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Positive periodic solutions of second-order difference equations with weak singularities

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Abstract

We study the existence of positive periodic solutions of the second-order difference equation

$$\Delta^2 u(t-1) + a(t)u(t) = f(t, u(t)) + c(t), \ t \in \mathbb{Z}$$

via Schauder's fixed point theorem, where $a, c : \mathbb{Z} \to \mathbb{R}_+$ are T-periodic functions, $f \in C(\mathbb{Z} \times (0, \infty), \mathbb{R})$ is T-periodic with respect to t and singular at $u = 0$.
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1 Introduction and the main results

Let $\mathbb{Z}$ denote the integer set, for $a, b \in \mathbb{Z}$ with $a < b$, $[a, b]_{\mathbb{Z}} := \{a, a+1, ..., b\}$ and $\mathbb{R}_+ :=$ $[0; \infty)$. In this article, we are concerned with the existence of positive periodic solutions of the second-order difference equation

$$\Delta^2 u(t-1) + a(t)u(t) = f(t, u(t)) + c(t), \quad t \in \mathbb{Z}, \tag{1.1}$$

where $a, c : \mathbb{Z} \to \mathbb{R}_+$ are T-periodic functions, $f \in C(\mathbb{Z} \times (0, \infty), \mathbb{R})$ is T-periodic with respect to t and singular at $u = 0$.

Positive periodic solutions of second-order difference equations have been studied by many authors, see [1-6]. However, in these therein, the nonlinearities are nonsingular, what would happen if the nonlinearity term is singular? It is of interest to note here that singular boundary value problems in the continuous case have been studied in great detail in the literature [7-20]. In 1987, Lazer and Solimini [7] firstly investigated the existence of the positive periodic solutions of the problem

$$u'' = \frac{1}{u^\lambda} + c(t), \tag{1.2}$$

where $c \in C(\mathbb{R}, \mathbb{R})$ is T-periodic. They proved that for $\lambda \geq 1$ (called *strong force condition* in a terminology first introduced by Gordon [8,9]), a necessary and sufficient condition for the existence of a positive periodic solution of (1.2) is that the mean value of c is negative,

$$\bar{c} := \frac{1}{T}\int_0^T c(t)dt < 0.$$

Moreover, if $0 < \lambda < 1$ (*weak force condition*) they found examples of functions c with negative mean values and such that periodic solutions do not exist. Subsequently, many authors studied the existence of positive solutions of the problem

$$u'' + a(t)u = f(t, u) + c(t), \tag{1.3}$$

where $a \in L^1(\mathbb{R}/T\mathbb{Z}; \mathbb{R}_+)$, $c \in L^1(\mathrm{R}/T\mathbb{Z}; \mathbb{R})$, $f \in \mathrm{Car}(\mathbb{R}/T\mathbb{Z} \times (0, \infty), \mathbb{R})$ and is singular at $u = 0$, see [7-20]. The first existence result with weak force condition appears in Rachunková et al. [16]. Since then, the Equation (1.3) with f has weak singularities has been studied by several authors, see Torres [17,18], Franco and Webb [19], Chu and Li [20].

Recently, Torres [18] showed how a weak singularity can play an important role if Schauder's fixed point theorem is chosen in the proof of the existence of positive periodic solution for (1.3). For convenience, for a given function $\xi \in L^\infty[0, T]$, we denote the essential supremum and infimum of ξ by ξ^* and ξ_*, respectively. We write $\xi \succ 0$ if $\xi \geq 0$ for a.e. $t \in [0, T]$ and it is positive in a set of positive measure. Under the assumption

(H1) The linear equation $u'' + a(t)u = 0$ is nonresonant and the corresponding Green's function

$$G(t, s) \geq 0, \qquad (t, s) \in [0, T] \times [0, T].$$

Torres showed the following three results

Theorem A. [[18], Theorem 1] Let (H1) hold and define

$$\gamma(t) = \int_0^T G(t, s)c(s)ds.$$

Assume that

(H2) there exist $b \in L^1(0, T)$ with $b \succ 0$ and $\lambda > 0$ such that

$$0 \leq f(t, u) \leq \frac{b(t)}{u^\lambda}, \quad \text{for all } u > 0, \quad a.e.\ t \in [0, T].$$

If $\gamma_* > 0$, then there exists a positive T-periodic solution of (1.3).

