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Properties of convolutions for hypergeometric series with univalent functions

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Abstract

The purpose of the present paper is to investigate various mapping and inclusion properties involving subclasses of analytic and univalent functions for a linear operator defined by means of Hadamard product (or convolution) with the Gaussian hypergeometric function.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^t(A,B)$ if

$$\left| \frac{f'(z) - 1}{t(A-B) - B(f'(z) - 1)} \right| < 1 \quad \bigl(-1 \le B < A \le 1; t \in \mathbb{C}\backslash\{0\}; z \in \mathbb{U}\bigr), \tag{1.2}$$

Clearly, a function f belongs to $\mathcal{R}^t(A,B)$ if and only if there exists a function w regular in $\mathbb{U}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$1 + \frac{1}{t}\bigl(f'(z) - 1\bigr) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}). \tag{1.3}$$

The class $\mathcal{R}^t(A,B)$ was introduced by Dixit and Pal [1]. By giving specific values to t, A and B in (1.2), we obtain the following subclasses studied by various researchers in earlier works:

(i) For $t = e^{-i\eta}\cos\eta$ ($|\eta| < \pi/2$), $A = 1 - 2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$, we obtain the class of functions f satisfying the condition:

$$\left|\frac{e^{i\eta}(f'(z) - 1)}{2(1-\alpha)\cos\eta + e^{i\eta}(f'(z) - 1)}\right| < 1 \quad (z \in U). \tag{1.4}$$

In this case, the class $\mathcal{R}^t(A, B)$ is equivalent to the class $\mathcal{R}_\eta(\alpha)$ which is studied by Ponnusamy and Rønning [2]. Here, $\mathcal{R}_\eta(\alpha)$ is the class of functions $f \in \mathcal{A}$ satisfying the condition:

$$\operatorname{Re}\bigl(e^{i\eta}\bigl(f'(z) - \alpha\bigr)\bigr) > 0 \qquad \bigl(|\eta| < \pi/2; 0 \le \alpha < 1; z \in \mathbb{U}\bigr).$$

(ii) For $t = e^{i\eta}\cos\eta$ ($|\eta| < \pi/2$), we obtain the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\left|\frac{e^{i\eta}(f'(z) - 1)}{Be^{i\eta}f'(z) - (A\cos\eta + iB\sin\eta)}\right| < 1 \quad (z \in \mathbb{U}),$$

which was studied by Dashrath [3].

(iii) For $t = 1$, $A = \beta$ and $B = -\beta$ $(0 < \beta \le 1)$, we obtain the class of functions f satisfying the condition:

$$\left|\frac{f'(z) - 1}{f'(z) + 1}\right| < \beta \quad (0 < \beta \le 1; z \in \mathbb{U}),$$

which was studied by Caplinger and Cauchy [4] and Padmanabhan [5].

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of starlike and convex functions of order α $(0 \le \alpha < 1)$ in $\mathbb{U}$, respectively. It is well known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^*$, $\mathcal{C}(\alpha) \subset \mathcal{C}(0) \equiv \mathcal{C}$ and $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$. For $\lambda > 0$, define

$$\mathcal{S}^*_\lambda = \left\{f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda, z \in \mathbb{U}\right\}$$

and

$$\mathcal{C}_\lambda = \bigl\{f \in \mathcal{A} : zf'(z) \in \mathcal{S}^*_\lambda\bigr\}.$$

It is a known fact that a sufficient condition for $f \in \mathcal{A}$ of the form (1.1) to belong to the class $\mathcal{S}^*$ is that $\sum_{n=2}^{\infty} na_n \le 1$. A simple extension of this result is the following [6]:

$$\sum_{n=2}^{\infty}(n + \lambda - 1)|a_n| \le \lambda \quad \Longrightarrow \quad f \in \mathcal{S}^*_\lambda. \tag{1.5}$$

For $\lambda = 1/2$, this was previously proved by Schild [2]. Since $f \in \mathcal{C}_\lambda$ if and only if $zf'(z) \in \mathcal{S}^*_\lambda$, we have a corresponding results for $\mathcal{C}_\lambda$,

$$\sum_{n=2}^{\infty} n(n + \lambda - 1)|a_n| \le \lambda \quad \Longrightarrow \quad f \in \mathcal{C}_\lambda. \tag{1.6}$$

Now we introduce the class $\mathcal{UST}$ (resp., $\mathcal{UCV}$) of uniformly starlike (resp., convex) functions. We say [7, 8] that $f \in \mathcal{A}$ is in $\mathcal{UST}$ (resp., $\mathcal{UCV}$) if for each $\xi \in \mathbb{U}$ and each circular arc γ in $\mathbb{U}$ with center η, the image arc $f(\gamma)$ is starlike with respect to $f(\xi)$ (resp., is a convex curve).

