Mahmudov and Keleshteri *Advances in Difference Equations* 2013, **2013**:115
http://www.advancesindifferenceequations.com/content/2013/1/115

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

On a class of generalized q-Bernoulli and q-Euler polynomials

Nazim I Mahmudov* and M Eini Keleshteri

*Correspondence:
nazim.mahmudov@emu.edu.tr
Eastern Mediterranean University,
Gazimagusa, T.R. North Cyprus,
via Mersiin 10, Turkey

Abstract

The main purpose of this paper is to introduce and investigate a new class of generalized q-Bernoulli and q-Euler polynomials. The q-analogues of well-known formulas are derived. A generalization of the Srivastava-Pintér addition theorem is obtained.

1 Introduction

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \neq 1); \qquad [0]_q! = 1; \qquad [n]_q! = [1]_q[2]_q \cdots [n]_q, \quad n \in \mathbb{N}, a \in \mathbb{C},$$

respectively. The q-polynomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0.$$

The q-binomial formula is known as

$$(1-a)_q^n = \prod_{j=0}^{n-1} (1-q^j a) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k.$$

In the standard approach to the q-calculus, two exponential functions are used:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, |z| < \frac{1}{|1-q|},$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q)q^k z\right), \quad 0 < |q| < 1, z \in \mathbb{C}.$$

From this form, we easily see that $e_q(z)E_q(-z) = 1$. Moreover,

$$D_q e_q(z) = e_q(z), \qquad D_q E_q(z) = E_q(qz),$$

where D_q is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, 0 \neq z \in \mathbb{C}.$$

The above q-standard notation can be found in [1].

Carlitz firstly extended the classical Bernoulli and Euler numbers and polynomials, introducing them as q-Bernoulli and q-Euler numbers and polynomials [2–4]. There are numerous recent investigations on this subject by, among many other authors, Cenki *et al.* [5–7], Choi *et al.* [8] and [9], Kim *et al.* [10–13], Ozden and Simsek [14], Ryoo *et al.* [15], Simsek [16, 17] and [18], and Luo and Srivastava [19], Srivastava *et al.* [20], Mahmudov [21, 22].

Recently, Natalini and Bernardini [23], Bretti *et al.* [24], Kurt [25, 26], Tremblay *et al.* [27, 28] studied the properties of the following generalized Bernoulli and Euler polynomials:

$$\begin{aligned} &\left(\frac{t^m}{e^t - \sum_{k=0}^{m-1} \frac{t^k}{k!}}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x)\frac{t^n}{n!}, \\ &\left(\frac{t^m}{e^t + \sum_{k=0}^{m-1} \frac{t^k}{k!}}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x)\frac{t^n}{n!}, \quad \alpha \in \mathbb{C}, 1^{\alpha} := 1. \end{aligned} \tag{1}$$

Motivated by the generalizations in (1) of the classical Bernoulli and Euler polynomials, we introduce and investigate here the so-called generalized two-dimensional q-Bernoulli and q-Euler polynomials, which are defined as follows.

Definition 1 Let $q, \alpha \in \mathbb{C}$, $m \in \mathbb{N}$, $0 < |q| < 1$. The generalized two-dimensional q-Bernoulli polynomials $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!},$$

where $T_{m-1,q}(t) = \sum_{k=0}^{m-1} \frac{t^k}{[k]_q!}$.

Definition 2 Let $q, \alpha \in \mathbb{C}$, $0 < |q| < 1$, $m \in \mathbb{N}$. The generalized two-dimensional q-Euler polynomials $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating functions

$$\left(\frac{2^m}{e_q(t) + T_{m-1,q}(t)}\right)^{\alpha} e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}.$$

It is obvious that

$$\lim_{q\to1^-}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)=B_n^{[m-1,\alpha]}(x+y),$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}=\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,0),\qquad \lim_{q\to1^-}\mathfrak{B}_{n,q}^{[m-1,\alpha]}=B_n^{[m-1,\alpha]},$$

$$\lim_{q\to1^-}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)=E_n^{[m-1,\alpha]}(x+y),$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}=\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,0),\qquad \lim_{q\to1^-}\mathfrak{E}_{n,q}^{[m-1,\alpha]}=E_n^{[m-1,\alpha]},$$

$$\lim_{q\to1^-}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)=B_n^{[m-1,\alpha]}(x),\qquad \lim_{q\to1^-}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)=B_n^{[m-1,\alpha]}(y),$$

$$\lim_{q\to1^-}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)=E_n^{[m-1,\alpha]}(x),\qquad \lim_{q\to1^-}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)=E_n^{[m-1,\alpha]}(y).$$