Theorem B. [[18], Theorem 2] Let (H1) hold. Assume that

(H3) there exist two functions $b, \hat{b} \in L^1(0, T)$ with $b, \hat{b} \succ 0$ and a constant $\lambda \in (0, 1)$ such that

$$0 \leq \frac{\hat{b}(t)}{u^\lambda} \leq f(t, u) \leq \frac{b(t)}{u^\lambda}, u \in (0, \infty), \text{ a.e. } t \in [0, T].$$

If $\gamma_* = 0$. Then (1.3) has a positive T-periodic solution.

Theorem C. [[18], Theorem 4] Let (H1) and (H3) hold. Let

$$\hat{\beta}_* = \min_{t\in[0,T]}\left(\int_0^T G(t, s)\hat{b}(s)ds\right), \quad \beta^* = \max_{t\in[0,T]}\left(\int_0^T G(t, s)b(s)ds\right).$$

If $\gamma^* \le 0$ and

$$\gamma_* \ge \left(\frac{\hat{\beta}_*}{(\beta^*)^\lambda}\lambda^2\right)^{\frac{1}{1-\lambda^2}}\left(1-\frac{1}{\lambda^2}\right).$$

Then (1.3) has a positive T-periodic solution.

However, the discrete analogue of (1.3) has received almost no attention. In this article, we will discuss in detail the singular discrete problem (1.1) with our goal being to fill the above stated gap in the literature. For other results on the existence of positive solution for the other singular discrete boundary value problem, see [21-24] and their references. From now on, for a given function $\xi \in l^\infty(0, \infty)$, we denote the essential supremum and infimum of ξ by ξ^* and ξ_*, respectively. We write $\xi \succ 0$ if $\xi \ge 0$ for t $[0, T]_{\mathbb{Z}}$ and it is positive in a set of positive measure.

Assume that

(A1) The linear equation $\Delta^2 u(t-1) + a(t)u(t) = 0$ is nonresonant and the corresponding Green's function

$$G(t,\ s) \ge 0, \qquad (t,\ s) \in [0,\ T]_{\mathbb{Z}} \times [0,\ T]_{\mathbb{Z}}.$$

(A2) There exist $b, e : [1, T]_{\mathbb{Z}} \to \mathbb{R}_+$ with $b, e \succ 0$, $\alpha, \beta \in (0, \infty)$, $m \le 1 \le M$, such that

$$0 \le f(t,\ u) \le \frac{b(t)}{u^\alpha}, \qquad u \in (M,\ \infty), \qquad t \in [1,\ T]_{\mathbb{Z}},$$

and

$$0 \le f(t,\ u) \le \frac{e(t)}{u^\beta}, \qquad u \in (0,\ m), \qquad t \in [1,\ T]_{\mathbb{Z}}.$$

(A3) There exist $b_1, b_2, e : [1, T]_{\mathbb{Z}} \to \mathbb{R}_+$ with $b_1, b_2, e \succ 0$, $\alpha, \beta, \mu, \nu \in (0, 1)$, such that

$$0 \le \frac{b_1(t)}{u^\alpha} \le f(t,\ u) \le \frac{b_2(t)}{u^\beta}, \quad u \in [1,\infty),\ t \in [1,\ T]_{\mathbb{Z}},$$

and

$$0 \le \frac{b_1(t)}{u^\mu} \le f(t,\ u) \le \frac{e(t)}{u^\nu}, \quad u \in [0, 1), \qquad t \in [1,\ T]_{\mathbb{Z}}.$$

To prove the main results, we will use the following notations.

$$\gamma(t) := \sum_{s=1}^{T} G(t,\ s)c(s), \qquad E(t) := \sum_{s=1}^{T} G(t,\ s)e(s);$$
$$B(t) := \sum_{s=1}^{T} G(t,\ s)b(s), \qquad B_i(t) := \sum_{s=1}^{T} G(t,\ s)b_i(s), \qquad i = 1, 2;$$
$$\rho^* := E^* + B_2^*, \qquad \sigma := \max\{\mu,\ \alpha\}, \qquad \delta := \max\{\nu,\ \beta\}.$$

Our main results are the following

Theorem 1.1. Let (A1) and (A2) hold. If $\gamma_* > 0$. Then (1.1) has a positive T-periodic solution.

