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# Existence of solutions for nonlinear second-order *q*-difference equations with first-order *q*-derivatives

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# Abstract

In this paper, we establish the existence of solutions for a boundary value problem with the nonlinear second-order *q*-difference equation

 $\begin{cases} D_q^2 u(t) = f(t, u(t), D_q u(t)), & t \in I, \\ D_q u(0) = 0, & D_q u(1) = \alpha u(1). \end{cases}$ 

The existence and uniqueness of solutions for the problem are proved by means of the Leray-Schauder nonlinear alternative and some standard fixed point theorems. Finally, we give two examples to demonstrate the use of the main results. The nonlinear team f contains  $D_a u(t)$  in the equation.

**Keywords:** *q*-difference equations; Leray-Schauder nonlinear alternative; boundary value problem; fixed point theorem

# **1** Introduction

In this paper, we study the existence of solutions for a boundary value problem with nonlinear second-order *q*-difference equations

$$\begin{cases} D_q^2 u(t) = f(t, u(t), D_q u(t)), & t \in I, \\ D_q u(0) = 0, & D_q u(1) = \alpha u(1), \end{cases}$$
(1.1)

where  $f \in C(I \times R^2, R)$ ,  $I = \{q^n : n \in N\} \cup \{0, 1\}$ ,  $q \in (0, 1)$ , and  $\alpha \neq 0$  is a fixed real number.

The *q*-difference equations initiated at the beginning of the twentieth century [1-4] is a very interesting field in difference equations. In the last few decades, it has evolved into a multidisciplinary subject and plays an important role in several fields of physics such as cosmic strings and black holes [5], conformal quantum mechanics [6], nuclear and high energy physics [7]. However, the theory of boundary value problems (BVPs) for nonlinear *q*-difference equations is still in the initial stages and many aspects of this theory need to be explored. To the best of our knowledge, for the BVPs of nonlinear *q*-difference equations, a few works were done; see [8–13] and the references therein. In particular, the study of BVPs for nonlinear *q*-difference equation with first-order *q*-difference is yet to be initiated.

The main aim of this paper is to develop some existence and uniqueness results for BVP (1.1). Our results are based on a variety of fixed point theorems such as the Banach



© 2013 Yu and Wang; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. contraction mapping principle, the Leray-Schauder nonlinear alternative and the Leray-Schauder continuous theorem. Some examples and special cases are also discussed.

# 2 Preliminary results

In this section, firstly, let us recall some basic concepts of *q*-calculus [14, 15].

**Definition 2.1** For 0 < q < 1, we define the *q*-derivative of a real-value function *f* as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

Note that  $\lim_{q\to 1^-} D_q f(t) = f'(t)$ .

**Definition 2.2** The higher-order *q*-derivatives are defined inductively as

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in N.$$

For example,  $D_q(t^k) = [k]_q t^{k-1}$ , where k is a positive integer and the bracket  $[k]_q = (q^k - 1)/(q-1)$ . In particular,  $D_q(t^2) = (1+q)t$ .

**Definition 2.3** The *q*-integral of a function *f* defined in the interval [*a*, *b*] is given by

$$\int_a^x f(t) d_q t := \sum_{n=0}^\infty x(1-q)q^n f\left(xq^n\right) - af\left(q^n a\right), \quad x \in [a,b],$$

and for a = 0, we denote

$$I_{q}f(x) = \int_{0}^{x} f(t) d_{q}t = \sum_{n=0}^{\infty} x(1-q)q^{n}f(xq^{n}).$$

Then

$$\int_{a}^{b} f(t) \, d_{q}t = \int_{0}^{b} f(t) \, d_{q}t - \int_{0}^{a} f(t) \, d_{q}t.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \qquad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in N.$$

Observe that

$$D_q I_q f(x) = f(x),$$

and if *f* is continuous at x = 0, then  $I_q D_q f(x) = f(x) - f(0)$ .

In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = D_q g(t)h(t) + g(qt)D_q h(t),$$
(2.1)

$$\int_{0}^{x} f(t) D_{q} g(t) d_{q} t = \left[ f(t) g(t) \right]_{0}^{x} - \int_{0}^{x} D_{q} f(t) g(qt) d_{q} t.$$
(2.2)

**Remark 2.4** In the limit  $q \rightarrow 1^-$ , the above results correspond to their counterparts in standard calculus.