In this paper, we consider the Gaussian hypergeometric function $F(a,b;c;z)$ defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad (a,b \in \mathbb{C}; c \neq 0,-1,-2,\ldots; z \in \mathbb{U}),$$

where $(\nu)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\nu)_n := \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } n = 0 \text{ and } \nu \in \mathbb{C}\backslash\{0\}, \\ \nu(\nu+1)\cdots(\nu+n-1) & \text{if } n \in \mathbb{N} \text{ and } \nu \in \mathbb{C}. \end{cases}$$

We note that $F(a,b;c;z) = F(b,a;c;z)$ and

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \quad \bigl(\mathrm{Re}(c-a-b) > 0\bigr). \tag{1.7}$$

We also recall (see [4]) that the function $F(a,b;c;z)$ is bounded if $\mathrm{Re}\{c-a-b\} > 0$, and has a pole at $z = 1$ if $\mathrm{Re}\{c-a-b\} \leq 0$. Moreover, univalence, starlikeness and convexity properties of $zF(a,b;c;z)$ have been studied extensively in Ponnusamy and Vuorinen [9] and Ruscheweyh and Singh [10].

For $f \in \mathcal{A}$, we define the operator $I_{a,b;c}f$ by

$$I_{a,b;c}f(z) = zF(a,b;c;z) * f(z), \tag{1.8}$$

where $*$ denotes the usual Hadamard product (or convolution) of power series. If f equals to the convex function $z/(1-z)$, then the operator $I_{a,b;c}f(z)$ becomes $zF(a,b;c;z)$. For a survey of special cases of this operator and also more general operators, we can refer to the article by Srivastava [11–13] and Swaminathan [14], where also a long list of other references can be found. Thus, the operator $I_{a,b;c}f$ and hence the Gaussian hypergeometric function is a natural object for studying inclusion properties related to the convolution product. In the present paper, we find a condition for univalency of the operator $I_{a,b;c}f$. We also investigate conditions such that $I_{a,b;c}f \in \mathcal{R}^t(A,B)$ ($\mathcal{UST}, \mathcal{UCV}, \mathcal{S}^*_\lambda$ and $\mathcal{C}_\lambda$), whenever $f \in \mathcal{R}^t(A,B)$.

2 A set of lemmas

Now we introduce several lemmas which are needed for the proof of our main results.

Lemma 2.1 [1] *Let a function f of the form* (1.1) *be in $\mathcal{R}^t(A,B)$. Then*

$$|a_n| \leq \frac{(A-B)|t|}{n}.$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(A-B)tz^{n-1}}{1+Bz^{n-1}}\right) dz \quad (n \geq 2; z \in \mathbb{U}).$$

Lemma 2.2 [1] *Let a function f of the form* (1.1) *be in $\mathcal{A}$. If*

$$\sum_{n=2}^{\infty}\bigl(1+|B|\bigr)n|a_n| \leq (A-B)|t| \quad (-1 \leq B < A \leq 1; t \in \mathbb{C})$$

then $f \in \mathcal{R}^t(A,B)$. The result is sharp for the function

$$f(z) = z + \frac{(A-B)t}{(1+|B|)n}z^n \quad (n \geq 2; z \in \mathbb{U}).$$

Lemma 2.3 [15] *Let $w(z)$ be regular in the unit disk $\mathbb{U}$ with $w(0) = 0$. Then, if $|w(z)|$ attains a maximum value on the circle $|z| = r$ $(0 \leq r < 1)$ at a point z, we can write*

$$z_1 w'(z_1) = mw(z_1),$$

where m is real and $m \geq 1$.

