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On the generalized Apostol-type Frobenius-Euler polynomials

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Abstract

The aim of this paper is to derive some new identities related to the Frobenius-Euler polynomials. We also give relation between the generalized Frobenius-Euler polynomials and the generalized Hurwitz-Lerch zeta function at negative integers. Furthermore, our results give generalized Carliz's results which are associated with Frobenius-Euler polynomials.

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1 Introduction, definitions and notations

Throughout this presentation, we use the following standard notions: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \ldots\}$. Also, as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Furthermore, $(\lambda)_0 = 1$ and

$$(\lambda)_k = \lambda(\lambda + 1)(\lambda + 2)\cdots(\lambda + k - 1),$$

where $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$.

The classical Frobenius-Euler polynomial $H_n^{(\alpha)}(x; u)$ of order α is defined by means of the following generating function:

$$\left(\frac{1-u}{e^t - u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x; u)\frac{t^n}{n!}, \tag{1}$$

where u is an algebraic number and $\alpha \in \mathbb{Z}$.

Observe that $H_n^{(1)}(x; u) = H_n(x; u)$, which denotes the Frobenius-Euler polynomials and $H_n^{(\alpha)}(0; u) = H_n^{(\alpha)}(u)$, which denotes the Frobenius-Euler numbers of order α. $H_n(x; -1) = E_n(x)$, which denotes the Euler polynomials (*cf.* [1–24]).

Definition 1.1 (for details, see [16, 17]) Let $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x \in \mathbb{R}$. The generalized Apostol-type Frobenius-Euler polynomials are defined by means of the following generating function:

$$\left(\frac{a^t - u}{\lambda b^t - u}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x; u; a, b, c; \lambda)\frac{t^n}{n!}. \tag{2}$$

Remark 1.2 If we set $x = 0$ and $\alpha = 1$ in (2), we get

$$\frac{a^t - u}{\lambda b^t - u} = \sum_{n=0}^{\infty} \mathcal{H}_n(u; a, b, c; \lambda)\frac{t^n}{n!}, \tag{3}$$

where $\mathcal{H}_n(u; \lambda; a, b, c)$ denotes the generalized Apostol-type Frobenius-Euler numbers (*cf.* [17]).

2 New identities

In this section, we derive many new identities related to the generalized Apostol-type Frobenius-Euler numbers and polynomials of order α.

Theorem 2.1 *Let $\alpha, \beta \in \mathbb{Z}$. Each of the following relationships holds true*:

$$\mathcal{H}_n^{(\alpha)}(x; u; a, b, c; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k^{(\alpha)}(u; a, b, c; \lambda)(x \ln c)^{n-k}, \tag{4}$$

$$\mathcal{H}_n^{(\alpha+\beta)}(x + y; u; a, b, c; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k^{(\alpha)}(x; u; a, b, c; \lambda)\mathcal{H}_{n-k}^{(\beta)}(y; u; a, b, c; \lambda), \tag{5}$$

$$\bigl((x + y) \ln c\bigr)^n = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{(\alpha)}(y; u; a, b, c; \lambda)\mathcal{H}_k^{(-\alpha)}(x; u; a, b, c; \lambda), \tag{6}$$

and

$$\mathcal{H}_n^{(-\alpha)}\bigl(x; u^2; a^2, b^2, c^2; \lambda^2\bigr) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k^{(-\alpha)}(x; u; a, b, c; \lambda)\mathcal{H}_{n-k}^{(-\alpha)}(x; -u; a, b, c; \lambda). \tag{7}$$

Proof of (6) From (2),

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(-\alpha)}(x; u; a, b, c; \lambda)\frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(y; u; a, b, c; \lambda)\frac{t^n}{n!} = c^{(x+y)t}. \tag{8}$$

Therefore,

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{(\alpha)}(y; u; a, b, c; \lambda)\mathcal{H}_k^{(-\alpha)}(x; u; a, b, c; \lambda) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (x \ln c)^n \frac{t^n}{n!}.$$

Thus, by using the Cauchy product in (8) and then equating the coefficients of $\frac{t^n}{n!}$ on both sides of the resulting equation, we obtain the desired result.

