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Global behavior for a strongly coupled model of plankton allelopathy

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Abstract

In this article, we consider a strongly coupled model of plankton allelopathy. Using the energy estimates and Gagliardo-Nirenberg-type inequalities, the existence and uniform boundedness of global solutions for the model are proved. Meanwhile, the sufficient conditions for global asymptotic stability of the positive equilibrium for this model are given by constructing a Lyapunov function.

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1 Introduction

The effects of toxic substances on ecological communities is an important problem from an environmental point of view. In 1996, Chattopadhyay [1] modified two species Lotka-Volterra competitive system by considering that each species produces a substance toxic to the other but only when the other is present and the modified model takes the following form

$$\begin{cases} \frac{du}{dt} = u(a_1 - b_{11}u - b_{12}v - e_1uv), \\ \frac{dv}{dt} = v(a_2 - b_{21}u - b_{22}v - e_2uv), \end{cases} \tag{1.1}$$

where a_i, b_{ij}, and $e_i (i, j = 1, 2)$ are positive constants, $u(t)$, $v(t)$ denote the population density of two competing species; a_1, a_2 are the intrinsic growth rates of two competing species; b_{11}, b_{22} are the rates of intra-specific competition of the first and the second species, respectively; b_{12}, b_{21} are the rates of inter-specific competition of the first and the second species, respectively; e_1 and e_2 are, respectively, the rates of toxic inhibition of the first species by the second and vice versa. For more details on the backgrounds about this system, see [1].

The system (1.1) has a positive equilibrium $E^* = (u^*, v^*)$ if and only if

$$\frac{b_{12}}{b_{22}} < \frac{a_1}{a_2} < \frac{b_{11}}{b_{21}}, \quad \frac{b_{12}}{b_{22}} < \frac{e_1}{e_2} < \frac{b_{11}}{b_{21}}, \tag{1.2}$$

where

$$u^* = \left(-q_{12} - \sqrt{q_{12}^2 - 4p_{12}r_{12}}\right)/(2p_{12}), \qquad v^* = \left(-q_{21} - \sqrt{q_{21}^2 - 4p_{21}r_{21}}\right)/(2p_{21}),$$

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and

$$p_{ij} = b_{ij}e_i - b_{ii}e_j,\ q_{ij} = a_ie_j - a_je_i - b_{ii}b_{jj} + b_{ij}b_{ji},\ r_{ij} = a_ib_{jj} - a_jb_{ij},\ i,\ j = 1, 2.$$

Chattopadhyay [1] proved that the equilibrium (u^*, v^*) is globally asymptotically stable if

$$4(b_{11} + e_1v)(b_{22} + e_2u) \geq (b_{12} + b_{21} + e_1u^* + e_2v^*)^2. \tag{1.3}$$

The corresponding weakly coupled reaction-diffusion system for (1.1) is as follows

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_{11}u - b_{12}v - e_1uv), & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta v + v(a_2 - b_{21}u - b_{22}v - e_2uv), & x \in \Omega,\ t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,\ 0) = u_0(x),\ v(x,\ 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is bounded smooth domain, n is the outward unit normal vector of the boundary $\partial\Omega$, $\partial_n = \partial/\partial n$. The, constants d_1 and d_2, called diffusion coefficients, are positive, and $u_0(x)$ and $v_0(x)$ are non-negative functions which are not identically zero.

It is obvious that (u^*, v^*) is the unique positive equilibrium of the system (1.4) if (1.2) holds. Tian et al. [2] proved that the equilibrium (u^*, v^*) of the system (1.4) is locally asymptotically stable if (1.2) holds.

In recent years, the SKT-type cross-diffusion systems have attracted the attention of a great number of investigators and have been successfully developed on the theoretical backgrounds. The above work mainly concentrate on (1) The instability and stability induced by cross-diffusion, and the existence of non-constant positive steady-state solutions [3-5]; (2) the global existence of strong solutions [6-13]; (3) the global existence of weak solutions based on semi-discretization or finite element approximation [14-17]; and (4) the dynamical behaviors [9,10], etc.