**Definition 2.5**  $f: I \times R^2 \rightarrow R$  is called an S-Carathéodory function if and only if

- (i) for each  $(u, v) \in \mathbb{R}^2$ ,  $t \mapsto f(t, u, v)$  is measurable on *I*;
- (ii) for a.e.  $t \in I$ ,  $(u, v) \mapsto f(t, u, v)$  is continuous on  $\mathbb{R}^2$ ;
- (iii) for each r > 0, there exists  $\varphi_r(t) \in L^1(I, R^+)$  with  $t\varphi_r(t) \in L^1(I, R^+)$  on I such that  $\max\{|u|, |v|\} \le r$  implies  $|f(t, u, v)| \le \varphi_r(t)$ , for a.e. I, where  $L^1(I, R^+) = \{u \in C_q : \int_0^1 u(t) d_q t$  exists}, and normed by  $||u||_{L^1} = \int_0^1 |u(t)| d_q t$  for all  $u \in L^1(I, R^+)$ .

**Theorem 2.6** (Nonlinear alternative for single-valued maps [16]) Let *E* be a Banach space, let *C* be a closed and convex subset of *E*, and let *U* be an open subset of *C* and  $0 \in U$ . Suppose that  $F : \overline{U} \to C$  is a continuous, compact (that is,  $F(\overline{U})$  is a relatively compact subset of *C*) map. Then either

- (i) *F* has a fixed point in  $\overline{U}$ , or
- (ii) there is a  $u \in \partial U$  (the boundary of U in C) and  $\lambda \in (0,1)$  with  $u = \lambda F(u)$ .

**Lemma 2.7** Let  $y \in C[0,1]$ , then the BVP

$$\begin{cases} D_q^2 u(t) = y(t), & t \in I, \\ D_q u(0) = 0, & D_q u(1) = \alpha u(1), \end{cases}$$
(2.3)

has a unique solution

$$u(t) = \int_0^t (t - qs)y(s) \, d_q s + \int_0^1 \left(\frac{1}{\alpha} - 1 + qs\right) y(s) \, d_q s$$
  
=  $\int_0^1 G(t, s; q)y(s) \, d_q s,$  (2.4)

where

$$G(t,s;q) = \frac{1}{\alpha} \begin{cases} 1 - \alpha + \alpha t, & s \le t, \\ 1 - \alpha + \alpha qs, & t \le s. \end{cases}$$
(2.5)

*Proof* Integrating the *q*-difference equation from 0 to *t*, we get

$$D_{q}u(t) = \int_{0}^{t} y(s) d_{q}s + a_{1}.$$
(2.6)

Integrating (2.6) from 0 to t and changing the order of integration, we have

$$u(t) = \int_0^t (t - qs)y(s) \, d_q s + a_1 t + a_0, \tag{2.7}$$

where  $a_1$ ,  $a_0$  are arbitrary constants. Using the boundary conditions  $D_q u(0) = 0$ ,  $D_q u(1) = \alpha u(1)$  in (2.7), we find that  $a_1 = 0$ , and

$$a_1 = \int_0^1 \left(\frac{1}{\alpha} - 1 + qs\right) y(s) \, d_q s.$$

Substituting the values of  $a_0$  and  $a_1$  in (2.7), we obtain

$$u(t) = \int_0^t (t - qs)y(s) \, d_q s + \int_0^1 \left(\frac{1}{\alpha} - 1 + qs\right)y(s) \, d_q s.$$

This completes the proof.

**Remark 2.8** For  $q \rightarrow 1$ , equation (2.4) takes the form

$$u(t) = \int_0^t (t - qs)y(s) \, d_q s + a_1 t + a_0,$$

which is the solution of a classical second-order ordinary differential equation u''(t) = y(t)and the associated form of Green's function for the classical case is

$$G(t,s) = \frac{1}{\alpha} \begin{cases} 1 - \alpha + \alpha t, & s \le t, \\ 1 - \alpha + \alpha s, & t \le s. \end{cases}$$

We consider the Banach space  $C_q = C(I, R)$  equipped with the standard norm  $||u|| = \max\{||u||_{\infty}, ||D_qu||_{\infty}\}$ , and  $|| \cdot ||_{\infty} = \sup\{|| \cdot ||, t \in I\}$ ,  $u \in C_q$ .

