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Homoclinic solutions for a forced generalized Liénard system

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Abstract

In this article, we find a special class of homoclinic solutions which tend to 0 as $t \rightarrow \pm\infty$, for a forced generalized Liénard system $\ddot{x} + f_1(x)\dot{x} + f_2(x)x^2 + g(x) = p(t)$. Since it is not a small perturbation of a Hamiltonian system, we cannot employ the well-known Melnikov method to determine the existence of homoclinic solutions. We use a sequence of periodically forced systems to approximate the considered system, and find their periodic solutions. We prove that the sequence of those periodic solutions has an accumulation which gives a homoclinic solution of the forced Liénard type system.

MSC: 34A34; 34C99

Keywords: homoclinic; bounded solution; non-Hamiltonian; accumulation

1 Introduction

As a special bounded solution, homoclinic solution is one of important subject in the study of qualitative theory of differential equations. In recent decades, many works (see *e.g.*, [1–5]) contributed to homoclinic solutions and heteroclinic solutions for small perturbation of integrable systems, where either the Melnikov method or the Liapunov-Schmidt reduction was used.

As indicated in [6, 7], an orbit is referred to as a *heteroclinic orbits* if it connects two different equilibria. It is called a *homoclinic orbit* if the two equilibria coincide. For autonomous Hamiltonian systems homoclinic (heteroclinic) orbits can be found from the invariant surfaces (curves) of identical energy containing saddles. In 1990, Rabinowitz [8] considered a nonautonomous Hamiltonian system

$$\ddot{q} + V_q(t, q) = 0, \tag{1}$$

where $t \in \mathbf{R}$, $q : \mathbf{R} \rightarrow \mathbf{R}^n$ and $V : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$ is a differentiable function such that $V(t, 0) \equiv 0$, and gave the existence of its homoclinic solutions. His strategy is to construct a sequence of periodic auxiliary systems to approximate the Hamiltonian system (1), and apply the variational method (see *e.g.*, [9, 10]) to obtain periodic solutions for those auxiliary equations, and prove that the desired homoclinic solution is just an accumulation of those periodic solutions. Later, several different types of Hamiltonian system were also studied for homoclinic orbits (see *e.g.*, [11, 12]). Based on these works, some efforts were made to find homoclinic orbits for nonlinear systems with a time-dependent force. Izy-

dorek and Janczewska [13] considered (1) with a bounded time-dependent force $f(t)$, i.e.,

$$\ddot{q} + V_q(t, q) = f(t), \tag{2}$$

where $t \in \mathbf{R}$, $q : \mathbf{R} \rightarrow \mathbf{R}^n$ and $V : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$ and $f : \mathbf{R} \rightarrow \mathbf{R}^n$, and found a solution $q_0(t)$ which satisfies

$$(q_0(t), \dot{q}_0(t)) \rightarrow (0, 0) \quad \text{as } t \rightarrow \pm\infty. \tag{3}$$

In addition, they also found in [14] such a kind of special solutions for a similar equation to (2). As pointed in [14], the limit $(0, 0)$ is not a solution of the system, they called the solution in (3) is *homoclinic* to zero. Later, some authors studied the existence of this special solution of some Hamiltonian systems (see e.g., [14–17]).

To deal with some non-Hamiltonian systems with a time-dependent force, which is independent of the state variable but cannot be regarded as a small perturbation, the topological degree theory [18, 19] and the fixed point theory [20, 21] are also applied to give the existence of periodic solutions and almost periodic solutions. Applying the Rabinowitz's strategy and the fixed point theory, Zhang [22] considered the existence of homoclinic solutions to the equation

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t),$$

by studying the convergence of a series of periodic solutions to the auxiliary periodic systems, he got a homoclinic solution which is an accumulation of the series of periodic solutions.

