, $\xi : X \to X$ and $H : X \to Y$, then we get stability results for the equation

$$\Phi(t) = b(t)\Phi\left(\xi(t)\right) + H(t)$$

corresponding to [40, Theorem 2.1 and Corollary 2.4], from which we can easily derive stability results analogous to [41, Theorem 2.1] (provided $0 \notin b(X)$) for the equation

$$\Phi\left(\xi(t)\right) = \frac{1}{b(t)}\Phi(t) - \frac{H(t)}{b(t)}.$$

We also can use Theorem 1 and Corollary 1 to deal with the stability of a bit more involved equations with any number of unknown functions, *e.g.*, as follows.

Corollary 4 *Let* $k, m \in \mathbb{N}$, $L_1, \ldots, L_k : X \to \mathbb{R}_+$, $G : X \times Y^m \to Y$, $f_1, \ldots, f_k, g_1, \ldots, g_m : X \to X$, $\Phi : X \times Y^k \to Y$ *and*

$$\left|\Phi(t, u_1, \ldots, u_k) - \Phi(t, v_1, \ldots, v_k)\right| \leq \sum_{i=1}^{k} L_i(t)\left|u_i - v_i\right|, \quad u_i, v_i \in Y, t \in X. \tag{28}$$

Suppose that functions $\varphi, \varepsilon, \mu_1, \ldots, \mu_m : X \to Y$ satisfy the inequality

$$\left| \varphi(t) - \Phi\big(t, \varphi\big(f_1(t)\big), \ldots, \varphi\big(f_k(t)\big)\big) - G\big(t, \mu_1\big(g_1(t)\big), \ldots, \mu_m\big(g_m(t)\big)\big) \right| \le \varepsilon(t), \quad t \in X,$$

and (5) holds with Λ given by (3). Then there exists a unique $\psi : X \to Y$ satisfying inequality (24), which solves the equation

$$\psi(t) = \Phi\big(t, \psi\big(f_1(t)\big), \ldots, \psi\big(f_k(t)\big)\big) + G\big(t, \mu_1\big(g_1(t)\big), \ldots, \mu_m\big(g_m(t)\big)\big). \tag{29}$$

Moreover,

$$\psi(t) = \lim_{n \to \infty} {}^{u}\big(T_0^n \varphi\big)(t), \quad t \in X, \tag{30}$$

where $(T_0 h)(t) := \Phi(t, h(f_1(t)), \ldots, h(f_k(t))) + G(t, \mu_1(g_1(t)), \ldots, \mu_m(g_m(t)))$ for $h \in Y^X$, $t \in X$.

Proof It is enough to use Theorem 1 with $\mathcal{J} = T_0$. □

Corollary 5 *Let $k, m \in \mathbb{N}$, $L_1, \ldots, L_k : X \to \mathbb{R}_+$, $G : X \times Y^m \to Y$, $f_1, \ldots, f_k, g_1, \ldots, g_m : X \to X$, and $\Phi : X \times Y^k \to Y$ satisfy (28). Suppose that functions $\varphi, \mu_1, \ldots, \mu_m : X \to Y$ and $\varepsilon : X \to \mathbb{R}_+$ are such that*

$$\left\| \varphi(t) - \Phi\big(t, \varphi\big(f_1(t)\big), \ldots, \varphi\big(f_k(t)\big)\big) - G\big(t, \mu_1\big(g_1(t)\big), \ldots, \mu_m\big(g_m(t)\big)\big) \right\|_u \le \varepsilon(t), \quad t \in X,$$

and (18) holds with Λ_0 given by (16). Then there exists a unique $\psi : X \to Y$ satisfying inequality (27) and equation (29). Moreover, ψ is given by (30).

Proof It is enough to use Corollary 1 with $\mathcal{J} = T_0$. □

In the next part we supply an example of a result concerning the Hyers-Ulam stability of functional equations in several variables.