The proofs of (4), (5) and (7) are the same as that of (2), thus we omit them. □

Observe that in (6) we have

$$\bigl((x + y) \ln c\bigr)^n = \bigl(\mathcal{H}^{(\alpha)}(y; u; a, b, c; \lambda) + \mathcal{H}^{(-\alpha)}(x; u; a, b, c; \lambda)\bigr)^n,$$

where $(\mathcal{H}^{(\alpha)}(y; u; a, b, c; \lambda))^n$ is replaced by $\mathcal{H}_n^{(\alpha)}(y; u; a, b, c; \lambda)$.

Theorem 2.2 *Let $\alpha \in \mathbb{N}$. Then we have*

$$\sum_{k=0}^{\alpha}\binom{\alpha}{k}(-u)^{\alpha-k}(x\ln c+k\ln a)^n=\sum_{p=0}^{n}\sum_{k=0}^{\alpha}\binom{n}{p}\binom{\alpha}{k}(-u)^{\alpha-k}(k\ln b)^p\mathcal{H}_{n-p}^{(\alpha)}(x;u;a,b,c;\lambda).$$

Proof By using (2), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\alpha}\binom{\alpha}{k}(-u)^{\alpha-k}(x\ln c+k\ln a)^n\right)\frac{t^n}{n!}\\&\quad=\sum_{n=0}^{\infty}\left(\sum_{p=0}^{n}\sum_{k=0}^{\alpha}\binom{n}{p}\binom{\alpha}{k}(-u)^{\alpha-k}(k\ln b)^p\mathcal{H}_{n-p}^{(\alpha)}(x;u;a,b,c;\lambda)\right)\frac{t^n}{n!}.\end{aligned}$$

By equating the coefficients of $\frac{t^n}{n!}$ on both sides of the resulting equation, we obtain the desired result. □

Theorem 2.3 *The following relationship holds true*:

$$\begin{aligned}&(2u-1)\sum_{r=0}^{n}\binom{n}{r}\mathcal{H}_r(x;u;a,b,c;\lambda)\mathcal{H}_{n-r}(y;1-u;a,b,c;\lambda)\\&\quad=(u-1)\mathcal{H}_n(x+y;u;a,b,c;\lambda)+u\mathcal{H}_n(x+y;1-u;a,b,c;\lambda)\\&\qquad+\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k(x+y;u;a,b,c;\lambda)\\&\qquad-\sum_{k=0}^{n}\binom{n}{k}(\ln a)_k^{n-k}\mathcal{H}(x+y;1-u;a,b,c;\lambda).\end{aligned}\tag{9}$$

Proof We set

$$\begin{aligned}&(2u-1)\frac{a^t-u}{\lambda b^t-u}c^{xt}\frac{a^t-(1-u)}{\lambda b^t-(1-u)}c^{yt}\\&\quad=\left(a^t-u\right)\left(a^t-(1-u)\right)c^{(x+y)t}\left(\frac{1}{\lambda b^t-u}-\frac{1}{\lambda b^t-(1-u)}\right).\end{aligned}$$

From the above equation, we see that

$$\begin{aligned}&(2u-1)\left(\sum_{n=0}^{\infty}\mathcal{H}_n(x;u;a,b,c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{H}_n(y;1-u;a,b,c;\lambda)\frac{t^n}{n!}\right)\\&\quad=\left(a^t-1+u\right)\sum_{n=0}^{\infty}\mathcal{H}_n(x+y;u;a,b,c;\lambda)\frac{t^n}{n!}-\left(a^t-u\right)\sum_{n=0}^{\infty}\mathcal{H}_n(x+y;1-u;a,b,c;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Therefore,

$$\begin{aligned}&(2u-1)\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\mathcal{H}_r(x;u;a,b,c;\lambda)\mathcal{H}_{n-r}(y;1-u;a,b,c;\lambda)\frac{t^n}{n!}\\&\quad=(u-1)\sum_{n=0}^{\infty}\mathcal{H}_n(x+y;u;a,b,c;\lambda)\frac{t^n}{n!}+u\sum_{n=0}^{\infty}\mathcal{H}_n(x+y;1-u;a,b,c;\lambda)\frac{t^n}{n!}\end{aligned}$$