Lemma 2.4 [2] (i) *For $a, b \in \mathbb{C}\backslash\{0,1\}$ and $c \in \mathbb{C}\backslash\{1\}$ with $c > \max\{0, a+b-1\}$,*

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} = \frac{1}{(a-1)(b-1)}\left(\frac{\Gamma(c+1-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - (c-1)\right).$$

(ii) *For $a, b \in \mathbb{C}\backslash\{0\}$ with $a > 0$ and $b > 0$ and $c > a+b+1$,*

$$\sum_{n=0}^{\infty}\frac{(n+1)(a)_n(b)_n}{(c)_n(1)_n} = \left(\frac{ab}{c-a-b-1}+1\right)\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}.$$

Lemma 2.5 [16] *A function f of the form* (1.1) *is in $\mathcal{UCV}$ if*

$$\sum_{n=2}^{\infty} n(2n-1)|a_n| \leq 1.$$

Lemma 2.6 [16] *A function f of the form* (1.1) *is in $\mathcal{UST}$ if*

$$\sum_{n=2}^{\infty}(3n-2)|a_n| \leq 1.$$

3 Main results

Theorem 3.1 *Let $f \in \mathcal{A}$. If*

$$\bigl|\bigl(I_{a,b;c}f(z)\bigr)' - 1\bigr|^{1-\beta}\left|\frac{z(I_{a,b;c}f(z))''}{(I_{a,b;c}f(z))'}\right|^{\beta} < \frac{1}{2^{\beta}} \quad (\beta \geq 0), \tag{3.1}$$

then $I_{a,b;c}f$ is univalent in $\mathbb{U}$.

Proof We note that

$$I_{a,b;c}(f) = z + \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n z^n$$

in $\mathcal{A}$. Define w by

$$w(z) = \left(I_{a,b;c}f(z)\right)' - 1$$

for $z \in \mathbb{U}$. Then it follows that w is analytic in $\mathbb{U}$ with $w(0) = 0$. By (3.1),

$$\begin{aligned} &\left|w(z)\right|^{1-\beta}\left|\frac{zw'(z)}{1+w(z)}\right|^{\beta} \\ &\quad = \left|w(z)\right|\left|\frac{zw'(z)}{w(z)}\frac{1}{1+w(z)}\right|^{\beta} < \frac{1}{2^{\beta}}. \end{aligned} \tag{3.2}$$

Suppose that there exists a point $z_1 \in \mathbb{U}$ such that

$$\max_{|z|\le|z_1|}\left|w(z)\right| = \left|w(z_1)\right| = 1.$$

Then, by Lemma 2.3, we can put

$$\frac{z_1w'(z_1)}{w(z_1)} = m \ge 1.$$

Therefore, we obtain

$$\left|w(z_1)\right|\left|\frac{z_1w'(z_1)}{w(z_1)}\frac{1}{1+w(z)}\right|^{\beta} \ge \left(\frac{m}{2}\right)^{\beta} \ge \frac{1}{2^{\beta}},$$

which contradicts the condition (3.2). This shows that

$$\left|w(z)\right| = \left|\left(I_{a,b;c}f(z)\right)' - 1\right| < 1,$$

which implies that $\mathrm{Re}(I_{a,b;c}f(z))' > 0$ for $z \in \mathbb{U}$. Therefore, by the Noshiro-Warschawski theorem [17], $I_{a,b;c}f$ is univalent in $\mathbb{U}$. □

Theorem 3.2 *Let $a, b \in \mathbb{C}\setminus\{0\}$ and $c > |a| + |b|$. Suppose that $f \in \mathcal{R}^t(A,B)$ and satisfy the condition*

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} \le \frac{1}{1+|B|} + 1. \tag{3.3}$$

Then the operator $I_{a,b;c}f$ maps $\mathcal{R}^t(A,B)$ into $\mathcal{R}^t(A,B)$.

Proof Let $a, b \in \mathbb{C}\setminus\{0\}$, $c > |a| + |b|$ and suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}^t(A,B)$. Then, by Lemma 2.2, it suffices to show that

$$T_1 := \sum_{n=2}^{\infty}\left(1+|B|\right)n|A_n| \le (A-B)|t|, \tag{3.4}$$

where

$$A_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n.$$

From Lemma 2.1 and the fact that $|(a)_n| \le (|a|)_n$, we have

$$\begin{aligned} T_1 &\le \sum_{n=2}^{\infty} (A-B)|t|\bigl(1+|B|\bigr)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &\quad \times (A-B)|t|\bigl(1+|B|\bigr)\left(\sum_{n=0}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} - 1\right). \end{aligned}$$

Using the formula (1.7) and the assumption, we find that

$$\begin{aligned} T_1 &\le (A-B)|t|\bigl(1+|B|\bigr)\left(\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right) \\ &\le (A-B)|t|, \end{aligned}$$

which implies that the operator $I_{a,b;c}f$ maps $\mathcal{R}^t(A,B)$ into $\mathcal{R}^t(A,B)$.