In this article, we consider the existence of homoclinic solutions of the forced generalized Liénard type system

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = p(t), \tag{4}$$

where f_1, f_2 and g are continuous functions on \mathbf{R} and p is a bounded continuous function on \mathbf{R} . This generalized equation is frequently encountered as a mathematical model of most dynamics processes in electromechanical systems of physics and engineering [23]. When $f_2(x) \equiv 0$, the equation becomes the equation in [22].

Equation (4) is equivalent to the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -[f_1(x) + f_2(x)y]y - g(x) + p(t), \end{aligned} \tag{5}$$

or

$$\begin{aligned} \dot{x} &= \frac{1}{a(x)}[y - b(x)], \\ \dot{y} &= -a(x)[g(x) - p(t)], \end{aligned} \tag{6}$$

where

$$a(x) = \exp\left(\int_0^x f_2(s) ds\right), \quad b(x) = \int_0^x a(\tau)f_1(\tau) d\tau.$$

There are some results of boundedness and the oscillation of the solutions to Equation (4) [24–30]. In 2007, Hesaaraki and Moradifam [31] studied the global asymptotic and oscillation and existence of periodic solution to a type of generalized Liénard system

$$\begin{aligned}\dot{x} &= \frac{1}{a(x(t))} [h(y(t)) - b(x(t))], \\ \dot{y} &= -a(x(t))g(x(t)).\end{aligned}$$

In 2009, by using the theory of topological degree, Zhou *et al.* [32] got the uniqueness of periodic solution to the system

$$\begin{aligned}\dot{x} &= \frac{1}{a(x(t))} [h(y(t)) - b(x(t))], \\ \dot{y} &= -a(x(t))[g(x(t)) - e(t)].\end{aligned}$$

To study the existence of homoclinic solutions to Equation (4), we still use a sequence of periodic forced systems to approximate Equation (4), and find their periodic solutions. Because our system and those approximating systems are not Hamiltonian, we use the fixed point theory and Massera's theorem (see [33]) instead of the variational method in finding those periodic solutions. We prove that the sequence of those periodic solutions has an accumulation which gives an homoclinic solution of the forced Liénard type system. We need the following hypotheses:

- (H₁) p is nonzero and continuous bounded function on \mathbf{R} and $\|p\| := \sup_{t \in \mathbf{R}} |p(t)| = b > 0$, where $b \leq a, 1 \leq a := \sup_{t \in \mathbf{R}} a(x) < +\infty$, such that $k \in \mathbf{Z}, p(-k) = p(k)$.
- (H₂) For each $x, y \in \mathbf{R}$, there exists a constant $r > 0$ such that $|f_i(x) - f_i(y)| \leq r|x - y|$ ($i = 1, 2$) and $f_1(x) - |f_2(x)|(M + \frac{aM}{2}) \geq a$, where $M = \max\{M_0, 2, b\}, M_0 = g^{-1}(b)$.
- (H₃) $g \in C^1(\mathbf{R}, \mathbf{R}), g(0) = 0, g(\pm\infty) = \pm\infty$ and $0 < g'(x) \leq \frac{a}{M}$ for each $x \in \mathbf{R}$.

Our main result will be given as follows.

Theorem 1.1 *Suppose that conditions (H₁)-(H₃) hold. Then system (4) has a nontrivial homoclinic solution $x(t)$, which satisfies that $(x(t), \dot{x}(t)) \rightarrow (0, 0)$ as $t \rightarrow \pm\infty$.*

When $f_2(x) \equiv 0$, the conditions of Theorem 1.1 are not equivalent to the conditions of the results in [22]. Obviously, let

$$f_1(x) \equiv 8, \quad f_2(x) = e^{-|x|},$$

and

$$g(x) = \frac{1}{2}x, \quad p(t) = \cos \pi t.$$

Then we get

$$a = e, \quad b = 1, \quad M = 2,$$

and the conditions (H₁)-(H₃) hold.

