

REVIEW

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# Higher-order Bernoulli, Euler and Hermite polynomials

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## Abstract

In (Kim and Kim in *J. Inequal. Appl.* 2013:111, 2013; Kim and Kim in *Integral Transforms Spec. Funct.*, 2013, doi:10.1080/10652469.2012.754756), we have investigated some properties of higher-order Bernoulli and Euler polynomial bases in  $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ . In this paper, we derive some interesting identities of higher-order Bernoulli and Euler polynomials arising from the properties of those bases for  $\mathbb{P}_n$ .

## 1 Introduction

For  $r \in \mathbb{R}$ , let us define the Bernoulli polynomials of order  $r$  as follows:

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [1-18]}). \quad (1)$$

In the special case,  $x = 0$ ,  $B_n^{(r)}(0) = B_n^{(r)}$  are called the  $n$ th Bernoulli numbers of order  $r$ . As is well known, the Euler polynomials of order  $r$  are defined by the generating function to be

$$\left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [1-10]}). \quad (2)$$

For  $\lambda (\neq 1) \in \mathbb{C}$ , the Frobenius-Euler polynomials of order  $r$  are also given by

$$\left(\frac{1-\lambda}{e^t - \lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!} \quad (\text{see [1, 7]}). \quad (3)$$

The Hermite polynomials are defined by the generating function to be:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \quad (\text{see [8-10, 19]}). \quad (4)$$

Thus, by (4), we get

$$H_n(x) = (H + 2x)^n = \sum_{l=0}^n \binom{n}{l} H_{n-l} 2^l x^l \quad (\text{see [14]}), \quad (5)$$

where  $H_n = H_n(0)$  are called the  $n$ th Hermite numbers. Let  $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ . Then  $\mathbb{P}_n$  is an  $(n + 1)$ -dimensional vector space over  $\mathbb{Q}$ . In [8, 10], it is called that  $\{E_0^{(r)}(x), E_1^{(r)}(x), \dots, E_n^{(r)}(x)\}$  and  $\{B_0^{(r)}(x), B_1^{(r)}(x), \dots, B_n^{(r)}(x)\}$  are bases for  $\mathbb{P}_n$ . Let  $\Omega$  denote the space of real-valued differential functions on  $(-\infty, \infty) = \mathbb{R}$ . We define four linear operators on  $\Omega$  as follows:

$$I(f)(x) = \int_x^{x+1} f(x) dx, \quad \Delta(f)(x) = f(x + 1) - f(x), \tag{6}$$

$$\tilde{\Delta}(f)(x) = f(x + 1) + f(x), \quad D(f)(x) = f'(x). \tag{7}$$

Thus, by (6) and (7), we get

$$I^n(f)(x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f_n(x+k) \quad (\text{see [8, 10, 12, 13]}), \tag{8}$$

where  $f'_1 = f, f'_2 = f_1, \dots, f'_n = f_{n-1}, n \in \mathbb{N}$ .

In this paper, we derive some new interesting identities of higher-order Bernoulli, Euler and Hermite polynomials arising from the properties of bases of higher-order Bernoulli and Euler polynomials for  $\mathbb{P}_n$ .

## 2 Some identities of higher-order Bernoulli and Euler polynomials

First, we introduce the following theorems, which are important in deriving our results in this paper.

**Theorem 1** [8] For  $r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ , let  $p(x) \in \mathbb{P}_n$ . Then we have

$$p(x) = \frac{1}{2^r} \sum_{k=0}^n \sum_{j=0}^r \frac{1}{k!} \binom{r}{j} D^k p(j) E_k^{(r)}(x).$$

**Theorem 2** [10] For  $r \in \mathbb{Z}_+$ , let  $p(x) \in \mathbb{P}_n$ :

(a) If  $r > n$ , then we have

$$p(x) = \sum_{k=0}^n \sum_{j=0}^k \frac{1}{k!} (-1)^{k-j} \binom{k}{j} (I^{r-k} p(j)) B_k^{(r)}(x).$$

(b) If  $r \leq n$ , then

$$p(x) = \sum_{k=0}^{r-1} \sum_{j=0}^k \frac{1}{k!} (-1)^{k-j} \binom{k}{j} (I^{r-k} p(j)) B_k^{(r)}(x) + \sum_{k=r}^n \sum_{j=0}^r \frac{1}{k!} (-1)^{r-j} \binom{k}{j} (D^{k-r} p(j)) B_k^{(r)}(x).$$

Let us take  $p(x) = H_n(x) \in \mathbb{P}_n$ .