Here $B_n^{[m-1,\alpha]}(x)$ and $E_n^{[m-1,\alpha]}(x)$ denote the generalized Bernoulli and Euler polynomials defined in (1). Notice that $B_n^{[m-1,\alpha]}(x)$ was introduced by Natalini [23], and $E_n^{[m-1,\alpha]}(x)$ was introduced by Kurt [25].

In fact Definitions 1 and 2 define two different types $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)$ of the generalized q-Bernoulli polynomials and two different types $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)$ of the generalized q-Euler polynomials. Both polynomials $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)$ ($\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)$) coincide with the classical higher-order generalized Bernoulli polynomials (Euler polynomials) in the limiting case $q\to1^-$.

2 Preliminaries and lemmas

In this section we provide some basic formulas for the generalized q-Bernoulli and q-Euler polynomials to obtain the main results of this paper in the next section. The following result is a q-analogue of the addition theorem for the classical Bernoulli and Euler polynomials.

Lemma 3 *For all $x,y\in\mathbb{C}$ we have*

$$\begin{aligned}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x+y)_q^{n-k},\\ \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{E}_{k,q}^{[m-1,\alpha]}(x+y)_q^{n-k},\end{aligned}\tag{2}$$

$$\begin{aligned}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)y^{n-k}\\ &=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)x^{n-k},\end{aligned}\tag{3}$$

$$\begin{aligned}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{(n-k)(n-k-1)/2}\mathfrak{E}_{k,q}^{[m-1,\alpha]}(x,0)y^{n-k}\\ &=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{E}_{k,q}^{[m-1,\alpha]}(0,y)x^{n-k}.\end{aligned}\tag{4}$$

In particular, setting $x = 0$ and $y = 0$ in (3) and (4), we get the following formulae for the generalized q-Bernoulli and q-Euler polynomials, respectively,

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]} x^{n-k},$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{B}_{k,q}^{[m-1,\alpha]} y^{n-k},$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]} x^{n-k},$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{E}_{k,q}^{[m-1,\alpha]} y^{n-k}.$$

Setting $y = 1$ and $x = 1$ in (3) and (4), we get, respectively,

$$\begin{aligned} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,1) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0), \\ \mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y), \end{aligned} \tag{5}$$

$$\begin{aligned} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,1) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{E}_{k,q}^{(\alpha)}(x,0), \\ \mathfrak{E}_{n,q}^{[m-1,\alpha]}(1,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(0,y). \end{aligned} \tag{6}$$

Clearly, (5) and (6) are the generalization of q-analogues of

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x), \qquad E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x),$$

respectively.

Lemma 4 *The generalized q-Bernoulli and q-Euler polynomials satisfy the following relations*:

$$\mathfrak{B}_{n,q}^{[m-1,\alpha+\beta]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0) \mathfrak{B}_{k,q}^{[m-1,\beta]}(0,y),$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha+\beta]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(x,0) \mathfrak{E}_{k,q}^{[m-1,\beta]}(0,y).$$

Lemma 5 *We have*

$$D_{q,x} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = [n]_q \mathfrak{B}_{n-1,q}^{[m-1,\alpha]}(x,y), \qquad D_{q,y} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = [n]_q \mathfrak{B}_{n-1,q}^{[m-1,\alpha]}(x,qy),$$
$$D_{q,x} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = [n]_q \mathfrak{E}_{n-1,q}^{[m-1,\alpha]}(x,y), \qquad D_{q,y} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = [n]_q \mathfrak{E}_{n-1,q}^{[m-1,\alpha]}(x,qy).$$

Lemma 6 *The generalized q-Bernoulli and q-Euler polynomials satisfy the following relations*:

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y) = \frac{[n]_q!}{[n-m]_q!} \mathfrak{B}_{n-m,q}^{[m-1,\alpha-1]}(0,y), \quad n \geq m, \tag{7}$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(1,y) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y) = 2^m \mathfrak{E}_{n,q}^{[m-1,\alpha-1]}(0,y),$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,-1) = \frac{[n]_q!}{[n-m]_q!} \mathfrak{B}_{n-m,q}^{[m-1,\alpha-1]}(x,-1), \quad n \geq m,$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,-1) = 2^m \mathfrak{E}_{n,q}^{[m-1,\alpha-1]}(x,-1).$$

Proof We prove only (7). The proof is based on the following equality:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y) \right) \frac{t^n}{[n]_q!} \\
&\quad = \left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)} \right)^{\alpha} e_q(t) E_q(ty) - T_{m-1,q}(t) \left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \\
&\quad = \left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \bigl(e_q(t) - T_{m-1,q}(t) \bigr) \\
&\quad = t^m \left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)} \right)^{\alpha-1} E_q(ty) = \sum_{n=0}^{\infty} \frac{[n+m]_q!}{[n]_q!} \mathfrak{B}_{n,q}^{[m-1,\alpha-1]}(0,y) \frac{t^{n+m}}{[n+m]_q!}.
\end{aligned}$$

Here we used the following relation:

$$\begin{aligned}
&T_{m-1,q}(t) \left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \\
&\quad = \sum_{n=0}^{m-1} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \\
&\quad = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y) \left(\frac{t^n}{[n]_q!} + \frac{t^{n+1}}{[n]_q!} + \frac{t^{n+2}}{[n]_q![2]_q!} + \cdots + \frac{t^{n+m-1}}{[n]_q![m-1]_q!} \right) \\
&\quad = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty} [n]_q \mathfrak{B}_{n-1,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \\
&\qquad + \sum_{n=0}^{\infty} \frac{[n]_q [n-1]_q}{[2]_q!} \mathfrak{B}_{n-2,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \\
&\qquad + \cdots + \sum_{n=0}^{\infty} \frac{[n]_q \cdots [n-m+2]_q}{[m-1]_q!} \mathfrak{B}_{n-m+1,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \\
&\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!}.
\end{aligned}$$

□

Corollary 7 *Taking $q \to 1^-$, we have*

$$\mathfrak{B}_n^{[m-1,\alpha]}(y+1) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k}^{[m-1,\alpha]}(y) = \frac{[n]_q!}{[n-m]_q!} \mathfrak{B}_{n-m}^{[m-1,\alpha-1]}(y), \quad n \geq m,$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(y+1) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_n^{[m-1,\alpha]}(y) = 2^m \mathfrak{E}_n^{[m-1,\alpha-1]}(y).$$

Lemma 8 *The generalized q-Bernoulli polynomials satisfy the following relations*:

$$\begin{aligned} &\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y) \\ &\quad = [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y) \mathfrak{B}_{n-1-k,q}^{[0,-1]}. \end{aligned} \tag{8}$$

Proof Indeed,

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y) \right) \frac{t^n}{[n]_q!} \\ &\quad = \left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)} \right)^{\alpha} e_q(t) E_q(ty) - T_{m-1,q}(t) \left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \\ &\quad = \left(\frac{t^m}{e_q(t) - T_{m-1,q}(t)} \right)^{\alpha} E_q(ty) \frac{e_q(t) - T_{m-1,q}(t)}{t} t \\ &\quad = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[0,-1]} \frac{t^{n+1}}{[n]_q!} \\ &\quad = \sum_{n=1}^{\infty} [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y) \mathfrak{B}_{n-1-k,q}^{[0,-1]} \frac{t^n}{[n]_q!}. \end{aligned}$$

□

Remark 9 Notice taking limit in (8) as $q \to 1^-$, we get

$$\mathfrak{B}_n^{[m-1,\alpha]}(y+1) - \sum_{k=0}^{\min(n,m-1)} \binom{n}{k} \mathfrak{B}_{n-k}^{[m-1,\alpha]}(y) = n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathfrak{B}_k^{[m-1,\alpha]}(y) \mathfrak{B}_{n-1-k}^{[0,-1]}.$$

It is a correct form of formula (2.7) from [27] for $\lambda = 1$.