2 Proof of theorem

Lemma 2.1 *Suppose that the conditions (H_1) - (H_3) hold, there is a region D surrounded by a Jordan curve, such that every solution of (4) which starts from the point of D is bounded uniformly.*

Proof Let D be the closure of the region surrounded by the closed curve

$$\Gamma := \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA'} \cup \overline{A'B'} \cup \overline{B'C'} \cup \overline{C'D'} \cup \overline{D'A}$$

as shown in Figure 1, where

$$\overline{AB} = \left\{ (x, y) \in \mathbf{R}^2 \mid y = -\frac{a}{2}(x + M) - M, x \in [-M, 0], y \in \left[-M - \frac{a}{2}M, -M\right] \right\},$$

$$\overline{BC} = \left\{ (x, y) \in \mathbf{R}^2 \mid y = -a(x + M) - M, x \in \left[-M - \frac{a}{2}M, -M\right], y \in [-M, 0] \right\},$$

$$\overline{CD} = \left\{ (x, y) \in \mathbf{R}^2 \mid x = -M - \frac{M}{a}, y \in \left[0, M + \frac{a}{2}M\right] \right\},$$

$$\overline{DA'} = \left\{ (x, y) \in \mathbf{R}^2 \mid y = M + \frac{a}{2}M, x \in \left[-M - \frac{M}{a}, 0\right] \right\},$$

$$\overline{A'B'} = \left\{ (x, y) \in \mathbf{R}^2 \mid y = -\frac{a}{2}(x - M) + M, x \in [0, M], y \in \left[M, M + \frac{a}{2}M\right] \right\},$$

$$\overline{B'C'} = \left\{ (x, y) \in \mathbf{R}^2 \mid y = -a(x - M) + M, x \in \left[M, M + \frac{a}{2}M\right], y \in [0, M] \right\},$$

$$\overline{C'D'} = \left\{ (x, y) \in \mathbf{R}^2 \mid x = M + \frac{M}{a}, y \in \left[-M - \frac{a}{2}M, 0\right] \right\},$$

$$\overline{D'A} = \left\{ (x, y) \in \mathbf{R}^2 \mid y = -M - \frac{a}{2}M, x \in \left[0, M + \frac{M}{a}\right] \right\}.$$

Let $\Xi := \{(x, y, t) \mid (x, y) \in D, t \in \mathbf{R}\}$. For every $t_0 \in \mathbf{R}$, $(x_0, y_0) \in \Gamma$, we claim that $q_k(t, t_0, x_0, y_0) \in \Xi$ for all $t > t_0$. We only need to prove this claim for every solution starting with the cylindrical surface $\Gamma \times \mathbf{R}$.

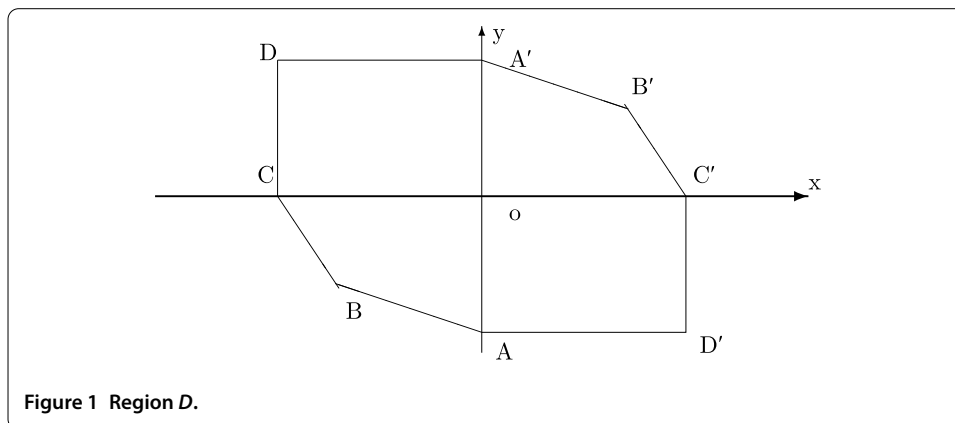


Figure 1 Region D .