Theorem 1.2. Let (A1) and (A3) hold. If $\gamma_* = 0$. Then (1.1) has a positive T-periodic solution.

Theorem 1.3. Let (A1) and (A3) hold. Assume that

$$\rho^* > \max\{(\delta\sigma B_{1_*})^{\delta}, (\delta\sigma B_{1_*})^{\frac{1}{\sigma}}\}. \tag{1.4}$$

If $\gamma^* \le 0$ and

$$\gamma_* \ge \left[\frac{B_{1_*}}{(\rho^*)^{\sigma}}\delta\sigma\right]^{\frac{1}{1-\delta\sigma}}\left(1-\frac{1}{\delta\sigma}\right). \tag{1.5}$$

Then (1.1) has a positive T-periodic solution.

Remark 1.1. Let us consider the function

$$f_0(t, u) = \begin{cases} \dfrac{1}{u^{\epsilon}}, & u \in [1, \infty), \\ \dfrac{1}{u^{\eta}}, & u \in (0, 1), \end{cases} \tag{1.6}$$

where $\varepsilon, \eta > 0$. Obviously, f_0 satisfies (A2) with $M = m = 1$, $b(t) = e(t) \equiv 1$. However, it is fail to satisfy (H2) since it can not be bounded by a single function $\frac{h(t)}{u^{\gamma}}$ for any $\gamma \in (0, \infty)$ and any $h \succ 0$. □

Remark 1.2. If $\varepsilon, \eta \in (0, 1)$, then the function f_0 defined by (1.6) satisfies (A3) with $\nu = \mu = \eta$, $\alpha = \beta = \varepsilon$, and $b_1(t) \equiv b_2(t) \equiv e(t) \equiv 1$. However, it is fail to satisfy (H3). □

2 Proof of Theorem 1.1

Let

$$X := \{u : \mathbb{Z} \to \mathbb{R} | u(t) = u(t+T)\}$$

under the norm $\|u\| = \max_{t\in[1,T]_{\mathbb{Z}}} |u(t)|$. Then $(X, ||\cdot||)$ is a Banach space.

A T-periodic solution of (1.1) is just a fixed point of the completely continuous map $A : X \to X$ defined as

$$(Au)(t) := \sum_{s=1}^{T} G(t, s)[f(s, u(s)) + c(s)] = \sum_{s=1}^{T} G(t, s) f(s, u(s)) + \gamma(t).$$

By Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set defined as

$$K = \{u \in X : r \le u(t) \le R, \quad \text{for all } t \in [0, T]_{\mathbb{Z}}\}$$

into itself, where $R > r > 0$ are positive constants to be fixed properly.

For given $u \in K$, let us denote

$$I_1 := \{t \in [0, T]_{\mathbb{Z}} | r \le u(t) < m\},$$
$$I_2 := \{t \in [0, T]_{\mathbb{Z}} | R \ge u(t) > M\},$$
$$I_3 := [0, T]_{\mathbb{Z}} \backslash (I_1 \cup I_2).$$

Given $u \in K$, by the nonnegative sign of G and f, we have

$$\begin{aligned}(Au)(t) &= \sum_{s=1}^{T} G(t,s)f(s,u(s)) + \gamma(t)\\ &= \sum_{s\in I_1} G(t,s)f(s,u(s)) + \sum_{s\in I_2} G(t,s)f(s,u(s))\\ &\quad + \sum_{s\in I_3} G(t,s)f(s,u(s)) + \gamma(t)\\ &\geq \gamma(t) \geq \gamma_* =: r.\end{aligned}$$

Let

$$\Lambda := \sup\left\{\max_{t\in[0,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)f(s,u(s)) | m \leq u(s) \leq M\right\}.$$