If, in the proof of Theorem 3.2, we take $b = \bar{a}$, then we have the following theorem under a weaker condition on the parameter c. □

Theorem 3.3 *Let $a \in \mathbb{C}\backslash\{0\}$ and $c > 2\operatorname{Re}\{a\}$. Suppose that $f \in \mathcal{R}^t(A,B)$ and satisfy the condition*

$$\frac{\Gamma(c-2\operatorname{Re} a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})} \le \frac{1}{1+|B|} + 1.$$

Then the operator $I_{a,\bar{a};c}(f)$ maps $\mathcal{R}^t(A,B)$ into $\mathcal{R}^t(A,B)$.

Proof The proof of Theorem 3.3 follows in the similar lines on the proof of Theorem 3.2 and so we omit the details. □

Theorem 3.4 *Let $a,b \in \mathbb{C}\backslash\{0\}$ and $\lambda \in (0,1]$. Suppose that $f \in \mathcal{R}^t(A,B)$, $|a| \ne 1$, $|b| \ne 1$ and $c \ne 1$ such that $c > |a| + |b|$ and satisfy the condition*

$$\begin{aligned} &\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(1 + \frac{(\lambda-1)(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right) \\ &\quad \le \lambda\left(1 + \frac{1}{(A-B)|t|}\right) + \frac{(\lambda-1)(c-1)}{(|a|-1)(|b|-1)}. \end{aligned} \tag{3.5}$$

Then the operator $I_{a,b;c}f$ maps $\mathcal{R}^t(A,B)$ into $\mathcal{S}^*_\lambda$.

Proof Let $a,b \in \mathbb{C}\backslash\{0\}$ and $c > |a| + |b|$ with $|a| \ne 1$, $|b| \ne 1$ and $c \ne 1$. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}^t(A,B)$. Then, by (1.5), it is sufficient to show that

$$T_2 := \sum_{n=2}^{\infty}(n+\lambda-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \le \lambda.$$

By using Lemma 2.1 and (i) of Lemma 2.4, we observe that

$$\begin{aligned}
T_2 &\leq \sum_{n=2}^{\infty}(n+\lambda-1)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{(A-B)|t|}{n} \\
&= (A-B)|t|\left[\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} + (\lambda-1)\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n+1}}\right] \\
&= (A-B)|t|\left[\left(\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right)\right. \\
&\quad \left. + (\lambda-1)\left\{\frac{1}{(|a|-1)(|b|-1)}\left(\frac{\Gamma(c+1-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}-(c-1)\right)-1\right\}\right] \\
&= (A-B)|t|\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(1+\frac{(\lambda-1)(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right)\right. \\
&\quad \left. - \frac{(\lambda-1)(c-1)}{(|a|-1)(|b|-1)} - \lambda\right] \leq \lambda,
\end{aligned}$$

by (3.5), which completes the proof of Theorem 3.4. □

Taking $\lambda = 1$ and $b = \bar{a}$ in Theorem 3.4, we have the following result.

Corollary 3.1 *Let $a \in \mathbb{C}\backslash\{0\}$ and $c > \max\{0, 2\operatorname{Re}\{a\}\}$. Suppose that $f \in \mathcal{R}^t(A,B)$ and satisfy the condition*

$$\frac{\Gamma(c-2\operatorname{Re}\{a\})\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})} \leq 1 + \frac{1}{(A-B)|t|}.$$

Then $I_{a,\bar{a};c}f \in \mathcal{S}_1^*$.

By using the same method as in the proof of Theorem 3.4, we have the following result.

Theorem 3.5 *Let $a, b \in \mathbb{C}\backslash\{0\}$ and $c > 1 + |a| + |b|$, $\lambda \in (0,1]$ and $f \in \mathcal{R}^t(A,B)$. Suppose that*

$$\frac{\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(\frac{|ab|}{c-|a|-|b|-1}+\lambda\right) \leq \lambda\left(1+\frac{1}{(A-B)|t|}\right). \tag{3.6}$$

Then the operator $I_{a,b;c}f$ maps $\mathcal{R}^t(A,B)$ into $\mathcal{C}_\lambda$.