Lemma 10 *We have*

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} \mathfrak{B}_{n-k,q}^{[m-1,1]}(x,0), \qquad y^n = \frac{1}{q^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[k+m]_q!} \mathfrak{B}_{n-k,q}^{[m-1,1]}(0,y),$$

$$x^n = \frac{1}{2^m} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,1]}(x,0) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,1]}(x,0) \right),$$

$$y^n = \frac{1}{2^m q^{\frac{n(n-1)}{2}}} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,1]}(0,y) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n,q}^{[m-1,1]}(0,y) \right).$$

From Lemma 10 we obtain the list of generalized q-Bernoulli polynomials as follows

$$\mathfrak{B}_{0,q}^{[m-1,1]}(x,0) = [m]_q!, \qquad \mathfrak{B}_{0,q}^{[m-1,1]}(0,y) = [m]_q!,$$

$$\mathfrak{B}_{1,q}^{[m-1,1]}(x,0) = [m]_q!\left(x - \frac{1}{[m+1]_q}\right), \qquad \mathfrak{B}_{1,q}^{[m-1,1]}(0,y) = [m]_q!\left(y - \frac{1}{[m+1]_q}\right),$$

$$\mathfrak{B}_{2,q}^{[m-1,1]}(x,0) = x^2 - \frac{[2]_q[m]_q!}{[m+1]_q}x + \frac{[2]_q q^{m+1}[m]_q!}{[m+1]_q^2[m+2]_q},$$

$$\mathfrak{B}_{2,q}^{[m-1,1]}(0,y) = qy^2 - \frac{[2]_q[m]_q!}{[m+1]_q}y + \frac{[2]_q q^{m+1}[m]_q!}{[m+1]_q^2[m+2]_q}.$$

3 Explicit relationship between the q-Bernoulli and q-Euler polynomials

In this section, we give some generalizations of the Srivastava-Pintér addition theorem. We also obtain new formulae and their some special cases below.

We present natural q-extensions of the main results of the papers [29, 30].

Theorem 11 *The relationships*

$$\begin{aligned}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) &= \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left[\frac{1}{l^{n-k}}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0) + \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{1}{l^{k-j}}\mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0)\right]\mathfrak{E}_{n-k,q}(0,ly), \end{aligned} \tag{9}$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{l^{n-k}}\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y) + \mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\right]\mathfrak{E}_{n-k,q}(lx,0) \tag{10}$$

hold true between the generalized q-Bernoulli polynomials and q-Euler polynomials.

Proof First we prove (9). Using the identity

$$\begin{aligned}&\left(\frac{t^m}{e_q(t) - \sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha} e_q(tx)E_q(ty) \\ &\quad = \frac{2}{e_q(\frac{t}{l})+1}\cdot E_q\left(\frac{t}{l}ly\right)\cdot\frac{e_q(\frac{t}{l})+1}{2}\cdot\left(\frac{t^m}{e_q(t) - \sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha} e_q(tx),\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,ly)\frac{t^n}{l^n[n]_q!}\sum_{k=0}^{\infty}\frac{t^k}{l^k[k]_q!}\sum_{j=0}^{\infty}\mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0)\frac{t^j}{[j]_q!} \\ &\quad + \frac{1}{2}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!} \\ &=: I_1 + I_2.\end{aligned}$$

It is clear that

$$\begin{aligned}I_2 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!}\\&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\mathfrak{E}_{n-k,q}(0,ly)\frac{t^n}{[n]_q!}.\end{aligned}$$

On the other hand,

$$\begin{aligned}I_1 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!}\sum_{j=0}^{\infty}\frac{t^j}{l^j[j]_q!}\\&= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q\mathfrak{E}_{j,q}(0,ly)\frac{t^k}{l^k[k]_q!}\\&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\j\end{bmatrix}_q\frac{1}{l^{n-k}}\mathfrak{E}_{j,q}(0,ly)\frac{t^n}{[n]_q!}\\&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\mathfrak{E}_{j,q}(0,ly)\sum_{k=0}^{n-j}\begin{bmatrix}n-j\\k\end{bmatrix}_q\frac{1}{l^{n-k}}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!}\\&\quad= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left[\frac{1}{l^{n-k}}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)+\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q\frac{1}{l^{k-j}}\mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0)\right]\\&\qquad\times\mathfrak{E}_{n-k,q}(0,ly)\frac{t^n}{[n]_q!}.\end{aligned}$$