Then, it follows from the continuity of f that $\Lambda < \infty$, and consequently, for every $u \in K$,

$$\begin{aligned}(Au)(t) &= \sum_{s=1}^{T} G(t,s)f(s,u(s)) + \gamma(t)\\ &= \sum_{s\in I_1} G(t,s)f(s,u(s)) + \sum_{s\in I_2} G(t,s)f(s,u(s))\\ &\quad + \sum_{s\in I_3} G(t,s)f(s,u(s)) + \gamma(t)\\ &\leq \sum_{s\in I_1} G(t,s)\frac{e(s)}{u^{\beta}} + \sum_{s\in I_2} G(t,s)\frac{b(s)}{u^{\alpha}} + \Lambda + \gamma^*\\ &\leq \sum_{s=1}^{T} G(t,s)\frac{e(s)}{u^{\beta}} + \sum_{s\in I_2} G(t,s)b(s) + \Lambda + \gamma^*\\ &\leq \sum_{s=1}^{T} G(t,s)\frac{e(s)}{r^{\beta}} + \sum_{s=1}^{T} G(t,s)b(s) + \Lambda + \gamma^*\\ &\leq \frac{E^*}{r^{\beta}} + (B^* + \Lambda + \gamma^*)\\ &< \frac{E^*}{r^{\beta}} + (B^* + \Lambda + \gamma^*) =: R.\end{aligned}$$

Therefore, $A(K) \subset K$ if $r = \gamma_*$ and $R = \frac{E^*}{r^{\beta}} + (B^* + \Lambda + \gamma^*)$, and the proof is finished. □

3 Proof of Theorem 1.2

We follow the same strategy and notations as in the proof of Theorem 1.1. Define a closed convex set

$$K = \{u \in X : r \leq u(t) \leq R, \quad \text{for all } t \in [0,T]_{\mathbb{Z}}, R > 1\}.$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set K into itself, where R and r are positive constants to be fixed properly and they should satisfy $R > r > 0$ and $R > 1$.

For given $u \in K$, let us denote

$$
\begin{aligned}
J_1 &:= \{t \in [0, T]_{\mathbb{Z}} | r \le u(t) < 1\},\\
J_2 &:= \{t \in [0, T]_{\mathbb{Z}} | R \ge u(t) \ge 1\}.
\end{aligned}
$$

Then for given $u \in K$, by the nonnegative sign of G and f, it follows that

$$
\begin{aligned}
(Au)(t) &= \sum_{s=1}^{T} G(t, s) f(s, u(s)) + \gamma(t)\\
&= \sum_{s \in J_1} G(t, s) f(s, u(s)) + \sum_{s \in J_2} G(t, s) f(s, u(s)) + \gamma(t)\\
&\le \sum_{s \in J_1} G(t, s) \frac{e(s)}{u^{\nu}} + \sum_{s \in J_2} G(t, s) \frac{b_2(s)}{u^{\beta}} + \gamma^*\\
&\le \sum_{s=1}^{T} G(t, s) \frac{e(s)}{r^{\nu}} + \sum_{s \in J_2} G(t, s) b_2(s) + \gamma^*\\
&\le \sum_{s=1}^{T} G(t, s) \frac{e(s)}{r^{\nu}} + \sum_{s=1}^{T} G(t, s) b_2(s) + \gamma^*\\
&\le \frac{E^*}{r^{\nu}} + (B_2^* + \gamma^*),
\end{aligned}
$$

On the other hand, for every $u \in K$,

$$
\begin{aligned}
(Au)(t) &= \sum_{s=1}^{T} G(t, s) f(s, u(s)) + \gamma(t)\\
&= \sum_{s \in J_1} G(t, s) f(s, u(s)) + \sum_{s \in J_2} G(t, s) f(s, u(s)) + \gamma(t)\\
&\ge \sum_{s \in J_1} G(t, s) \frac{b_1(s)}{u^{\mu}} + \sum_{s \in J_2} G(t, s) \frac{b_1(s)}{u^{\alpha}} + \gamma_*\\
&\ge \sum_{s \in J_1} G(t, s) \frac{b_1(s)}{R^{\mu}} + \sum_{s \in J_2} G(t, s) \frac{b_1(s)}{R^{\alpha}}\\
&\ge \sum_{s \in J_1} G(t, s) \frac{b_1(s)}{R^{\sigma}} + \sum_{s \in J_2} G(t, s) \frac{b_1(s)}{R^{\sigma}}\\
&\ge \sum_{s=1}^{T} G(t, s) \frac{b_1(s)}{R^{\sigma}}\\
&\ge \frac{B_{1_*}}{R^{\sigma}}.
\end{aligned}
$$