If $(x_0, y_0) \in \overline{AB}$, by (H_2) , we have

$$\begin{aligned} \dot{x} &= y < 0, \\ \frac{dy}{dx} &= -f_1(x) - f_2(x)y - \frac{g(x)}{y} + \frac{p(t)}{y} \\ &\leq -f_1(x) - f_2(x)y + \frac{p(t)}{y} \\ &\leq -f_1(x) + |f_2(x)| \left(M + \frac{a}{2}M \right) + \frac{b}{M} \\ &\leq -a + \frac{b}{M} \leq -a + \frac{a}{M} \\ &\leq -\frac{a}{2}. \end{aligned}$$

Then the curve cannot leave Ξ from $\overline{AB} \times \mathbf{R}$ for $t > t_0$.

If $(x_0, y_0) \in \overline{BC}$, since $x \leq M$, by (H_2) , we have $|g(x)| \geq b$. Then

$$\begin{aligned} \dot{x} &= y < 0, \\ \frac{dy}{dx} &= -f_1(x) - f_2(x)y - \frac{g(x)}{y} + \frac{p(t)}{y} \\ &\leq -f_1(x) + |f_2(x)|M \\ &\leq -f_1(x) + |f_2(x)| \left(M + \frac{a}{2}M \right) \\ &\leq -a. \end{aligned}$$

Then the curve cannot leave Ξ from $\overline{BC} \times \mathbf{R}$ for $t > t_0$.

If $(x_0, y_0) \in \overline{CD}$, from $\dot{x} = y > 0$, so the curve cannot leave Ξ from $\overline{CD} \times \mathbf{R}$ for $t > t_0$.

If $(x_0, y_0) \in \overline{DA'}$, by $(H_2), (H_3)$, we have

$$\begin{aligned} \dot{y} &= -f_1(x)y - f_2(x)y^2 - g(x) + p(t) \\ &= \left(M + \frac{a}{2}M \right) \left[-f_1(x) - f_2(x) \left(M + \frac{a}{2}M \right) \right] - g(x) + p(t) \\ &\leq \left(M + \frac{a}{2}M \right) \left[-f_1(x) + |f_2(x)| \left(M + \frac{a}{2}M \right) \right] - g(x) + p(t) \\ &\leq -a \left(M + \frac{a}{2}M \right) - g \left(-M - \frac{M}{a} \right) + a \\ &\leq -a \left(M + \frac{a}{2}M \right) + \frac{a}{M} \left(M + \frac{M}{a} \right) + a \\ &\leq -a \left(M + \frac{a}{2}M \right) + 2a + 1 \\ &\leq -a(M - 2) - \frac{a^2M}{2} + 1 \\ &\leq 0. \end{aligned}$$

Then the curve cannot leave Ξ from $\overline{DA'} \times \mathbf{R}$ for $t > t_0$.

Since the region D is symmetrical, we can prove that the curve cannot leave Ξ from $\overline{A'B'} \cup \overline{B'C'} \cup \overline{C'D'} \cup \overline{D'A'} \times \mathbf{R}$ for $t > t_0$. Then we complete the proof of the claim and this lemma. \square

From the combination of Lemma 2.2 and the Massera's theorem (see [33]), we have the following lemma.

Lemma 2.2 *Suppose that (H_1) - (H_3) hold, and p is a T -periodic function. Then (4) has a T -periodic solution $x(t)$ satisfying $|x(t)| + |\dot{x}(t)| \leq B$, where B depends on D .*

Remark 2.3 From the construction of D , we can take $(2 + \frac{1}{a} + \frac{a}{2})M$ as a value of B in Lemma 2.2.