Define an integral operator  $T: C_p \to C_p$  by

$$Tu(t) = \int_{0}^{1} G(t,s;q) f(s,u(s), D_{q}u(s)) d_{q}s$$
  
=  $\int_{0}^{t} (t-qs) f(s,u(s), D_{q}u(s)) d_{q}s$   
+  $\int_{0}^{1} \left(\frac{1}{\alpha} - 1 + qs\right) f(s,u(s), D_{q}u(s)) d_{q}s, \quad t \in I, u \in C_{q}.$  (2.8)

Obviously, *T* is well defined and  $u \in C_q$  is a solution of BVP (1.1) if and only if *u* is a fixed point of *T*.

# 3 Existence and uniqueness results

In this section, we apply various fixed point theorems to BVP (1.1). First, we give the uniqueness result based on Banach's contraction principle.

**Theorem 3.1** Let  $f : I \times R^2 \to R$  be a continuous function, and there exists  $L_1(t), L_2(t) \in C([0,1], [0, +\infty))$  such that

$$\left|f(t, u_1, v_1) - f(t, u_2, v_2)\right| \le L_1(t)|u_1 - u_2| + L_2(t)|v_1 - v_2|, \quad t \in I, (u_1, v_1), (u_2, v_2) \in \mathbb{R}^2.$$

In addition, suppose either

(H<sub>1</sub>) 
$$\Lambda < |\alpha|$$
 for  $0 < |\alpha| < 1$ , or

(H<sub>2</sub>)  $\Lambda < 1$  for  $|\alpha| \ge 1$ 

holds, where  $\Lambda = \max_{t \in [0,1]} \{L_1(t) + L_2(t)\}$ . Then BVP (1.1) has a unique solution.

*Proof* Case 1:  $|\alpha| < 1$ . Let us set  $\sup_{t \in I} |f(t, 0, 0)| = M_0$  and choose

$$r \ge \frac{M_0}{|\alpha|(1-\delta)},\tag{3.1}$$

where  $\delta$  is such that  $\frac{\Lambda}{|\alpha|} \leq \delta \leq 1$ . Now we show that  $TB_r \subset B_r$ , where  $B_r = \{u \in C_q : ||u|| \leq r\}$ . For each  $u \in B_r$ , we have

$$\begin{split} |Tu(t)| &\leq \sup_{t \in I} \left| \int_{0}^{t} (t - qs) f(s, u(s), D_{q}u(s)) d_{q}s + \int_{0}^{1} \left(\frac{1}{\alpha} - 1 + qs\right) f(s, u(s), D_{q}u(s)) d_{q}s \\ &\leq \sup_{t \in I} \left| \int_{0}^{t} (t - qs) \left( \left| f(s, u(s), D_{q}u(s)\right) - f(s, 0, 0) \right| + \left| f(s, 0, 0) \right| \right) d_{q}s \\ &+ \int_{0}^{1} \left(\frac{1}{\alpha} - 1 + qs\right) \left( \left| f(s, u(s), D_{q}u(s)\right) - f(s, 0, 0) \right| + \left| f(s, 0, 0) \right| \right) d_{q}s \right| \\ &\leq \sup_{t \in I} \left| \int_{0}^{t} (t - qs) (L_{1}(s) |u(s)| + L_{2}(s) |D_{q}u(s)| + |f(s, 0, 0)|) d_{q}s \\ &+ \int_{0}^{1} \left(\frac{1}{\alpha} - 1 + qs\right) (L_{1}(s) |u(s)| + L_{2}(s) |D_{q}u(s)| + |f(s, 0, 0)|) d_{q}s \\ &\leq (\Lambda ||u|| + M_{0}) \sup_{t \in I} \left| \int_{0}^{t} (t - qs) d_{q}s + \int_{0}^{1} \left(\frac{1}{\alpha} - 1 + qs\right) d_{q}s \right| \\ &\leq (\Lambda ||u|| + M_{0}) \sup_{t \in I} \left| \left| \frac{t^{2}}{1 + q} + \frac{1}{\alpha} - 1 + \frac{q}{1 + q} \right| \right\} \\ &\leq (\Lambda ||u|| + M_{0}) \frac{1}{|\alpha|} \leq (\Lambda r + M_{0}) \frac{1}{|\alpha|} \leq \left(\frac{\Lambda}{|\alpha|} + (1 - \delta)\right) r \leq r, \end{split}$$

and

$$\begin{aligned} |D_q Tu(t)| &\leq |D_q Tu(t)| \leq \sup_{t \in I} \left| \int_0^t f(s, u(s), D_q u(s)) d_q s \right| \\ &\leq \sup_{t \in I} \int_0^t (|f(s, u(s), D_q u(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) d_q s \\ &\leq \sup_{t \in I} \int_0^t (L_1(s)|u(s)| + L_2(s)|D_q u(s)| + |f(s, 0, 0)|) d_q s \\ &\leq (\Lambda ||u|| + M_0) \leq (\Lambda r + |\alpha|(1 - \delta)r) \leq \left(\frac{\Lambda}{|\alpha|} + (1 - \delta)\right)r \leq r. \end{aligned}$$