$$+\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}\mathcal{H}_r(x+y;u;a,b,c;\lambda)\frac{t^n}{n!}$$
$$-\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}\mathcal{H}_r(x+y;1-u;a,b,c;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 2.4 By substituting $a=1$, $b=c=e$, $\lambda=1$ into Theorem 2.3, we get Carlitz's results (for details, see [1, Eq. 2.19]) as follows:

$$\begin{aligned}&(2u-1)\sum_{r=0}^{n}\binom{n}{r}H_r(x;u)H_{n-r}(y;1-u)\\&\quad=(u-1)H_n(x+y;u)+uH_n(x+y;1-u)+H_n(x+y;u)-H_n(x+y;1-u).\end{aligned}$$

We give the following generating function of the polynomials $Y_n(x;\lambda;a)$:

$$\frac{t}{\lambda a^t-1}a^{xt}=\sum_{n=0}^{\infty}Y_n(x;\lambda;a)\frac{t^n}{n!}\quad(a\geq 1)\tag{10}$$

(*cf.* [16, 17]). We also note that

$$Y_n(0;\lambda;a)=Y_n(\lambda;a).$$

If we substitute $x=0$ and $a=1$ into (10), we see that

$$Y_n(\lambda;1)=\frac{1}{\lambda-1}.$$

Theorem 2.5 *The generalized Apostol-type Frobenius-Euler polynomial holds true as follows*:

$$\begin{aligned}&n\bigl(\mathcal{H}_n(x;u;a,b,b;\lambda)-\ln\bigl(c^x\bigr)\mathcal{H}_n(x;u;a,b,c;\lambda)\bigr)\\&\quad=\ln a^{\frac{1}{u}}\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}\biggl(1;\frac{1}{u};a\biggr)\mathcal{H}_k(x;u;a,b,b;\lambda)\\&\qquad+\ln b^{\frac{\lambda}{u}}\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}\biggl(\frac{1}{u};a\biggr)\mathcal{H}_k^{(2)}(x;u;a,b,b;\lambda).\end{aligned}\tag{11}$$

Proof Substituting $c=b$ for $\alpha=1$ into (2) and taking derivative with respect to t, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{H}_{n+1}(x;u;a,b,b;\lambda)\frac{t^n}{n!}\\&\quad=\frac{a^t\ln a}{a^t-u}\frac{a^t-u}{\lambda b^t-u}b^{xt}+\frac{\ln b\lambda b^t}{a^t-u}\biggl(\frac{a^t-u}{\lambda b^t-u}\biggr)^2b^{xt}+\ln\bigl(b^x\bigr)\frac{a^t-u}{\lambda b^t-u}b^{xt}.\end{aligned}$$

Using (10), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{H}_{n+1}(x;u;a,b,b;\lambda)\frac{t^n}{n!} &= \frac{\ln(a^{\frac{1}{u}})}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}\left(1;\frac{1}{u};a\right)\mathcal{H}_k(x;u;a,b,b;\lambda)\frac{t^n}{n!}\\ &\quad+\frac{\ln(b^{\frac{\lambda}{u}})}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}\left(\frac{1}{u};a\right)\mathcal{H}_k^{(2)}(x;u;a,b,b;\lambda)\frac{t^n}{n!}\\ &\quad+\ln\bigl(b^x\bigr)\sum_{n=0}^{\infty}\mathcal{H}_n(x;u;a,b,b;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Thus, after some elementary calculations, we arrive at (11). □

Theorem 2.6 *Let $|u|<1$ and $m\in\mathbb{N}$. Then we have*

$$\mathcal{H}^{(-m)}(u;a,b,c;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{(-\alpha)}(-x;u;a,b,c;\lambda)\mathcal{H}_{n-k}^{(\alpha-m)}(x;u;a,b,c;\lambda). \tag{12}$$

Proof In (2), we replace α by $-\alpha$, then we set

$$\left(\frac{a^t-u}{\lambda b^t-u}\right)^{-\alpha}c^{(-x)t}\sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha-m)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}=\left(\frac{a^t-u}{\lambda b^t-u}\right)^{-m}.$$