Proof Let $a, b \in \mathbb{C}\backslash\{0\}$ and $c > |a| + |b| + 1$. Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}^t(A,B)$. To show that the operator $I_{a,b;c}f$ belongs to $\mathcal{C}_\lambda$, from (1.6), it is enough to show that

$$T_3 := \sum_{n=2}^{\infty} n(n+\lambda-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq \lambda.$$

From Lemma 2.1 and (1.7), we find that

$$\begin{aligned}
T_3 &\le (A-B)|t|\left[\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}}+\lambda\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right]\\
&=(A-B)|t|\left[\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}+\lambda\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right]\\
&=(A-B)|t|\left[\frac{|ab|}{c}\frac{\Gamma(c-|a|-|b|-1)\Gamma(c+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}+\lambda\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right]\\
&=(A-B)|t|\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(\frac{|ab|}{c-|a|-|b|-1}+\lambda\right)-\lambda\right]\\
&\le\lambda,
\end{aligned}$$

by (3.6) and the conclusion follows. □

Similarly, taking $\lambda = 1$ and $b = \bar{a}$ in Theorem 3.5, we have the following result.

Corollary 3.2 *Let* $a \in \mathbb{C}\backslash\{0\}$, $c > \max\{0, 1 + 2\operatorname{Re}\{a\}\}$ *and* $\lambda \in (0,1]$. *Suppose that*

$$\frac{\Gamma(c-2\operatorname{Re}\{a\})\Gamma(c)}{\Gamma(c-a)\Gamma(c-\bar{a})}\left(\frac{|a|^2}{c-1-2\operatorname{Re}\{a\}}+1\right)\le 1+\frac{1}{(A-B)|t|}.$$

Then $I_{a,\bar{a};c}f \in \mathcal{C}_1$.

By using Lemma 2.5 and Lemma 2.6, we have the following theorem for $\mathcal{UCV}$ and $\mathcal{UST}$.

Theorem 3.6 *Let* $a, b \in \mathbb{C}\backslash\{0\}$, $c > |a| + |b| + 1$ *and* $f \in \mathcal{R}^t(A,B)$. *Suppose that*

$$(A-B)|t|\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(\frac{2|ab|}{c-|a|-|b|-1}+1\right)-1\right]\le 1. \tag{3.7}$$

Then the operator $I_{a,b;c}f$ *maps* $\mathcal{R}^t(A,B)$ *into* $\mathcal{UCV}$.

Proof Let $a, b \in \mathbb{C}\backslash\{0\}$ and $c > |a| + |b| + 1$. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}^t(A,B)$. By Lemma 2.5, we need only to show that

$$T_4 := \sum_{n=2}^{\infty} n(2n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\le 1.$$

Then, from (1.7) and $(|a|)_n = |a|(|a|)_{n-1}$, we have

$$\begin{aligned}
T_4 &\le (A-B)|t|\left[\sum_{n=1}^{\infty}(2n+1)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right]\\
&=(A-B)|t|\left[2\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}}+\sum_{n=0}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}-1\right]\\
&=(A-B)|t|\left[\frac{2|a||b|}{c}\frac{\Gamma(c-|a|-|b|-1)\Gamma(c+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}+\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right]
\end{aligned}$$

$$= (A-B)|t|\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(\frac{2|ab|}{c-|a|-|b|-1}+1\right)-1\right]$$
$$\leq 1,$$

by (3.7), and so we have Theorem 3.6. □

Theorem 3.7 *Let $a, b \in \mathbb{C}\backslash\{0\}$, $c > |a| + |b|$ with $|a| \neq 1$, $|b| \neq 1$, and $c \neq 1$ and $f \in \mathcal{R}^t(A,B)$. Suppose that*

$$(A-B)|t|\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(3-\frac{2(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right)+\frac{2(c-1)}{(|a|-1)(|b|-1)}-1\right] \leq 1. \quad (3.8)$$

Then the operator $I_{a,b;c}f$ maps $\mathcal{R}^t(A,B)$ into $\mathcal{UST}$.