Then, by (5), we get

$$\begin{aligned} p^{(k)}(x) &= D^k p(x) = 2^k n(n-1) \cdots (n-k+1) H_{n-k}(x) \\ &= 2^k \frac{n!}{(n-k)!} H_{n-k}(x). \end{aligned} \tag{9}$$

From Theorem 1 and (9), we can derive the following equation (10):

$$\begin{aligned} H_n(x) &= \frac{1}{2^r} \sum_{k=0}^n \left\{ \sum_{j=0}^r \frac{1}{k!} \binom{r}{j} 2^k \frac{n!}{(n-k)!} H_{n-k}(j) \right\} E_k^{(r)}(x) \\ &= \frac{1}{2^r} \sum_{k=0}^n \binom{n}{k} 2^k \left[ \sum_{j=0}^r \binom{r}{j} H_{n-k}(j) \right] E_k^{(r)}(x). \end{aligned} \tag{10}$$

Therefore, by (10), we obtain the following theorem.

**Theorem 3** For  $n, r \in \mathbb{Z}_+$ , we have

$$H_n(x) = \frac{1}{2^r} \sum_{k=0}^n \binom{n}{k} 2^k \left[ \sum_{j=0}^r \binom{r}{j} H_{n-k}(j) \right] E_k^{(r)}(x).$$

We recall an explicit expression for Hermite polynomials as follows:

$$H_n(x) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^l n!}{l!(n-2l)!} (2x)^{n-2l}. \tag{11}$$

By (11), we get

$$H_{n-k}(j) = \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(-1)^l (n-k)!}{l!(n-k-2l)!} (2j)^{n-k-2l}. \tag{12}$$

Thus, by Theorem 3 and (12), we obtain the following corollary.

**Corollary 4** For  $n, r \in \mathbb{Z}_+$ , we have

$$H_n(x) = \frac{1}{2^r} \sum_{k=0}^n \left\{ \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{\binom{n}{k} \binom{r}{j} 2^k (-1)^l (n-k)! (2j)^{n-k-2l}}{l!(n-k-2l)!} \right\} E_k^{(r)}(x).$$

Now, we consider the identities of Hermite polynomials arising from the property of the basis of higher-order Bernoulli polynomials in  $\mathbb{P}_n$ .

For  $r > k$ , by (6) and (8), we get

$$\begin{aligned} l^{r-k} H_n(x) &= \sum_{l=0}^{r-k} \binom{r-k}{l} (-1)^{r-k-l} \frac{H_{n+r-k}(x+l)}{2^{r-k} (n+1) \cdots (n+r-k)} \\ &= \sum_{l=0}^{r-k} \binom{r-k}{l} (-1)^{r-k-l} \frac{n! H_{n+r-k}(x+l)}{2^{r-k} (n+r-k)!}. \end{aligned} \tag{13}$$

Therefore, by Theorem 2 and (13), we obtain the following theorem.

**Theorem 5** For  $n, r \in \mathbb{Z}_+$ , with  $r > n$ , we have

$$H_n(x) = n! \sum_{k=0}^n \left\{ \sum_{j=0}^k \sum_{l=0}^{r-k} \frac{\binom{r-k}{l} \binom{k}{j} (-1)^{r-j-l} H_{n+r-k}(j+l)}{2^{r-k} k! (n+r-k)!} \right\} B_k^{(r)}(x).$$

Let us assume that  $r, k \in \mathbb{Z}_+$ , with  $r \leq n$ . Then, by (b) of Theorem 2, we get

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^k \sum_{l=0}^{r-k} \frac{\binom{r-k}{l} \binom{k}{j} (-1)^{r-j-l} H_{n+r-k}(j+l)}{2^{r-k} k! (n+r-k)!} \right\} B_k^{(r)}(x) \\ &\quad + n! \sum_{k=r}^n \left\{ \sum_{j=0}^r \frac{(-1)^{r-j} \binom{r}{j} 2^{k-r} H_{n+r-k}(j)}{k! (n+r-k)!} \right\} B_k^{(r)}(x). \end{aligned} \tag{14}$$

Therefore, by (14), we obtain the following theorem.