Thus $Au \in K$ if r, R are chosen so that

$$
\frac{B_{1*}}{R^{\sigma}} \ge r, \qquad \frac{E^*}{r^{\nu}} + (B_2^* + \gamma^*) \le R.
$$

Note that B_{1^*}, $E^* > 0$ and taking $R = \frac{1}{r}$, it is sufficient to find $R > 1$ such that

$$
B_{1*} R^{1-\sigma} \ge 1, \qquad E^* R^{\nu} + (B_2^* + \gamma^*) \le R,
$$

and these inequalities hold for R big enough because $\sigma < 1$ and $\nu < 1$. □

Remark 3.1. It is worth remarking that Theorem 1.2 is also valid for the special case that $c(t) \equiv 0$, which implies that $\gamma_* = 0$. □

4 Proof of Theorem 1.3

Define a closed convex set

$$K = \{u \in X : r \leq u(t) \leq R, \quad \text{for all } t \in [0, T]_{\mathbb{Z}}, \quad 0 < r < 1 < R\}.$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set K into itself, where R and r are positive constants to be fixed properly and they should satisfy $R > 1 > r > 0$.

Recall that $\delta = \max\{\nu, \beta\}$ and $r < 1$, for given $u \in K$,

$$\begin{aligned}(Au)(t) &= \sum_{s=1}^{T} G(t, s) f(s, u(s)) + \gamma(t) \\ &= \sum_{s \in J_1} G(t, s) f(s, u(s)) + \sum_{s \in J_2} G(t, s) f(s, u(s)) + \gamma(t) \\ &\leq \sum_{s \in J_1} G(t, s) \frac{e(s)}{u^{\nu}} + \sum_{s \in J_2} G(t, s) \frac{b_2(s)}{u^{\beta}} + \gamma^* \\ &\leq \sum_{s \in J_1} G(t, s) \frac{e(s)}{r^{\nu}} + \sum_{s \in J_2} G(t, s) \frac{b_2(s)}{r^{\beta}} \\ &\leq \sum_{s=1}^{T} G(t, s) \frac{e(s)}{r^{\delta}} + \sum_{s=1}^{T} G(t, s) \frac{b_2(s)}{r^{\delta}} \\ &\leq \frac{\rho^*}{r^{\delta}},\end{aligned}$$

where J_i $(i = 1, 2)$ is defined as in Section 3 and $\rho^* = E^* + B_2^*$.

On the other hand, since $\sigma = \max\{\mu, \alpha\}$ and $R > 1$, for every $u \in K$,

$$\begin{aligned}(Au)(t) &= \sum_{s=1}^{T} G(t, s) f(s, u(s)) + \gamma(t) \\ &= \sum_{s \in J_1} G(t, s) f(s, u(s)) + \sum_{s \in J_2} G(t, s) f(s, u(s)) + \gamma(t) \\ &\geq \sum_{s \in J_1} G(t, s) \frac{b_1(s)}{u^{\mu}} + \sum_{s \in J_2} G(t, s) \frac{b_1(s)}{u^{\alpha}} + \gamma_* \\ &\geq \sum_{s \in J_1} G(t, s) \frac{b_1(s)}{R^{\sigma}} + \sum_{s \in J_2} G(t, s) \frac{b_1(s)}{R^{\sigma}} + \gamma_* \\ &\geq \frac{B_{1*}}{R^{\sigma}} + \gamma_*.\end{aligned}$$

In this case, to prove that $A(K) \subset K$ it is sufficient to find $r < R$ with $0 < r < 1 < R$ such that

$$\frac{B_{1*}}{R^{\sigma}} + \gamma_* \geq r, \quad \frac{\rho^*}{r^{\delta}} \leq R. \tag{4.1}$$