5 Stability of the equation of homomorphism for the square symmetric groupoids

Let T be a nonempty set. A binary operation $\star : T^2 \to T$ is square symmetric provided

$$(x \star y) \star (x \star y) = (x \star x) \star (y \star y), \quad x, y \in T$$

(see, *e.g.*, [18, 42–45]). If $(T, +)$ is a commutative semigroup, $\gamma_0 \in T$ and $\alpha, \beta : T \to T$ are endomorphisms with $\alpha \circ \beta = \beta \circ \alpha$ (for instance, $\alpha = \beta^n$ for an $n \in \mathbb{N}_0$), then it is easy to check that $* : T^2 \to T$, given by

$$x * y := \alpha(x) + \beta(y) + \gamma_0, \quad x, y \in T,$$

is square symmetric. So, it is easily seen that the following two very well-known functional equations

$$f(x + y) = f(x) + f(y) \quad \text{(Cauchy)},$$

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2} \quad \text{(Jensen)}$$

are particular examples of homomorphism equations for square symmetric groupoids. Moreover, we have a very particular case of a square symmetric operation when T is a linear space over a field $\mathbb{K}$ and $*$ is defined by

$$s * t := Cs + Dt + u, \quad s, t \in T,\tag{31}$$

with some fixed $C, D \in \mathbb{K}$ and $u \in T$. So, the general linear functional equation

$$f(Cx + Dy + u) = Af(x) + Bf(y) + c$$

(for instance, with some fixed $A, B \in \mathbb{K}$, $c \in T$ and for functions $f \in T^T$) is also an example of a homomorphism equation for some square symmetric groupoids.

The next corollary provides an example of stability result for the functional equation of homomorphism for square symmetric groupoids, which corresponds to some outcomes in [18, 42–44].

Corollary 6 *Let $C, D \in \mathbb{R}$, $C \neq -D$, $S, \varphi : X \to Y$, $\chi : X^2 \to Y$, $w \in Y$, $\star : X^2 \to X$ be a square symmetric binary operation, and let $\rho : X \to X$ be given by $\rho(t) := t \star t$ for $t \in X$,*

$$\left| \varphi(s \star t) - C\varphi(s) - D\varphi(t) - w \right| \le \chi(s, t), \quad s, t \in X,\tag{32}$$

$$S(t) \overset{u}{=} \sum_{n=0}^{\infty} |C + D|^{-n-1} \chi\left(\rho^n(t), \rho^n(t)\right), \quad t \in X,\tag{33}$$

and

$$\lim_{n \to \infty} {}^u |C + D|^{-n} \chi\left(\rho^n(t), \rho^n(s)\right) = 0, \quad s, t \in X.\tag{34}$$

Then there exists a unique solution $\Phi \in Y^X$ of the functional equation

$$\Phi(s \star t) = C\Phi(s) + D\Phi(t) + w\tag{35}$$

such that

$$\left| \varphi(t) - \Phi(t) \right| \le S(t), \quad t \in X.\tag{36}$$

Moreover,

$$\Phi(t) = \lim_{n \to \infty} {}^u \left[(C + D)^{-n} \varphi\left(\rho^n(t)\right) - \sum_{i=1}^{n} (C + D)^{-i} w \right], \quad t \in X.\tag{37}$$

Proof Take $s = t$ in (32). Then we get

$$\left| \varphi\left(\rho(s)\right) - (C + D)\varphi(s) - w \right| \le \chi(s, s), \quad s \in X,\tag{38}$$

and consequently

$$\left| (C + D)^{-1} \varphi\left(\rho(s)\right) - (C + D)^{-1} w - \varphi(s) \right| \le |C + D|^{-1} \chi(s, s), \quad s \in X.\tag{39}$$

Hence (21) and (23) hold with $\xi = \rho$ and

$$F(t, v) := (C + D)^{-1}(v - w), \qquad \varepsilon(t) := |C + D|^{-1}\chi(t, t), \qquad \lambda(t) := |C + D|^{-1} \qquad (40)$$

for $v \in Y$, $t \in X$. Now, if $S : X \to Y$ is defined by (22), then it is easily seen that (33) holds. Consequently, according to Corollary 2, there exists a unique solution $\Phi \in Y^X$ of the equation

$$\varphi\big(\rho(s)\big) = (C + D)\varphi(s) + w \qquad (41)$$

such that condition (36) is fulfilled. Moreover, Φ is given by (37).