Hence, we obtain that  $||Tu|| \leq r$ , so  $TB_r \subset B_r$ .

Now, for  $u, v \in C_q$  and for each  $t \in I$ , we have

$$\begin{aligned} |Tu(t) - Tv(t)| &\leq \sup_{t \in I} |Tu(t) - Tv(t)| \\ &\leq \sup_{t \in I} \left| \int_0^t (t - qs) |f(s, u(s), D_q u(s)) - f(s, v(s), D_q v(s))| \, d_q s \right| \\ &+ \int_0^1 \left( \frac{1}{\alpha} - 1 + qs \right) |f(s, u(s), D_q u(s)) - f(s, v(s), D_q v(s))| \, d_q s | \\ &\leq \sup_{t \in I} \left| \int_0^t (t - qs) (L_1(s) |u(s) - v(s)| + L_2(s) |D_q u(s) - D_q v(s)|) \, d_q s \right| \end{aligned}$$

$$+ \int_{0}^{1} \left( \frac{1}{\alpha} - 1 + qs \right) \left( L_{1}(s) | u(s) - v(s) | + L_{2}(s) | D_{q}u(s) - D_{q}v(s) | \right) d_{q}s$$
  
$$\leq \Lambda \sup_{t \in I} \left\{ \left| \frac{t^{2}}{1 + q} + \frac{1}{\alpha} - 1 + \frac{q}{1 + q} \right| \right\} \| u - v \|$$
  
$$\leq \frac{\Lambda}{|\alpha|} \| u - v \| < \| u - v \|,$$

and

$$\begin{aligned} \left| D_q T u(t) - D_q T v(t) \right| &\leq \sup_{t \in I} \left| D_q T u(t) - D_q T v(t) \right| \\ &\leq \sup_{t \in I} \left| \int_0^t \left| f\left(s, u(s), D_q u(s)\right) - f\left(s, v(s), D_q v(s)\right) \right| d_q s \right| \\ &\leq \sup_{t \in I} \left| \int_0^t \left( L_1(s) \left| u(s) - v(s) \right| + L_2(s) \left| D_q u(s) - D_q v(s) \right| \right) d_q s \right| \\ &\leq \Lambda \| u - v \| \leq \frac{\Lambda}{|\alpha|} \| u - v \| < \| u - v \|. \end{aligned}$$

Therefore, we obtain that ||Tu - Tv|| < ||u - v||, so *T* is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle.

Case 2:  $|\alpha| \ge 1$ . It is similar to the proof of case 1. This completes the proof of Theorem 3.1.

**Corollary 3.2** Assume that  $f : I \times R^2 \to R$  is a continuous function and there exist two positive constants  $L_1$ ,  $L_2$  such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le L_1 |u_1 - u_2| + L_2 |v_1 - v_2|, \quad t \in I, (u_1, v_1), (u_2, v_2) \in \mathbb{R}^2.$$

In addition, suppose either

 $\begin{array}{l} ({\rm H}_3) \ \ L_1 + L_2 < |\alpha| \ for \ 0 < |\alpha| < 1, \ or \\ ({\rm H}_4) \ \ L_1 + L_2 < 1 \ for \ |\alpha| \geq 1 \end{array}$ 

holds. Then BVP (1.1) has a unique solution.

**Corollary 3.3** Assume that  $f : I \times R^2 \to R$  is a continuous function and there exist two functions  $L_1(t), L_2(t) \in L^1(I, R^+)$  such that

 $|f(t, u_1, v_1) - f(t, u_2, v_2)| \le L_1 |u_1 - u_2| + L_2 |v_1 - v_2|, \quad t \in I, (u_1, v_1), (u_2, v_2) \in \mathbb{R}^2.$ 

In addition, suppose either

(H<sub>5</sub>)  $A + Bq|\alpha| < |\alpha|$  for  $0 < |\alpha| < 1$ , or (H<sub>6</sub>) A < 1 for  $|\alpha| \ge 1$ 

holds, where

$$A = \int_0^1 [L_1(s) + L_2(s)] d_q s, \qquad B = \int_0^1 s [L_1(s) + L_2(s)] d_q s.$$

Then BVP (1.1) has a unique solution.