Proof Let $a, b \in \mathbb{C}\backslash\{0\}$ and $c > |a| + |b|$ with $|a| \neq 1$, $|b| \neq 1$ and $c \neq 1$. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}^t(A,B)$. By Lemma 2.6, it suffices to show that

$$T_5 := \sum_{n=2}^{\infty}(3n-2)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq 1.$$

Then, from (1.7) and $(|a|)_n = |a|(|a|)_{n-1}$, we have

$$T_5 \leq (A-B)|t|\left[3\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} - 2\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}}\right]$$
$$= (A-B)|t|\left[3\left(\sum_{n=0}^{\infty}\frac{(|a|)_{n}(|b|)_{n}}{(c)_{n}(1)_{n}}-1\right) - 2\sum_{n=2}^{\infty}\frac{(c-1)(|a|-1)_{n}(|b|-1)_{n}}{(|a|-1)(|b|-1)(c-1)_{n}(1)_{n}}\right]$$
$$= (A-B)|t|\left[3\left(\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right)\right.$$
$$\left.-2\frac{c-1}{(|a|-1)(|b|-1)}\left(\frac{\Gamma(c-|a|-|b|+1)\Gamma(c-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}-\frac{(|a|-1)(|b|-1)}{c-1}-1\right)\right]$$
$$= (A-B)|t|\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(3-\frac{2(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right)+\frac{2(c-1)}{(|a|-1)(|b|-1)}-1\right]$$
$$\leq 1,$$

by (3.8), which completes the proof of Theorem 3.7. □

Next, we give the condition on the parameters a, b and c that the convolution of the odd function $zF(a,b;c;z^2)$ and $f \in \mathcal{R}^t(A,B)$ belongs to $\mathcal{R}^t(A,B)$.

Theorem 3.8 *Let $a, b \in \mathbb{C}\backslash\{0\}$, $c > |a| + |b|$ with $|a| \neq 1$ and $|b| \neq 1$ and $f \in \mathcal{R}^t(A,B)$. Suppose that*

$$(1+|B|)\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(2-\frac{c-|a|-|b|}{(|a|-1)(|b|-1)}\right)+\frac{(c-1)}{(|a|-1)(|b|-1)}-1\right] \leq 1. \quad (3.9)$$

Then the operator $zF(a,b;c;z^2) * f(z) \in R^t(A,B)$.

Proof Let $a, b \in \mathbb{C}\backslash\{0\}$ and $c > |a| + |b|$ with $|a| \neq 1$, $|b| \neq 1$. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}^t(A,B)$. We note that

$$zF(a,b;c;z^2) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^{2n-1}.$$

By Lemma 2.2, it is enough to show that

$$T_6 := \sum_{n=2}^{\infty} (1+|B|)(2n-1) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \right| \leq (A-B)|t|.$$

Then, by a similar proof as Theorem 3.7, we get

$$\begin{aligned} T_6 &\leq (A-B)|t|(1+|B|)\left[2\sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} - \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}} \right] \\ &= (A-B)|t|(1+|B|)\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left(2 - \frac{c-|a|-|b|}{(|a|-1)(|b|-1)} \right) \right. \\ &\quad \left. + \frac{(c-1)}{(|a|-1)(|b|-1)} - 1 \right] \\ &\leq (A-B)|t|, \end{aligned}$$

by (3.9), and hence we have the result. □

Finally, we establish the condition on the parameters a, b and c that the function $zF(a,b;c;z)$ belongs to the class $\mathcal{R}^t(A,B)$.

Theorem 3.9 *Let $a, b \in \mathbb{C}\backslash\{0\}$ and $c > |a| + |b| + 1$. Suppose that*

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left(\frac{|ab|}{c-|a|-|b|-1} + 1 \right) - 1 \leq \frac{(A-B)|t|}{1+|B|}. \tag{3.10}$$

Then the function $zF(a,b;c;z) \in \mathcal{R}^t(A,B)$.

Proof By Lemma 2.2, it is sufficient to show that

$$T_7 := \sum_{n=2}^{\infty} (1+|B|) n \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \right| \leq (A-B)|t|.$$

Then, by (ii) of Lemma 2.1, we observe that

$$\begin{aligned} T_7 &\leq \sum_{n=2}^{\infty} (1+|B|) n \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= (1+|B|)\left[\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left(\frac{|ab|}{c-|a|-|b|-1} + 1 \right) - 1 \right] \\ &\leq (A-B)|t|, \end{aligned}$$

by (3.10). This completes the proof of Theorem 3.9. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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