Now we show that Φ is a solution of equation (35). To this end, fix $s, t \in X$ and $n \in \mathbb{N}_0$. Then $\rho^n(s \star t) = \rho^n(s) \star \rho^n(t)$, whence by (32),

$$\big|C\mathcal{T}^n\varphi(s) + D\mathcal{T}^n\varphi(t) + w - \mathcal{T}^n\varphi(s \star t)\big|$$

$$= \left| C\left((C + D)^{-n}\varphi\big(\rho^n(s)\big) - \sum_{i=1}^{n}(C + D)^{-i}w \right) \right.$$

$$+ D\left((C + D)^{-n}\varphi\big(\rho^n(t)\big) - \sum_{i=1}^{n}(C + D)^{-i}w \right) + w$$

$$\left. - \left((C + D)^{-n}\varphi\big(\rho^n(s \star t)\big) - \sum_{i=1}^{n}(C + D)^{-i}w \right) \right|$$

$$\leq |C + D|^{-n}\big|C\varphi\big(\rho^n(s)\big) + D\varphi\big(\rho^n(t)\big) + w - \varphi\big(\rho^n(s) \star \rho^n(t)\big)\big|$$

$$\leq |C + D|^{-n}\chi\big(\rho^n(s), \rho^n(t)\big).$$

This means that Φ satisfies (35) (in view of (34) and (37)).

Since $\Phi \in X^S$ is the unique solution of functional equation (41) fulfilling (36), and every solution of (35) satisfies (41), we also obtain the statement concerning the uniqueness of Φ. □

Corollary 7 *Let $C, D \in \mathbb{R}$, $C \neq -D$, $\varphi : X \to Y$, $\chi_0 : X^2 \to \mathbb{R}_+$, $w \in Y$, $\star : X^2 \to X$ be a square symmetric binary operation, let $\rho : X \to X$ be given by $\rho(t) := t \star t$ for $t \in X$,*

$$\big\|\varphi(s \star t) - C\varphi(s) - D\varphi(t) - w\big\|_u \leq \chi_0(s, t), \quad s, t \in X, \qquad (42)$$

$$S_0(t) := \sum_{n=0}^{\infty} \frac{\chi_0(\rho^n(t), \rho^n(t))}{|C + D|^{n+1}} < \infty, \quad t \in X \qquad (43)$$

and

$$\lim_{n \to \infty} \frac{\chi_0(\rho^n(s), \rho^n(t))}{|C + D|^n} = 0, \quad s, t \in X. \qquad (44)$$

Then there is a unique solution $\Phi \in Y^X$ of equation (35) such that

$$\big\|\varphi(t) - \Phi(t)\big\|_u \leq S_0(t), \quad t \in X. \qquad (45)$$

Moreover, Φ is given by (37).

Proof We argue as in the proof of Corollary 1. Namely, define $S : X \to Y$ and $\chi : X^2 \to Y$ by $S(t) := S_0(t)u$ and $\chi(t) := \chi_0(t)u$ for $t \in X$. Then (42)-(44) imply (32)-(34). Hence, by Corollary 6, there exists a unique solution $\Phi \in Y^X$ of functional equation (35) such that (36) holds; in particular, Φ is given by (37). Clearly (36) and (45) are equivalent, so we also obtain the statement on the uniqueness of Φ. □

In the very particular case where $C = D = 1$ and functions χ and χ_0 are constant, Corollaries 6 and 7 yield analogues of the classical result of Hyers [27].

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