Proof It is similar to the proof of Theorem 3.1.

The next existence result is based on the Leray-Schauder nonlinear alternative theorem.

**Lemma 3.4** Let  $f : I \times R^2 \to R$  be an S-Carathéodory function. Then  $T : C_q \to C_q$  is completely continuous.

*Proof* The proof consists of several steps.

(i) *T* maps bounded sets into bounded sets in  $C_q$ . Let  $B_r = \{u \in C_q : ||u|| \le r\}$  be a bounded set in  $C_q$  and  $u \in B_r$ . Then we have

$$\begin{aligned} \left| Tu(t) \right| &\leq \int_0^t \left| t - qs \right| \left| f\left(s, u(s), D_q u(s)\right) \right| d_q s + \int_0^1 \left| \frac{1}{\alpha} - 1 + qs \right| \left| f\left(s, u(s), D_q u(s)\right) \right| d_q s \\ &\leq \left( 1 + \frac{1}{|\alpha|} \right) \int_0^1 \varphi_r(s) d_q s = \left( 1 + \frac{1}{|\alpha|} \right) \|\varphi_r\|_{L^1}, \end{aligned}$$

and

$$\left|D_{q}Tu(t)\right| \leq \int_{0}^{t} \left|f\left(s,u(s),D_{q}u(s)\right)\right| d_{q}s \leq \int_{0}^{1} \varphi_{r}(s) d_{q}s = \|\varphi_{r}\|_{L^{1}}.$$

Thus  $||Tu|| \le \max\{||Tu||_{\infty}, ||D_qTu||_{\infty}\} \le (1 + \frac{1}{|\alpha|}) ||\varphi_r||_{L^1}.$ 

(ii) *T* maps bounded sets into equicontinuous sets of  $C_q$ .

Let  $r_1, r_2 \in I$ ,  $r_1 < r_2$ , and let  $B_r$  be a bounded set of  $C_q$  as before. Then, for  $u \in B_r$ , we have

$$\begin{aligned} \left| Tu(r_2) - Tu(r_1) \right| &= \left| \int_0^{r_2} (r_2 - qs) f(s, u(s), D_q u(s)) d_q s \right| \\ &= \left| \int_0^{r_1} (r_1 - qs) f(s, u(s), D_q u(s)) d_q s \right| \\ &= \left| \int_0^{r_1} (r_2 - r_1) f(s, u(s), D_q u(s)) d_q s \right| \\ &+ \int_{r_1}^{r_2} (r_2 - qs) f(s, u(s), D_q u(s)) d_q s \right| \\ &\leq \int_0^{r_1} |r_2 - r_1| \varphi_r(s) d_q s + \int_{r_1}^{r_2} |r_2 - qs| \varphi_r(s) d_q s \to 0 \\ &(r_2 - r_1 \to 0), \end{aligned}$$

and

$$\begin{aligned} \left| d_{q}Tu(r_{2}) - D_{q}Tu(r_{1}) \right| &= \left| \int_{0}^{r_{2}} f\left(s, u(s), D_{q}u(s)\right) d_{q}s - \int_{0}^{r_{1}} f\left(s, u(s), D_{q}u(s)\right) d_{q}s \right| \\ &= \left| \int_{r_{1}}^{r_{2}} f\left(s, u(s), D_{q}u(s)\right) d_{q}s \right| \le \int_{r_{1}}^{r_{2}} \varphi_{r}(s) d_{q}s \to 0 \\ &(r_{2} - r_{1} \to 0). \end{aligned}$$

As a consequence of the Arzelá-Ascoli theorem, we can conclude that  $T: C_q \to C_q$  is completely continuous. This proof is completed.

**Theorem 3.5** Let  $f : I \times R^2 \to R$  be an S-Carathéodory function. Suppose further that there exists a real number M > 0 such that

$$\frac{|\alpha|M}{(1+|\alpha|)\|\varphi_r\|_{L^1}}>1$$

holds, where

$$\|\varphi_r\|_{L^1} = \int_0^1 \varphi_r(s) \, d_q s \neq 0.$$

Then BVP (1.1) has at least one solution.