Now we consider the following two periodic equations

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)x^2 + g(x) = p(t), \tag{7}$$

and

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)x^2 + g(x) = \phi(t), \tag{8}$$

where p is ω_1 -periodic function and ϕ is ω_2 -periodic function. Suppose that all conditions in Theorem 2.1 hold and

$$\sup_{t \in \mathbf{R}} |p(t)| \leq \sup_{t \in \mathbf{R}} |\phi(t)| < +\infty.$$

Then, we obtain two periodic solutions $(x_1(t), \dot{x}_1(t))$ of (7) and $(x_2(t), \dot{x}_2(t))$ of (8), from the construction of D in Theorem 2.1, we can obtain two Jordan domains D_p of (7) and D_ϕ of (8) such that $D_p \subset D_\phi$. The trajectory of $(x_1(t), \dot{x}_1(t))$ is contained in D_p and the trajectory of $(x_2(t), \dot{x}_2(t))$ is contained in D_ϕ . Hence, we can find a region D_0 which is independent of D_p and contains both the two trajectories.

For each $k \in \mathbf{N}$, let $p_k : \mathbf{R} \rightarrow \mathbf{R}$ be a $2k$ -periodic function such that $p_k(t) = p(t)$ for all $t \in [-k, k]$. Since $p(-k) = p(k)$ as indicated in (H_1) , we see that p_k is continuous on the whole \mathbf{R} . Now, we consider a series of periodic equations

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)x^2 + g(x) = p_k(t),$$

or periodic systems

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -[f_1(x) + f_2(x)y]y - g(x) + p_k(t). \end{aligned} \tag{9}$$

Noting that $\sup_{t \in \mathbf{R}} |p_k(t)| \leq \sup_{t \in \mathbf{R}} |p(t)| < +\infty$ for every $k \in \mathbf{N}$, and from the discussion above we can get results as follows.

Lemma 2.4 *Suppose that conditions (H_1) - (H_3) hold. Then system (9) possesses a $2k$ -periodic solution q_k for every $k \in \mathbf{N}$. Moreover, there exists a constant B independent of k such that $|q_k| := \max_{t \in [-k, k]} \{|x_k(t)|, |y_k(t)|\} \leq B$.*

Now, for the sequence of periodic solutions $\{q_k\}_{k \in \mathbb{N}}$ determined in Lemma 2.4, we obtain the following lemma by using the Ascoli-Arzelà theorem.

Lemma 2.5 *There exist a subsequence $\{q_i\}_{i \in \mathbb{N}}$ of $\{q_k\}_{k \in \mathbb{N}}$ and continuous functions $q_0 : \mathbb{R} \rightarrow \mathbb{R}^2$ and $\tilde{q}_0 : \mathbb{R} \rightarrow \mathbb{R}^2$ such that $q_i \rightarrow q_0$, $\dot{q}_i \rightarrow \tilde{q}_0$, as $i \rightarrow +\infty$, in $C_b(\mathbb{R}, \mathbb{R}^2)$.*

Proof For each $t, t_0 \in \mathbb{R}$, $t \geq t_0$, by Lemma 2.4, there exists a constant $B_1 > 0$ such that $|-f_1(x_k(s))y_k(s) - f_2(x(s))y_k^2(s) - g(x_k(s))| \leq B_1$. We note that

$$\begin{aligned} |x_k(t) - x_k(t_0)| &= \left| \int_{t_0}^t \frac{dx_k(t)}{dt} dt \right| \\ &\leq \int_{t_0}^t |y_k(s)| ds \leq B|t - t_0|, \\ |y_k(t) - y_k(t_0)| &\leq \int_{t_0}^t |-f_1(x_k(s))y_k(s) - f_2(x(s))y_k^2(s) - g(x_k(s))| dt \\ &\quad + \int_{t_0}^t |p_k(s)| ds \\ &\leq (B_1 + b)|t - t_0|, \end{aligned}$$