**Theorem 6** For  $n, r \in \mathbb{Z}_+$ , with  $r \leq n$ , we have

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^k \sum_{l=0}^{r-k} \frac{\binom{r-k}{l} \binom{k}{j} (-1)^{r-j-l} H_{n+r-k}(j+l)}{2^{r-k} k! (n+r-k)!} \right\} B_k^{(r)}(x) \\ &\quad + n! \sum_{k=r}^n \left\{ \sum_{j=0}^r \frac{(-1)^{r-j} \binom{r}{j} 2^{k-r} H_{n+r-k}(j)}{k! (n+r-k)!} \right\} B_k^{(r)}(x). \end{aligned}$$

**Remark** From (12), we note that

$$H_{n+r-k}(j+l) = \sum_{m=0}^{\lfloor \frac{n+r-k}{2} \rfloor} \frac{(-1)^m (n+r-k)!}{m! (n+r-k-2m)!} (2j+2l)^{n+r-k-2m} \tag{15}$$

and

$$H_{n+r-k}(j) = \sum_{m=0}^{\lfloor \frac{n+r-k}{2} \rfloor} \frac{(-1)^m (n+r-k)!}{m! (n+r-k-2m)!} (2j)^{n+r-k-2m}. \tag{16}$$

**Theorem 7** [10] For  $n, r \in \mathbb{Z}_+$ , with  $r > n$  and  $p(x) \in \mathbb{P}_n$ , we have

$$p(x) = \sum_{k=0}^n \left\{ \sum_{j=0}^k \sum_{l=0}^n \frac{(r-k)! S_2(l+r-k, r-k)}{(l+r-k)! k!} (-1)^{k-j} \binom{k}{j} p^{(l)}(j) \right\} B_k^{(r)}(x),$$

where  $S_2(l, n)$  is the Stirling number of the second kind and  $p^{(l)}(j) = D^l p(j)$ .

**Theorem 8** [10] For  $n, r \in \mathbb{Z}_+$ , with  $r \leq n$  and  $p(x) \in \mathbb{P}_n$ , we have

$$\begin{aligned} p(x) &= \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^k \sum_{l=0}^n \frac{(r-k)! S_2(l+r-k, r-k)}{(l+r-k)! k!} (-1)^{k-j} \binom{k}{j} p^{(l)}(j) \right\} B_k^{(r)}(x) \\ &\quad + \sum_{k=r}^n \left\{ \sum_{j=0}^r \frac{(-1)^{r-j}}{k!} \binom{r}{j} p^{(k-r)}(j) \right\} B_k^{(r)}(x). \end{aligned}$$

Let us take  $p(x) = H_n(x) \in \mathbb{P}_n$ . Then, by Theorem 7 and Theorem 8, we obtain the following corollary.

**Corollary 9** For  $n, r \in \mathbb{Z}_+$ :

(a) For  $r > n$ , we have

$$H_n(x) = n! \sum_{k=0}^n \left\{ \sum_{j=0}^k \sum_{l=0}^n \frac{(r-k)! S_2(l+r-k, r-k)}{(l+r-k)! k! (n-l)!} (-1)^{k-j} \binom{k}{j} 2^l H_{n-l}(j) \right\} B_k^{(r)}(x).$$

(b) For  $r \leq n$ , we have

$$H_n(x) = n! \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^k \sum_{l=0}^n \frac{(r-k)! S_2(l+r-k, r-k)}{(l+r-k)! k! (n-l)!} (-1)^{k-j} \binom{k}{j} 2^l H_{n-l}(j) \right\} B_k^{(r)}(x) \\ + n! \sum_{k=r}^n \left\{ \sum_{j=0}^r \frac{(-1)^{r-j} \binom{r}{j} 2^{k-r} H_{n-k+r}(j)}{k! (n-k+r)!} \right\} B_k^{(r)}(x).$$

**Theorem 10** [9] For  $p(x) \in \mathbb{P}_n$ , we have

$$p(x) = \frac{1}{(1-\lambda)^r} \sum_{k=0}^n \left\{ \sum_{j=0}^r \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} p^{(k)}(j) \right\} H_k^{(r)}(x|\lambda).$$