*Proof* In view of Lemma 3.4, we obtain that  $T : C_q \to C_q$  is completely continuous. Let  $\lambda \in (0, 1)$  and  $u = \lambda T u$ . Then, for  $t \in I$ , we have

$$\begin{split} u(t) &| = \left| \lambda T u(t) \right| \\ &\leq \int_0^t \left| t - qs \right| \left| f\left(s, u(s), D_q u(s)\right) \right| d_q s \\ &+ \int_0^1 \left| \frac{1}{\alpha} - 1 + qs \right| \left| f\left(s, u(s), D_q u(s)\right) \right| d_q s \\ &\leq \left( 1 + \frac{1}{|\alpha|} \right) \int_0^1 \varphi_r(s) d_q s = \left( 1 + \frac{1}{|\alpha|} \right) \|\varphi_r\|_{L^1} \end{split}$$

and

$$\left|D_{q}u(t)\right| = \left|D_{q}\lambda Tu(t)\right| \leq \int_{0}^{t} \left|f\left(s, u(s), D_{q}u(s)\right)\right| d_{q}s \leq \int_{0}^{1} \varphi_{r}(s) d_{q}s = \|\varphi_{r}\|_{L^{1}}.$$

Hence, consequently,

$$\frac{|\alpha| \|u\|}{(1+|\alpha|) \|\varphi_r\|_{L^1}} \le 1.$$

Therefore, there exists M > 0 such that  $||u|| \neq M$ . Let us set  $U = \{u \in C_q : ||u|| < M\}$ . Note that the operator  $T : \overline{U} \to C_q$  is completely continuous (which is known to be compact restricted to bounded sets). From the choice of U, there is no  $U \in \partial U$  such that  $u = \lambda T u$  for some  $\lambda \in (0,1)$ . Consequently, by Theorem 2.6, we deduce that T has a fixed point  $u \in \overline{U}$  which is a solution of problem (1.1). This completes the proof.

The next existence result is based on the Leray-Schauder continuation theorem.

**Theorem 3.6** Let  $f : I \times R^2 \to R$  be an S-Carathéodory function. Suppose further that there exist functions  $p(t), q(t), r(t) \in L^1(I, R^+)$  with  $tp(t) \in L^1(I, R^+)$  such that

$$|f(t, u, v)| \le p(t)|u| + q(t)|v| + r(t)$$
, for a.e.  $t \in I$  and  $(u, v) \in \mathbb{R}^2$ .

Then BVP (1.1) has at least one solution provided  $(N+1)P + P_1 + Q < 1$ , where  $N = \max\{\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\}$ .

*Proof* We consider the space

$$P = \left\{ u \in C_q : D_q u(0) = 0, D_q u(1) = \alpha u(1) \right\}$$

and define the operator  $T_1: P \times [0,1] \rightarrow P$  by

$$T_1(u,\lambda) = \lambda T u = \lambda \int_0^1 G(t,s;q) f\left(s,u(s), D_q u(s)\right) d_q s, \quad t \in I.$$
(3.2)

Obviously, we can see that  $P \subset C_q$ . In view of Lemma 3.4, it is easy to know that for each  $\lambda \in [0,1]$ ,  $T_1(u,\lambda)$  is completely continuous in P. It is clear that  $u \in P$  is a solution of BVP (1.1) if and only if u is a fixed point of  $T_1(\cdot, 1)$ . Clearly,  $T_1(u, 0) = 0$  for each  $u \in P$ . If for each  $\lambda \in [0,1]$  the fixed points of  $T_1(\cdot, 1)$  in P belong to a closed ball of P independent of  $\lambda$ , then the Leray-Schauder continuation theorem completes the proof.