Next we prove (10). Using the identity

$$\begin{aligned}&\left(\frac{t^m}{e_q(t)-\sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha}e_q(tx)E_q(ty)\\&\quad= \frac{2}{e_q(\frac{t}{l})+1}\cdot e_q\left(\frac{t}{l}lx\right)\cdot\frac{e_q(\frac{t}{l})+1}{2}\cdot\left(\frac{t^m}{e_q(t)-\sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha}E_q(ty),\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(lx,0)\frac{t^n}{l^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\frac{t^n}{[n]_q!}\\&\quad+\frac{1}{2}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\frac{t^k}{l^k[k]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\frac{t^n}{[n]_q!}\\&=: I_1+I_2.\end{aligned}$$

It is clear that

$$\begin{aligned} I_2 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\frac{t^k}{l^k[k]_q!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)\mathfrak{E}_{n-k,q}(lx,0)\frac{t^n}{[n]_q!}. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_1 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\frac{t^k}{m^k[k]_q!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\mathfrak{E}_{n-k,q}(lx,0)\frac{t^n}{[n]_q!}. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!} \\ &\quad = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q l^{k-n}\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y) + \mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\right]\mathfrak{E}_{n-k,q}(lx,0)\frac{t^n}{[n]_q!}. \end{aligned}$$

□

Next we discuss some special cases of Theorem 11.

Theorem 12 *The relationship*

$$\begin{aligned} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) &= \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y) + \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q\mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y)\right. \\ &\quad \left. + [k]_q\sum_{j=0}^{k-1}\begin{bmatrix} k-1 \\ j \end{bmatrix}_q\mathfrak{B}_{j,q}^{[m-1,\alpha]}(0,y)\mathfrak{B}_{k-1-j,q}^{[0,-1]}\right]\mathfrak{E}_{n-k,q}(x,0) \end{aligned}$$

holds true between the generalized q-Bernoulli polynomials and the q-Euler polynomials.

Remark 13 Taking $q \to 1^-$ in Theorem 12, we obtain the Srivastava-Pintér addition theorem for the generalized Bernoulli and Euler polynomials.

$$\begin{aligned} \mathfrak{B}_{n}^{[m-1,\alpha]}(x+y) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[\mathfrak{B}_{k}^{[m-1,\alpha]}(y) + \sum_{k=0}^{\min(n,m-1)}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(y)\right. \\ &\quad \left. + k\sum_{j=0}^{k-1}\binom{k-1}{j}\mathfrak{B}_{j}^{[m-1,\alpha]}(y)\mathfrak{B}_{k-1-j}^{[0,-1]}\right]\mathfrak{E}_{n-k}(x). \end{aligned} \tag{11}$$

Notice that the Srivastava-Pintér addition theorem for the generalized Apostol-Bernoulli polynomials and the Apostol-Euler polynomials was given in [27]. The formula (11) is a correct version of Theorem 3 [27] for $\lambda = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgements
Dedicated to Professor Hari M Srivastava.
The authors would like to thank the reviewers for their valuable comments and helpful suggestions that improved the note's quality.