If we fix $R = \frac{\rho^*}{r^{\delta}}$, then the first inequality holds if r verifies

$$\frac{B_{1*}}{(\rho^*)^{\sigma}} r^{\sigma\delta} + \gamma_* \geq r,$$

or equivalently,

$$\gamma_* \geq f(r) := r - \frac{B_{1*}}{(\rho^*)^\sigma} r^{\sigma\delta}.$$

The function $f(r)$ possesses a minimum in $r_0 := \left[\frac{B_{1*}}{(\rho^*)^\sigma}\delta\sigma\right]^{\frac{1}{1-\delta\sigma}}$. Taking $r = r_0$, (1.4) implies that $r < 1$. Then the first inequality in (4.1) holds if $\gamma_* \geq f(r_0)$, which is just condition (1.5). The second inequality in (4.1) holds directly by the choice of R, and it would remain to prove that $R = \frac{\rho^*}{r_0^\delta} > 1$. To the end, it follows from (1.4) that

$$\begin{aligned} R = \frac{\rho^*}{r_0^\delta} &> \frac{(\delta\sigma B_{1*})^\delta \cdot (\rho^*)^{\overline{\frac{\delta\sigma}{1-\delta\sigma}}}}{(\delta\sigma B_{1*})^{\overline{\frac{\delta}{1-\delta\sigma}}}} \\ &> \frac{(\delta\sigma B_{1*})^\delta \cdot (\delta\sigma B1_*)^{\overline{\frac{\delta^2\sigma}{1-\delta\sigma}}}}{(\delta\sigma B_{1*})^{\overline{\frac{B_1\delta}{1-\delta\sigma}}}} \\ &= 1. \end{aligned}$$

This completes the proof. □

Remark 4.1. Note that the condition (1.4), which is stated as

$$\rho^* > \max\{(\delta\sigma B_{1*})^\delta, \ (\delta\sigma B_{1*})^{\frac{1}{\sigma}}\}$$

is crucial to guarantee that $R > 1 > r_0$, and in the proof of Theorem 1.3 we require $R > 1 > r_0$ because the exponents in inequalities of (A3) is different. However, in the special case that

$$\lambda := \alpha = \beta = \mu = \nu,$$

if we define $\omega(t) := \max\{b_2(t), e(t)\}$, $t \in [0, T]_{\mathbb{Z}}$, then the condition (1.4) is needn't because $R > r_0$ can be easily verified by

$$b_1(t) \leq \omega(t), \ t \in [0, T]_{\mathbb{Z}}.$$

□

Example 4.1. Let us consider the second order periodic boundary value problem

$$\begin{aligned} &\Delta^2 u(t-1) + 4\sin^2\frac{\pi}{16}u = f(t, u) - c_0, \quad t \in [1, 4]_{\mathbb{Z}}, \\ &u(0) = u(4), \quad u(1) = u(5), \end{aligned} \tag{4.2}$$

where

$$f(t, u) = \frac{5-t}{u^{\frac{1}{5}}}, \qquad u \in (0, \infty), \ t \in [1, 4]_{\mathbb{Z}}$$

and $c_0 \in \left(0, \frac{3\cdot[8\sqrt{10}]^{-4/3}}{((4+3\sqrt{2})\sqrt{2-\sqrt{2}}+2\sqrt{2})^{1/3}}\right)$ is a constant.

It is easy to check that (4.2) is equivalent to the operator equation

$$u(t) = \sum_{s=1}^{4} G(t,\ s)f(s,\ u(s)) + \sum_{s=1}^{4} G(t,\ s)(-c_0)ds =:\ (Au)\ (t), \qquad t \in [0,4]_{\mathbb{Z}},$$

here

$$G(t,\ s) = \begin{cases} \frac{1}{\sin\frac{\pi}{8}}\left[\sin\frac{\pi(t-s)}{8} + \sin\frac{\pi(4-t+s)}{8}\right], 0 \le s \le t \le 4, \\ \frac{1}{\sin\frac{\pi}{8}}\left[\sin\frac{\pi(s-t)}{8} + \sin\frac{\pi(4-s+t)}{8}\right], 0 \le t \le s \le 4. \end{cases}$$

Clearly, $G(t, s) > 0$ for all $(t, s) \in\ 0[4]_{\mathbb{Z}} \times 0[4]_{\mathbb{Z}}$.
Let

$$b_1(t) \equiv 1,\ b_2(t) \equiv 4,\ e(t) \equiv 6,$$
$$\alpha = \nu = \frac{1}{2}, \quad \beta = \frac{1}{6}, \quad \mu = \frac{1}{7},$$