Next we show that the fixed point of  $T_1(\cdot, 1)$  has *a priori* bound *M*, which is independent of  $\lambda$ . Assume that  $u = T_1(u, \lambda)$ , and set

$$P = \int_0^1 p(s) \, d_q s, \qquad P_1 = \int_0^1 s p(s) \, d_q s,$$
$$Q = \int_0^1 q(s) \, d_q s, \qquad R = \int_0^1 r(s) \, d_q s.$$

By (2.5), it is clear that  $|G(t, s; q)| \le N$  for each  $\alpha \ne 0$ . For any  $u \in P$ , we have

$$\begin{aligned} \left| u(t) \right| &= \left| u(1) - \int_t^1 D_q u(s) \, d_q s \right| \le \left| \frac{1}{\alpha} D_q u(1) \right| + \left| \int_t^1 D_q u(s) \, d_q s \right| \\ &\le (N+1-t) \|D_q u\|_{\infty} \le (N+1+t) \|D_q u\|_{\infty}, \quad t \in I, \end{aligned}$$

and so it holds that

$$\|D_{q}u\|_{\infty} \leq \|\lambda f(s, u(s), D_{q}u(s))\|_{L^{1}} \leq \|f(s, u(s), D_{q}u(s))\|_{L^{1}}$$
  
$$\leq \|p(t)|u(s)| + q(t)|D_{q}u(s)| + r(s)\|_{L^{1}}$$
  
$$\leq ((N+1)P + P_{1} + Q)\|D_{q}u\|_{\infty} + R;$$

therefore,

$$||D_q u||_{\infty} \le \frac{R}{1 - ((N+1)P + P_1 + Q)} := M_1.$$

At the same time, we have

$$\begin{aligned} \left| u(t) \right| &\leq \lambda \left| \int_0^1 G(t,s,q) f\left(s,u(s),D_q u(s)\right) d_q s \right| \\ &\leq N \int_0^1 \left| f\left(s,u(s),D_q u(s)\right) \right| d_q s \end{aligned}$$

$$\leq N \int_0^1 (p(t)|u(s)| + q(t)|D_qu(s)| + r(s)) d_q s$$
  
$$\leq NP \|u\|_{\infty} + NQ \|D_qu\|_{\infty} + NR, \quad t \in I,$$

and so

$$\|u\|_{\infty} \leq \frac{N(QM_1+R)}{1-NP} := M_2.$$

Set  $M = \max{\{M_1, M_2\}}$ , which is independent of  $\lambda$ . So, BVP (1.1) has at least one solution. This completes the proof.

## 4 Example

**Example 4.1** Consider the following BVP:

$$\begin{cases} D_q^2 u(t) = e^t + \frac{1}{4} \sin(u(t)) + \frac{1}{8} \tan^{-1}(D_q u(t)), & t \in I, \\ D_q u(0) = 0, & D_q u(1) = \frac{1}{2} u(1). \end{cases}$$
(4.1)

Here,  $f(t, u(t), D_q u(t)) = e^t + \frac{1}{4} \sin(u(t)) + \frac{1}{8} \tan^{-1}(D_q u(t)), q = \frac{1}{2}, \alpha = \frac{1}{2}$ . Clearly,  $|f(t, u_1, v_1) - f(t, u_2, v_2)| \le \frac{1}{4} |u_1 - u_2| + \frac{1}{8} |v_1 - v_2|$ . Then  $L_1 = \frac{1}{4}, L_2 = \frac{1}{8}$  and  $L_1 + L_2 = \frac{3}{8} < \alpha$ . By Corollary 3.2, we obtain that BVP (4.1) has a unique solution.

**Example 4.2** Consider the following BVP:

$$\begin{cases} D_q^2 u(t) = \frac{t}{20} \sin(u(t)) + \frac{1}{(2+\sqrt{2})\sqrt{t}} (D_q u(t))^{\beta} + \frac{3}{2}t, \quad t \in I, \\ D_q u(0) = 0, \qquad D_q u(1) = \frac{1}{2} u(1). \end{cases}$$
(4.2)

Here,  $q = \frac{1}{2}, 0 < \beta < 1$ . It is obvious that  $|f(t, u, v)| \le \frac{t}{20}|u| + \frac{1}{(2+\sqrt{2})\sqrt{t}}|v| + \frac{3}{2}t$ , where  $p(t) = \frac{t}{20}$ ,  $q(t) = \frac{1}{(2+\sqrt{2})\sqrt{t}}, r(t) = \frac{3}{2}t$ . Then  $P = \frac{1}{30}, P_1 = \frac{1}{35}, Q = \frac{1}{2}, R = 1$ , so  $(N + 1)P + P_1 + Q = \frac{22}{35} < 1$ . By Theorem 3.6, we obtain that BVP (4.2) has at least one solution.

### **Competing interests**

The authors declare that they have no competing interests.

### Authors' contributions

Each of the authors, CLY and JFW contributed to each part of this work equally and read and approved the final version of the manuscript.

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