and

$$\begin{aligned} |\dot{x}_k(t) - \dot{x}_k(t_0)| &= |y_k(t) - y_k(t_0)| \leq (B_1 + b)|t - t_0|, \\ |\dot{y}_k(t) - \dot{y}_k(t_0)| &= |f_1(x_k(t))y_k(t) + f_2(x(t))y_k^2(t) + g(x_k(t)) - p_k(t) \\ &\quad - f_1(x_k(t_0))y_k(t_0) - f_2(x(t_0))y_k^2(t_0) - g(x_k(t_0)) + p_k(t_0)| \\ &\leq |f_1(x_k(t))y_k(t) - f_1(x_k(t_0))y_k(t_0)| + |f_2(x(t))y_k^2(t) \\ &\quad - f_2(x(t_0))y_k^2(t_0)| + |g(x_k(t)) - g(x_k(t_0))| + |p_k(t) - p_k(t_0)|. \end{aligned}$$

For $|\dot{y}_k(t) - \dot{y}_k(t_0)|$, since $x_k(t)$ is bounded, we can define

$$\max |f(x_k(t))| := r_1 \geq 0.$$

By the conditions (H_2) , we can get the estimation as follows.

$$\begin{aligned} &|f_1(x_k(t))y_k(t) - f_1(x_k(t_0))y_k(t_0)| \\ &= |f_1(x_k(t))y_k(t) - f_1(x_k(t))y_k(t_0) + f_1(x_k(t))y_k(t_0) - f_1(x_k(t_0))y_k(t_0)| \\ &\leq |f_1(x_k(t))y_k(t) - f_1(x_k(t))y_k(t_0)| + |f_1(x_k(t))y_k(t_0) - f_1(x_k(t_0))y_k(t_0)| \\ &\leq |f_1(x_k(t))| |y_k(t) - y_k(t_0)| + |y_k(t_0)| |f_1(x_k(t)) - f_1(x_k(t_0))| \\ &\leq r_1 |y_k(t) - y_k(t_0)| + Br |x_k(t) - x_k(t_0)| \\ &\leq [r_1(B_1 + b) + B^2r] |t - t_0|. \end{aligned}$$

Let

$$\max |f_2(x_k(t))(y_k(t) + y_k(t_0))| := r_2 \geq 0,$$

we have

$$\begin{aligned}
 & |f_2(x_k(t))y_k^2(t) - f_2(x_k(t_0))y_k^2(t_0)| \\
 &= |f_2(x_k(t))y_k^2(t) - f_2(x_k(t))y_k^2(t_0) + f_2(x_k(t))y_k^2(t) - f_2(x_k(t_0))y_k^2(t_0)| \\
 &\leq |f_2(x_k(t))y_k^2(t) - f_2(x_k(t))y_k^2(t_0)| + |f_2(x_k(t))y_k^2(t) - f_2(x_k(t_0))y_k^2(t_0)| \\
 &\leq |f_2(x_k(t))(y_k(t) + y_k(t_0))||y_k(t) - y_k(t_0)| + |y_k^2(t_0)||f_2(x_k(t)) - f_2(x_k(t_0))| \\
 &\leq r_2|y_k(t) - y_k(t_0)| + B^2r|x_k(t) - x_k(t_0)| \\
 &\leq [r_2(B_1 + b) + B^3r]|t - t_0|.
 \end{aligned}$$

Moreover, by (H_3) , we have

$$|g(x_k(t)) - g(x_k(t_0))| \leq \frac{a}{M}|x_k(t) - x_k(t_0)| \leq B\frac{a}{M}|t - t_0|.$$

Since p is continuous and bounded, there is a constant $r_3 > 0$ such that

$$|p_k(t) - p_k(t_0)| \leq r_3|t - t_0|.$$

Hence, we obtain

$$|\dot{y}_k(t) - \dot{y}_k(t_0)| \leq \left\{ [r_1(B_1 + b) + B^2r] + [r_2(B_1 + b) + B^3r] + B\frac{a}{M} + r_3 \right\} |t - t_0|.$$