Tian et al. [2] considered the following SKT-type cross-diffusion system

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_{11}u - b_{12}v - e_1uv), & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta(v + d_3uv) + v(a_2 - b_{21}u - b_{22}v - e_2uv), & x \in \Omega,\ t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,\ 0) = u_0(x),\ v(x,\ 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

and they proved that:

(1) If $\mu_2 < \tilde{\mu}$, then there exists a positive constant d_2^* such that the equilibrium (u^*, v^*) of the system (1.5) is unstable provided that $d_2 \geq d_2^*$, the (1.2) and the following condition

$$(1 + d_3u^*)(b_{11} + e_1v^*) < d_3v^*(b_{12} + e_1u^*) \tag{1.6}$$

hold, where $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ are the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, $\tilde{\mu} = v^*(b_{12} + e_1u^*)/d_1$;

(2) the steady-state system of the system (1.4) has non-constant positive solution, if one of the following conditions is satisfied:
(i) $d_1 \geq D_1$ for some positive $D_1(d_2)$;
(ii) $d_2 \geq D_2$ for some positive $D_2(d_1)$; and

(3) if $\tilde{\mu} \in (\mu_n, \mu_{n+1})$ for some $n \geq 1$, and the sum $\sigma_n = \sum_{i=2}^{n} \dim E(\mu_i)$ is odd, then there exists a positive number d_2^* such that the system (1.5) has at least one inhomogeneous positive steady-state solution if $d_2 \geq d_2^*$, (1.2) and (1.6) hold, where d_1 and d_3 are fixed.

We are concerned with the following plankton allelopathy model with full cross-diffusion

$$\begin{cases} u_t = \Delta(d_1 u + \alpha_{11} u^2 + \alpha_{12} uv) + u(a_1 - b_{11} u - b_{12} v - e_1 uv), & x \in \Omega,\ t > 0, \\ v_t = \Delta(d_2 v + \alpha_{21} uv + \alpha_{22} v^2) + v(a_2 - b_{21} u - b_{22} v - e_2 uv), & x \in \Omega,\ t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,\ 0) = u_0(x),\ v(x,\ 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$

where Ω is a bounded domain $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, n is the outward unit normal vector of the boundary $\partial\Omega$. d_i, α_{ij}, a_i, b_{ij}, e_i $(i, j = 1, 2)$ are positive constants and the initial data u_0 and v_0 are continuous non-negative functions which are not identically zero. The homogeneous Neumann boundary condition indicates that the system is self-contained with zero population flux across the boundary. The parameters d_1, d_2 are the diffusion rates, $\alpha_{ii}(i = 1, 2)$ are referred as self-diffusion pressures, and α_{ij} $(i, j = 1, 2, i \neq j)$ are cross-diffusion pressures. For more details on the backgrounds about self-diffusion and cross-diffusion, one can see [8].

The local existence of solutions for the system (1.7) is an immediate consequence of a series of important articles by Amann [18-20]. Roughly speaking, if $u_0(x)$ and $v_0(x)$ in $W_p^1(\Omega)$ with $p > n$, then (1.7) has a unique non-negative solution $u, v \in C([0, T), W_P^1(\Omega)) \cap C^\infty((0, T), C^\infty(\Omega))$, where $T \in (0, \infty]$ is the maximal existence time for the solution. If the solution (u, v) satisfies the estimate

$$\sup\{||u(\cdot,\ t)||_{W_p^1(\Omega)},\ ||v(\cdot,\ t)||_{W_p^1(\Omega)}\ :\ 0 < t < T\} < \infty,$$

then $T = +\infty$. Moreover, if $u_0(x),\ v_0(x) \in W_P^2(\Omega)$, then $u, v \in C([0, \infty), W_P^2(\Omega))$.

For the following SKT system

$$\begin{cases} u_t = d_1 \Delta[(1 + \alpha v + \gamma u)u] + au(1 - u - cv), & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta[(1 + \delta v)v] + bv(1 - du - v), & x \in \Omega,\ t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,\ 0) = u_0(x),\ v(x,\ 0) = v_0(x), & x \in \Omega \end{cases} \tag{P}$$

Yamada [13] proposed four open problems:

(1) The global existence of solutions of (P) in the case $\delta > 0$ and the space dimension $N \geq 6$;
(2) the global existence in the case $\gamma = 0$;
(3) in order to study the asymptotic behavior of u, v as $t \to \infty$, need to establish the uniform boundedness of global solutions; and

(4) the global existence of solutions for the following full SKT system

$$\begin{cases} u_t = d_1\Delta[(1+\alpha v+\gamma u)u] + au(1-u-cv), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta[(1+\beta u+\delta v)v] + bv(1-du-v), & x\in\Omega,\ t>0,\\ \partial_n u = \partial_n v = 0, & x\in\partial\Omega,\ t>0,\\ u(x,\ 0)=u_0(x),\ v(x,\ 0)=v_0(x), & x\in\Omega \end{cases}$$

with α, γ, β, $\delta > 0$.