Then

$$\sigma = \delta = \frac{1}{2},$$

and

$$0 < \frac{1}{u^{\frac{1}{2}}} \le \frac{4-t}{u^{\frac{1}{5}}} \le \frac{4}{u^{\frac{1}{6}}}, \quad u \in [1,\infty), \quad t \in [0,T]_{\mathbb{Z}},$$
$$0 < \frac{1}{u^{\frac{1}{7}}} \le \frac{4-t}{u^{\frac{1}{5}}} \le \frac{6}{u^{\frac{1}{2}}}, \quad u \in [0,1), \quad t \in [0,T]_{\mathbb{Z}}.$$

Thus, the condition (A3) is satisfied. By simple computations, we get

$$B_1(t) = \sum_{s=1}^{4} G(t,s)\cdot\frac{1}{2} = \frac{(4+3\sqrt{2})\sqrt{2-\sqrt{2}}}{2} + \sqrt{2};$$

$$B_2(t) = \sum_{s=1}^{4} G(t,s)\cdot 4 = (16+12\sqrt{2})\sqrt{2-\sqrt{2}} + 8\sqrt{2};$$

$$E(t) = \sum_{s=1}^{4} G(t,s)\cdot 6 = (24+18\sqrt{2})\sqrt{2-\sqrt{2}} + 12\sqrt{2};$$

$$B_{1*} = B_1^* = \frac{(4+3\sqrt{2})\sqrt{2-\sqrt{2}}}{2} + \sqrt{2};$$

$$B_{2*} = B_2^* = (16+12\sqrt{2})\sqrt{2-\sqrt{2}} + 8\sqrt{2};$$

$$E_* = E^* = (24+18\sqrt{2})\sqrt{2-\sqrt{2}} + 12\sqrt{2};$$

$$(\delta\sigma B_{1*})^{\delta} = \left[\frac{(4+3\sqrt{2})\sqrt{2-\sqrt{2}} + 2\sqrt{2}}{8}\right]^{\frac{1}{2}};$$

$$(\delta\sigma B_{1*})^{\frac{1}{\sigma}} = \left[\frac{(4+3\sqrt{2})\sqrt{2-\sqrt{2}} + 2\sqrt{2}}{8}\right]^{2};$$

$$\rho^* = E^* + B_2^* = (40+30\sqrt{2})\sqrt{2-\sqrt{2}} + 20\sqrt{2};$$

$$\max\{(\delta\sigma B_{1*})^{\delta}, (\delta\sigma B_{1*})^{\frac{1}{\sigma}}\} = \frac{(12+8\sqrt{2})\sqrt{2-\sqrt{2}} + 7\sqrt{2} + 14}{32};$$

and $\rho^* > \max\{(\delta\sigma B_{1*})^{\delta}, (\delta\sigma B_{1*})^{\frac{1}{\sigma}}\}$. So the condition (1.4) is satisfied. Moreover,

$$\gamma(t) = \sum_{s=1}^{4} G(t,s)(-c_0) = -(4+3\sqrt{2})\sqrt{2-\sqrt{2}} \cdot c_0 - 2\sqrt{2}c_0,$$

and so

$$\gamma^* = \gamma_* = -(4+3\sqrt{2})\sqrt{2-\sqrt{2}} \cdot c_0 - 2\sqrt{2}c_0 < 0.$$

Finally, since $c_0 \in \left(0, \frac{3.[8\sqrt{10}]^{-4/3}}{\left((4+3\sqrt{2})\sqrt{2-\sqrt{2}}+2\sqrt{2}\right)^{1/3}}\right)$, it follows that

$$\gamma_* \geq -3\left[\frac{\left((4+3\sqrt{2})\sqrt{2-\sqrt{2}}+2\sqrt{2}\right)^{\frac{1}{2}}}{8\sqrt{10}}\right]^{\frac{4}{3}} = \left[\frac{B_{1*}}{(\rho^*)^{\sigma}}\delta\sigma\right]^{\frac{1}{1-\delta\sigma}}\left(1-\frac{1}{\delta\sigma}\right).$$

Consequently, Theorem 1.3 yields that (4.2) has a positive solution. □

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Authors' contributions
RM completed the main study, carried out the results of this article and drafted the manuscript. YL checked the proofs and verified the calculation. All the authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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