By using (2), we get

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{(-\alpha)}(-x;u;a,b,c;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_n^{(\alpha-m)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{H}_n^{(-m)}(u;a,b,c;\lambda)\frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{(-\alpha)}(-x;u;a,b,c;\lambda)\mathcal{H}_{n-k}^{(\alpha-m)}(x;u;a,b,c;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{H}_n^{(-m)}(u;a,b,c;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at (12). □

3 Interpolation function

In this section, we give a recurrence relation between the generalized Frobenius-Euler polynomials and the Hurwitz-Lerch zeta function. Recently, many authors have studied not only the Hurwitz-Lerch zeta function, but also its generalizations, for example (among others), Srivastava [19], Srivastava and Choi [24] and also Garg *et al.* [6]. The generalization of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ is given as follows:

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a):=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s}$$

($\mu \in \mathbb{C}$, $a, \upsilon \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\rho, \sigma \in \mathbb{R}^+$, $\rho < \sigma$ when $s, z \in \mathbb{C}$ ($|z| < 1$); $\rho = \sigma$ and $\Re(s - \mu + \nu) > 0$ when $|z| = 1$). It is obvious that

$$\Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi_{\mu}^{*}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s} \tag{13}$$

and

$$\Phi_{n}^{*}(z,s,a) = \sum_{n=0}^{\infty} \frac{(n)_n}{n!} \frac{z^n}{(n+a)^s} = \Phi(z,s,a),$$

where $\Phi(z,s,a)$ denotes the Lerch-Zeta function (*cf.* [6, 19, 21, 24]).

Relation between the generalized Apostol-type Frobenius-Euler polynomials and the Hurwitz-Lerch zeta function is given as follows.

Theorem 3.1 *Let* $|\frac{\lambda}{u}| < 1$. *We have*

$$\mathcal{H}_n^{(\alpha)}(x;u;a,b,c;\lambda) = \sum_{k=0}^{\alpha} \binom{\alpha}{k} (-u)^{\alpha-k-1} \mathfrak{G}\left(-n;x,\frac{\lambda}{u};a,b,c;\alpha,k\right), \tag{14}$$

where

$$\mathfrak{G}(s;x,\beta;a,b,c;\alpha,j) = \sum_{m=0}^{\infty} \binom{m+\alpha-1}{m} \frac{\beta^m}{(x\ln c + j\ln a + m\ln b)^s}, \quad |\beta| < 1.$$

Proof From (2), we have

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^n}{n!} = \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-u)^{\alpha-j-1} \sum_{m=0}^{\infty} \binom{m+\alpha-1}{m} \left(\frac{\lambda}{u}\right)^m e^{\alpha(x\ln c + k\ln a + m\ln b)}.$$

Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)}(x;u;a,b,c;\lambda)\frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \sum_{k=0}^{\alpha} \binom{\alpha}{k} (-u)^{\alpha-k-1} \sum_{m=0}^{\infty} \binom{m+\alpha-1}{m} \left(\frac{\lambda}{u}\right)^m (x\ln c + k\ln a + m\ln b)^n \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have arrive at (14). □

Remark 3.2 By substituting $a = 1$, $b = c = e$ into (14), we have

$$\mathcal{H}_n^{(\alpha)}(x;u;\lambda) = -\frac{(1-u)^{\alpha}}{u} \mathfrak{G}\left(-n;x,\frac{\lambda}{u};1,e,e;\alpha,1\right) = -\frac{(1-u)^{\alpha}}{u} \Phi\left(\frac{\lambda}{u},-n,x\right),$$

where

$$\mathfrak{G}\left(-n;x,\frac{\lambda}{u};1,e,e;\alpha,1\right) = \Phi\left(\frac{\lambda}{u},-n,x\right).$$

Remark 3.3 The function $\mathfrak{G}(s;x,\beta;a,b,c;\alpha,j)$ is an interpolation function of the generalized Apostol-type Frobenius-Euler polynomials of order α at negative integers, which is given by the analytic continuation of the $\mathfrak{G}(s;x,\beta;a,b,c;\alpha,j)$ for $s=-n$, $n\in\mathbb{N}$.