Very few global existence results for (1.7) are known. The main purpose of this article is to establish the uniform boundedness of global solutions for the system (1.7) in one space dimension. For convenience, we consider the following system

$$\begin{cases} u_t = (d_1u+\alpha_{11}u^2+\alpha_{12}uv)_{xx} + u(a_1-b_{11}u-b_{12}v-e_1uv), & 0<x<1,\ t>0,\\ v_t = (d_2v+\alpha_{21}uv+\alpha_{22}v^2)_{xx} + v(a_2-b_{21}u-b_{22}v-e_2uv), & 0<x<1,\ t>0,\\ u_x(x,\ t)=v_x(x,\ t)=0, & x=0,1,\ t>0,\\ u(x,\ 0)=u_0(x),\ v(x,\ 0)=v_0(x), & 0<x<1. \end{cases} \tag{1.8}$$

We firstly investigate the global existence and the uniform boundedness of the solutions for (1.8), then prove the global asymptotic stability of the positive equilibrium (u^*, v^*) of (1.8) by an important lemma from [21]. The proof is complete and complement to the uniform convergence theorems in [22-24].

It is obvious that (u^*, v^*) is the unique positive equilibrium of the system (1.8) if (1.2) holds.

For simplicity, we denote $\|\cdot\|_{W_p^k(0,1)}$ by $|\cdot|_{k,p}$ and $\|\cdot\|_{L^p(0,1)}$ by $|\cdot|_p$. Our main results are as follows.

Theorem 1.1. Let $u_0, v_0 \in W_2^2(0,1)$, (u, v) is the unique non-negative solution of system (1.2) in the maximal existence interval [0, T). Assume that

$$8\alpha_{11}\alpha_{21} > \alpha_{12}^2, \qquad 8\alpha_{22}\alpha_{12} > \alpha_{21}^2. \tag{1.9}$$

Then there exist $t_0 > 0$ and positive constants M, M' which depend on d_i, α_{ij}, a_i, b_{ij}, $e_i(i, j = 1, 2)$, such that

$$\sup\{|u(\cdot,\ t)|_{1,2},\ |v(\cdot,\ t)|_{1,2} : t\in(t_0,\ T)\} \le M', \tag{1.10}$$

$$\max\{u(x,\ t),\ v(x,\ t)\ :\ (x,\ t)\in[0,1]\times(t_0,\ T)\} \le M, \tag{1.11}$$

and $T = +\infty$. Moreover, in the case that d_1, $d_2 \ge 1$, $d_2/d_1 \in [\underline{d}, \bar{d}]$, where $\underline{d}$ and $\bar{d}$ are positive constants, M', M depend on $\underline{d}, \bar{d}$, but do not depend on d_1, d_2.

Remark 1.1. Since the continuous embedding $H^1(\Omega)$ ⊠ $L^\infty(\Omega)$ holds only in one space dimension, we can only establish the uniform maximum-norm estimates about time for the solution in one space dimension.

Theorem 1.2 Assume that all conditions in Theorem 1.1 are satisfied. Assume further that,

$$4d_1d_2u^*v^* > M^2(\alpha_{12}u^* + \alpha_{21}v^*)^2, \tag{1.12}$$

$$4b_{11}b_{22} > (b_{12} + b_{21} + e_1u^* + e_2v^*)^2 \tag{1.13}$$

and (1.2) hold, M is given by (1.11). Then the unique positive equilibrium (u^*, v^*) of (1.8) is globally asymptotically stable.

Remark 1.2. The system (1.8) has no non-constant positive steady-state solution if all conditions of Theorem 1.2 hold.

2 Global solutions

In order to establish the uniform W_2^1-estimates of the solutions for system (1.2), the following Gagliardo-Nirenberg- type inequalities and the corresponding corollary play important roles (see [25]).

Theorem 2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \lfloor C^m$. For every function $u \in W_r^m(\Omega)$, $1 \le q, r \le \infty$, the derivative $D^j u$ $(0 \le j < m)$ satisfies the inequality

$$|D^j u|_p \le C(|D^m u|_r^a |u|_q^{1-a} + |u|_q),$$

provided one of the following three conditions is satisfied: (1) $r \le q$, (2) $0 < n(r-q)/(mrq) < 1$, or (3) $n(r-q)/(mrq) = 1$ and $m - n/q$ is not a non-negative integer, where $1/p = j/n + a(1/r - m/n) + (1 - a)/q$ for all $a \in [j/m, 1)$, and the positive constant C depends on n, m, j, q, r, a.