Let us take  $p(x) = H_n(x) \in \mathbb{P}_n$ . Then

$$H_n(x) = \frac{1}{(1-\lambda)^r} \sum_{k=0}^n \left\{ \sum_{j=0}^r \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} 2^k \frac{n!}{(n-k)!} H_{n-k}(j) \right\} H_k^{(r)}(x|\lambda) \\ = \frac{1}{(1-\lambda)^r} \sum_{k=0}^n \binom{n}{k} 2^k \left\{ \sum_{j=0}^r \binom{r}{j} (-\lambda)^{r-j} H_{n-k}(j) \right\} H_k^{(r)}(x|\lambda) \\ = \frac{1}{(1-\lambda)^r} \sum_{k=0}^n \left\{ \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{\binom{n}{k} 2^k \binom{r}{j} (-\lambda)^{r-j} (-1)^l (n-k)! (2j)^{n-k-2l}}{l! (n-k-2l)!} \right\} H_k^{(r)}(x|\lambda). \quad (17)$$

Therefore, by (17), we obtain the following corollary.

**Corollary 11** For  $n \in \mathbb{Z}_+$ , we have

$$H_n(x) = \frac{1}{(1-\lambda)^r} \sum_{k=0}^n \left\{ \sum_{j=0}^r \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{\binom{n}{k} 2^k \binom{r}{j} (-\lambda)^{r-j} (-1)^l (n-k)! (2j)^{n-k-2l}}{l! (n-k-2l)!} \right\} H_k^{(r)}(x|\lambda).$$

For  $r = 1$ , the Frobenius-Euler polynomials are defined by the generating function to be

$$\left( \frac{1-\lambda}{e^t - \lambda} \right) e^{xt} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{t^n}{n!} \quad (\text{see [9]}). \quad (18)$$

Thus, by (18), we get

$$\frac{d}{d\lambda} H_n(x|\lambda) = \frac{1}{1-\lambda} (H_n^{(2)}(x|\lambda) - H_n(x|\lambda)). \quad (19)$$

For  $n \in \mathbb{Z}_+$ , let  $p(x) \in \mathbb{P}_n$ . Then we note that

$$(1 - \lambda)p(x) = \sum_{k=0}^n \frac{1}{k!} \{p^{(k)}(1) - \lambda p^{(k)}(0)\} H_k(x|\lambda) \quad (\text{see [9]}). \tag{20}$$

Let us take  $p(x) = H_n(x)$ . Then, by (20), we get

$$\begin{aligned} (1 - \lambda)H_n(x) &= \sum_{k=0}^n \frac{1}{k!} \left\{ 2^k \frac{n!}{(n-k)!} H_{n-k}(1) - \lambda 2^k \frac{n!}{(n-k)!} H_{n-k} \right\} H_k(x|\lambda) \\ &= \sum_{k=0}^n \binom{n}{k} 2^k (H_{n-k}(1) - \lambda H_{n-k}) H_k(x|\lambda) \quad (\text{see [9]}). \end{aligned} \tag{21}$$

Therefore, by (21), we obtain the following theorem.

**Theorem 12** For  $n \in \mathbb{Z}_+$ , we have

$$(1 - \lambda)H_n(x) = \sum_{k=0}^n \binom{n}{k} 2^k (H_{n-k}(1) - \lambda H_{n-k}) H_k(x|\lambda).$$

Let us take  $\frac{d}{d\lambda}$  on the both sides of Theorem 12.

Then, we have

$$\begin{aligned} -H_n(x) &= - \sum_{k=0}^n \binom{n}{k} 2^k H_{n-k} H_k(x|\lambda) \\ &\quad + \sum_{k=0}^n \binom{n}{k} 2^k (H_{n-k}(1) - \lambda H_{n-k}) \left( \frac{d}{d\lambda} H_k(x|\lambda) \right). \end{aligned} \tag{22}$$

By (22), we get

$$\begin{aligned} H_n(x) &= \sum_{k=0}^n \binom{n}{k} 2^k H_{n-k} H_k(x|\lambda) \\ &\quad + \sum_{k=0}^n \binom{n}{k} (\lambda H_{n-k} - H_{n-k}(1)) 2^k \left( \frac{d}{d\lambda} H_k(x|\lambda) \right). \end{aligned} \tag{23}$$

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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