So $\{q_k\}_{k \in \mathbf{N}}$ and $\{\dot{q}_k\}_{k \in \mathbf{N}}$ are both equicontinuous. On the other hand, from Lemma 2.4, $\{q_k\}_{k \in \mathbf{N}}$ is bounded uniformly, and so does $\{\dot{q}_k\}_{k \in \mathbf{N}}$. Hence we obtain the existence of a subsequence $\{q_i\}_{i \in \mathbf{N}}$ convergent to a certain $q_0 := (x_0(t), y_0(t))$ in $C_b(\mathbf{R}, \mathbf{R}^2)$ by Ascoli-Arzelà theorem. \square

Remark 2.6 We cannot conclude that the convergence of the subsequence $\{q_i\}_{i \in \mathbf{N}}$ is uniform for $t \in \mathbf{R}$. However, for every $a, b \in \mathbf{R}$, $a < b$, the uniform convergence holds on $[a, b]$.

To be convenient, we still represent $\{q_i\}_{i \in \mathbf{N}}$ by $\{q_k\}$. For each $k \in \mathbf{N}$, let $E_{2k} \subset C_b(\mathbf{R}, \mathbf{R})$, denote the Banach space of continuous $2k$ -periodic functions on \mathbf{R} with values in \mathbf{R} under the norm

$$\|x\|_{E_k} = \left(\int_{-k}^k |x(t)|^2 dt \right)^{\frac{1}{2}}.$$

Now, we prove the main result that q_0 is the desired homoclinic solution of (5).

Lemma 2.7 *The function q_0 determined by Lemma 2.5 is a nontrivial homoclinic solution of (5).*

Proof The proof will be divided into three steps.

Step 1. We show that q_0 is a solution of (9). For every $k \in \mathbf{N}$ and $t \in \mathbf{R}$ we have

$$\begin{aligned}
 \dot{x}_k &= y_k, \\
 \dot{y}_k &= -[f_1(x_k) + f_2(x_k)y_k]y_k - g(x_k) + p_k(t).
 \end{aligned} \tag{10}$$

For each fixed $a, b \in \mathbf{R}$, $a < b$, since $q_k \rightarrow q_0$ and $p_k \rightarrow p$ on $[a, b]$ uniformly, we have

$$\begin{aligned} \dot{q}_k &= \left(\frac{dx_k(t)}{dt}, \frac{dy_k(t)}{dt} \right) \rightarrow w \\ &:= (y_0(t), -[f_1(x_0(t)) + f_2(x_0(t))y_0(t)]y_0(t) - g(x_0(t)) + p(t)) \end{aligned} \quad (11)$$

on $[a, b]$ uniformly. Hence there exists a constant $k_0 \in \mathbf{N}$ such that (10) can be transformed into

$$\begin{aligned} \dot{x}_k &= y_k, \\ \dot{y}_k &= -[f_1(x_k) + f_2(x_k)y_k]y_k - g(x_k) + p(t) \end{aligned}$$

for each $k \geq k_0$ and $t \in [a, b]$. So \dot{q}_k is continuous on $[a, b]$ for $k \geq k_0$. Note the fact that \dot{q}_k is a derivative of q_k in $[a, b]$ for every $k \geq k_0$ and \dot{q}_k converges to \dot{q}_0 uniformly in $[a, b]$ by Lemma 2.5. Since (11) holds and $q_k \rightarrow q_0$ uniformly on $[a, b]$, we have $w = \dot{q}_0$ in (a, b) . Because a and b are arbitrary, we conclude that $w = \dot{q}_0$ in \mathbf{R} and q_0 satisfies (5). Moreover, we have actually proved that $\{q_k\}$ converges to q_0 in $C_b(\mathbf{R}, \mathbf{R}^2)$.