4 Relations between Array-type polynomials, Apostol-Bernoulli polynomials and generalized Apostol-type Frobenius-Euler polynomial

In [17], Simsek constructed the generalized λ-Stirling type numbers of the second kind $\mathcal{S}(n,v;a,b;\lambda)$ by means of the following generating function:

$$f_{S,v}(t;a,b;\lambda)=\frac{(\lambda b^t-a^t)^v}{v!}=\sum_{n=0}^{\infty}\mathcal{S}(n,v;a,b;\lambda)\frac{t^n}{n!}. \tag{15}$$

The generating function for these polynomials $\mathcal{S}_v^n(x;a,b;\lambda)$ is given by

$$g_v(x,t;a,b;\lambda)=\frac{1}{v!}\left(\lambda b^t-a^t\right)^v b^{xt}=\sum_{n=0}^{\infty}\mathcal{S}_v^n(x;a,b;\lambda)\frac{t^n}{n!} \tag{16}$$

(*cf.* [17]).

The generalized Apostol-Bernoulli polynomials were defined by Srivastava *et al.* [22, p.254, Eq. (20)] as follows.

Let $a,b,c\in\mathbb{R}^+$ with $a\neq b$, $x\in\mathbb{R}$ and $n\in\mathbb{N}_0$. Then the generalized Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$ of order $\alpha\in\mathbb{Z}$ are defined by means of the following generating functions:

$$f_B(x,a,b,c;\lambda;\alpha)=\left(\frac{t}{\lambda b^t-a^t}\right)^{\alpha}c^{xt}=\sum_{n=0}^{\infty}\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \tag{17}$$

where

$$\left|t\ln\left(\frac{a}{b}\right)+\ln\lambda\right|<2\pi.$$

We note that $\mathfrak{B}_n^{(1)}(x;\lambda;a,b,c)=\mathfrak{B}_n(x;\lambda;a,b,c)$ and also $\mathfrak{B}_n(x;\lambda;1,e,e)=B_n(x;\lambda)$, which denotes the Apostol-Bernoulli polynomials (*cf.* [1–24]).

Theorem 4.1 *Let v be an integer. Then we have*

$$\mathcal{H}_{n-v}^{(-v)}(x;u;a,b,c;\lambda)=\frac{v!}{u^{2v}(n)_v}\sum_{k=0}^{n}\binom{n}{k}\mathcal{S}_v^n\left(x,1,b;\frac{\lambda}{u}\right)Y_{n-k}^{(v)}\left(\frac{1}{u};a\right).$$

Proof Replacing c by b in (2) and after some calculations, we have

$$\sum_{n=0}^{\infty}\mathcal{H}_n^{(-v)}(x;u;a,b,b;\lambda)\frac{t^{n+v}}{n!}=\frac{v!}{u^{2v}}\sum_{n=0}^{\infty}\mathcal{S}_v^n\left(x,1,b;\frac{\lambda}{u}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}Y_n^{(v)}\left(\frac{1}{u};a\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Corollary 4.2

$$\mathcal{H}_{n-\nu}^{(-\nu)}(x;u;a,b,c;\lambda) = \frac{\nu!}{u^{2\nu}(n)_\alpha}\sum_{k=0}^{n}\binom{n}{k}\mathcal{S}\left(k,\nu,1,b;\frac{\lambda}{u}\right)\mathfrak{B}_{n-k}\left(x,a,b;\frac{\lambda}{u}\right).$$

Proof Replacing c by b in (2) and after some calculations, we have

$$\sum_{n=0}^{\infty}\mathcal{H}_{n-\nu}^{(-\nu)}(x;u;a,b,b;\lambda)\frac{t^{n+\nu}}{n!} = \frac{\nu!}{u^{2\nu}}\sum_{n=0}^{\infty}\mathcal{S}\left(n,\nu,1,b;\frac{\lambda}{u}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathfrak{B}_n\left(x,a,b;\frac{\lambda}{u}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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