Received: 8 December 2012 Accepted: 4 March 2013 Published: 22 April 2013

References
1. Andrews, GE, Askey, R, Roy, R: Special Functions. Encyclopedia of Mathematics and Its Applications, vol. 71. Cambridge University Press, Cambridge (1999)
2. Carlitz, L: q-Bernoulli numbers and polynomials. Duke Math. J. **15**, 987-1000 (1948)
3. Carlitz, L: q-Bernoulli and Eulerian numbers. Trans. Am. Math. Soc. **76**, 332-350 (1954)
4. Carlitz, L: Expansions of q-Bernoulli numbers. Duke Math. J. **25**, 355-364 (1958)
5. Cenkci, M, Can, M: Some results on q-analogue of the Lerch zeta function. Adv. Stud. Contemp. Math. **12**, 213-223 (2006)
6. Cenkci, M, Can, M, Kurt, V: q-Extensions of Genocchi numbers. J. Korean Math. Soc. **43**, 183-198 (2006)
7. Cenkci, M, Kurt, V, Rim, SH, Simsek, Y: On (i,q)-Bernoulli and Euler numbers. Appl. Math. Lett. **21**, 706-711 (2008)
8. Choi, J, Anderson, PJ, Srivastava, HM: Some q-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n, and the multiple Hurwitz zeta function. Appl. Math. Comput. **199**, 723-737 (2008)
9. Choi, J, Anderson, PJ, Srivastava, HM: Carlitz's q-Bernoulli and q-Euler numbers and polynomials and a class of q-Hurwitz zeta functions. Appl. Math. Comput. **215**, 1185-1208 (2009)
10. Kim, T: Some formulae for the q-Bernoulli and Euler polynomial of higher order. J. Math. Anal. Appl. **273**, 236-242 (2002)
11. Kim, T: q-Generalized Euler numbers and polynomials. Russ. J. Math. Phys. **13**, 293-298 (2006)
12. Kim, T, Kim, YH, Hwang, KW: On the q-extensions of the Bernoulli and Euler numbers, related identities and Lerch zeta function. Proc. Jangjeon Math. Soc. **12**, 77-92 (2009)
13. Kim, T, Rim, SH, Simsek, Y, Kim, D: On the analogs of Bernoulli and Euler numbers, related identities and zeta and L-functions. J. Korean Math. Soc. **45**, 435-453 (2008)
14. Ozden, H, Simsek, Y: A new extension of q-Euler numbers and polynomials related to their interpolation functions. Appl. Math. Lett. **21**, 934-939 (2008)
15. Ryoo, CS, Seo, JJ, Kim, T: A note on generalized twisted q-Euler numbers and polynomials. J. Comput. Anal. Appl. **10**, 483-493 (2008)
16. Simsek, Y: q-Analogue of the twisted l-series and q-twisted Euler numbers. J. Number Theory **110**, 267-278 (2005)
17. Simsek, Y: Twisted (h,q)-Bernoulli numbers and polynomials related to twisted (h,q)-zeta function and L-function. J. Math. Anal. Appl. **324**, 790-804 (2006)
18. Simsek, Y: Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions. Adv. Stud. Contemp. Math. **16**, 251-278 (2008)
19. Luo, QM, Srivastava, HM: q-Extensions of some relationships between the Bernoulli and Euler polynomials. Taiwan. J. Math. **15**, 241-257 (2011)
20. Srivastava, HM, Kim, T, Simsek, Y: q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series. Russ. J. Math. Phys. **12**, 241-268 (2005)
21. Mahmudov, NI: q-Analogues of the Bernoulli and Genocchi polynomials and the Srivastava-Pintér addition theorems. Discrete Dyn. Nat. Soc. **2012**, Article ID 169348 (2012). doi:10.1155/2012/169348
22. Mahmudov, NI: On a class of q-Bernoulli and q-Euler polynomials. Adv. Differ. Equ. **2013**, 108 (2013)
23. Natalini, P, Bernardini, A: A generalization of the Bernoulli polynomials. J. Appl. Math. **3**, 155-163 (2003)
24. Bretti, G, Natalini, P, Ricci, PE: Generalizations of the Bernoulli and Appell polynomials. Abstr. Appl. Anal. **7**, 613-623 (2004)
25. Kurt, B: A further generalization of the Euler polynomials and on the 2D-Euler polynomials (in press)
26. Kurt, B: A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials $B_n^2(x,y)$. Appl. Math. Sci. **47**, 2315-2322 (2010)
27. Tremblay, R, Gaboury, S, Fugère, BJ: A new class of generalized Apostol-Bernoulli polynomials and some analogues of the Srivastava-Pintér addition theorem. Appl. Math. Lett. **24**, 1888-1893 (2011)
28. Tremblay, R, Gaboury, S, Fugère, BJ: Some new classes of generalized Apostol-Euler and Apostol-Genocchi polynomials. Int. J. Math. Math. Sci. **2012**, Article ID 182785 (2012). doi:10.1155/2012/182785
29. Srivastava, HM, Pintér, Á: Remarks on some relationships between the Bernoulli and Euler polynomials. Appl. Math. Lett. **17**, 375-380 (2004)
30. Luo, QM: Some results for the q-Bernoulli and q-Euler polynomials. J. Math. Anal. Appl. **363**, 7-18 (2010)

doi:10.1186/1687-1847-2013-115
Cite this article as: Mahmudov and Keleshteri: **On a class of generalized q-Bernoulli and q-Euler polynomials.** *Advances in Difference Equations* 2013 **2013**:115.