Step 2. We prove that $x_0(t) \rightarrow 0$, as $t \rightarrow \pm\infty$. We note that

$$x_k \ddot{x}_k = -g(x_k)x_k - f_1(x_k)(\dot{x}_k)^2 - f_2(x_k)(\dot{x}_k)^3 + p_k(t)\dot{x}_k, \quad (12)$$

by Lemmas 2.1 and 2.2, we have

$$\dot{x}_k \leq M + \frac{aM}{2},$$

from (H_2) , we obtain

$$f_1(x_k) + f_2(x_k)\dot{x}_k \geq f_1(x_k) - |f_2(x_k)| \left(M + \frac{aM}{2} \right) \geq a.$$

Integrating (12) from $-k$ to k , we have

$$\int_{-k}^k (f_1(x_k) + f_2(x_k)(\dot{x}_k))(\dot{x}_k)^2 dt = \int_{-k}^k p_k(t)\dot{x}_k dt.$$

So

$$\begin{aligned} a \|\dot{x}_k\|_{E_k}^2 &\leq \left| \int_{-k}^k (f_1(x_k) + f_2(x_k)(\dot{x}_k))(\dot{x}_k)^2 dt \right| \\ &\leq \|p_k\|_{E_k} \|\dot{x}_k\|_{E_k}. \end{aligned}$$

Hence, by (H_1) , there is a constant $d_1 > 0$ independent of k such that

$$\|\dot{x}_k\|_{E_k} = \|y_k\|_{E_k} \leq d_1. \quad (13)$$

On the other hand, we have

$$x_k \ddot{x}_k = -g(x_k)x_k - f_1(x_k)\dot{x}_k x_k - f_2(x_k)(\dot{x}_k)^2 x_k + p_k(t)x_k. \quad (14)$$

From Lemmas 2.1 and 2.2, there exists a constant $m_1 > 0$ independent of k such that

$$|f_2(x_k)(\dot{x}_k)^2| \leq m_1.$$

Let $m_2 = \min_{(x,\dot{x}) \in D} g'(x)$, since $g'(x) > 0$, we have $m_2 > 0$ and $|g(x_k)| \geq m_2|x_k|$. Integrating (14) from $-k$ to k , we have

$$d_1^2 = \|\dot{x}_k\|_{E_k}^2 = \int_{-k}^k (\dot{x}_k)^2 dt \geq m_2 \|x_k\|_{E_k}^2 - (m_1 + b)\|x_k\|.$$

Then there exists a constant $d_2 > 0$ independent of k such that

$$\|x_k\|_{E_k} \leq d_2. \tag{15}$$

We note that

$$\|x_0\|_{L^2}^2 = \lim_{i \rightarrow +\infty} \int_{-i}^i |x_0(t)|^2 dt = \lim_{i \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{-i}^i |x_k(t)|^2 dt.$$

Obviously,

$$\int_{-i}^i |x_k(t)|^2 dt \leq \int_{-k}^k |x_k(t)|^2 dt = \|x_k\|_{E_k}^2 \leq d_2^2$$

holds for every $i \in \mathbf{N}$ and $k \in \mathbf{N}$, $k \geq i$. Since $x_k \rightarrow x_0$ on $[-i, i]$ uniformly. Let $k \rightarrow +\infty$, we obtain

$$\int_{-i}^i |x_0(t)|^2 dt \leq d_2^2,$$

and let $i \rightarrow +\infty$, we have

$$\int_{-\infty}^{+\infty} |x_0(t)|^2 dt \leq d_2^2.$$

Hence,

$$\lim_{r \rightarrow \infty} \int_{|t| \geq r} |x_0(t)|^2 dt = 0. \tag{16}$$

This implies that $x_0(t) \rightarrow 0$ as $t \rightarrow \pm\infty$. Similarly, $y_0(t) \rightarrow 0$ as $t \rightarrow \pm\infty$.

Step 3. $p(t) \not\equiv 0$ implies that q_0 is nontrivial.

We complete this lemma. □

Finally, Theorem 1.1 is proved by summarizing the results in Lemmas 2.1 and 2.7.

Competing interests

The author declares that he has no competing interests.

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