Corollary 1. There exists a positive constant C such that

$$|u|_2 \le C(|u_x|_2^{1/3} |u|_1^{2/3} + |u|_1), \quad \forall u \in W_2^1(0,1), \tag{2.1}$$

$$|u|_4 \le C(|u_x|_2^{1/2} |u|_1^{1/2} + |u|_1), \quad \forall u \in W_2^1(0,1), \tag{2.2}$$

$$|u|_{\frac{7}{2}} \le C(|u_x|_2^{10/21} |u|_1^{11/21} + |u|_1), \quad \forall u \in W_2^1(0,1), \tag{2.3}$$

$$|u_x|_2 \le C(|u_{xx}|_2^{3/5} |u|_1^{2/5} + |u|_1), \quad \forall u \in W_2^2(0,1). \tag{2.4}$$

Throughout this article, we always denote that C is Sobolev embedding constant or other kind of universal constant, A_j, B_j, C_j are some positive constants which depend only on α_{ij}, a_i, b_{ij}, $e_i (i, j = 1, 2)$, K_j are positive constants depending on d_i, α_{ij}, a_i, b_{ij}, e_i $(i, j = 1, 2)$. When $d_1, d_2 \ge 1$, $d_2/d_1 \in [\underline{d}, \bar{d}]$, K_j depend on $\underline{d}, \bar{d}$, but do not depend on d_1 and d_2.

Proof of Theorem 1.1. Step 1, estimate $|u|_1$, $|v|_1$.

Taking integration of the first, the first and the second equations in (1.8) over the domain (0,1), respectively, we have

$$\frac{d}{dt}\int_0^1 u dx = \int_0^1 u(a_1 - b_{11}u - b_{12}v - e_1 uv)dx \le a_1 \int_0^1 u dx - b_{11}\left(\int_0^1 u dx\right)^2,$$

$$\frac{d}{dt}\int_0^1 v dx = \int_0^1 v(a_2 - b_{21}u - b_{22}v - e_2 uv)dx \le a_2 \int_0^1 v dx - b_{22}\left(\int_0^1 v dx\right)^2.$$

So, there exists a positive constant M_0 which depends on a_i, $b_{ij} (i, j = 1, 2)$, such that

$$\int_0^1 u\, dx, \int_0^1 v\, dx \le M_0, \quad t \ge \tau_0. \tag{2.5}$$

Moreover, there exists a positive constant M_0' which depends on a_i, $b_{ij}(i, j = 1, 2)$ and L^1-norm of u_0, v_0, such that

$$\int_0^1 u\, dx, \int_0^1 v\, dx \le M_0', \quad t \ge 0. \tag{2.5a}$$

Step 2, estimate $|u|_2$, $|v|_2$.

Multiplying the first two equations in system (1.2) by u, v, respectively, and integrating over (0, 1) we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx \le -d_1\int_0^1 u_x^2 dx - \int_0^1 [(2\alpha_{11}u + \alpha_{12}v)u_x^2 + \alpha_{12}v_x u u_x]dx + a_1\int_0^1 u^2 dx,$$
$$\frac{1}{2}\frac{d}{dt}\int_0^1 v^2 dx \le -d_2\int_0^1 v_x^2 dx - \int_0^1 [(\alpha_{21}u + 2\alpha_{22}v)v_x^2 + \alpha_{21}u_x v v_x]dx + a_2\int_0^1 v^2 dx,$$

from which it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2)dx &\le -d\int_0^1 (u_x^2+v_x^2)dx + a_1\int_0^1 u^2 dx + a_2\int_0^1 v^2 dx - \int_0^1 q(u_x,\ v_x)dx \\ &\le -d\int_0^1 (u_x^2+v_x^2)dx + (a_1+a_2)\int_0^1 (u^2+v^2)dx - \int_0^1 q(u_x,\ v_x)dx,\end{aligned}$$

where $d = \min\{d_1, d_2\}$. Some tedious calculations yield that

$$q(u_x,\ v_x) = (2\alpha_{11}u + \alpha_{12}v)u_x^2 + (\alpha_{12}u + \alpha_{21}v)u_x v_x + (\alpha_{21}u + 2\alpha_{22}v)v_x^2$$

is positive definite quadratic form of u_x, v_x if (1.9) holds. So (1.9) implies that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2)dx \le -d\int_0^1 (u_x^2+v_x^2)dx + (a_1+a_2)\int_0^1 (u^2+v^2)dx. \tag{2.6}$$

Now, we proceed in the following two cases.

(i) $t \ge \tau_0$.The inequality (2.1) implies that $|u|_2^6 \le C(|u_x|_2^2|u|_1^4 + |u|_1^6) \le CM_0^4(|u_x|_2^2 + M_0^2)$. So we have $\int_0^1 u_x^2 dx \ge \frac{1}{CM_0^4}\left(\int_0^1 u^2 dx\right)^3 - M_0^2$, and

$$-\int_0^1 (u_x^2+v_x^2)dx \le -\frac{1}{9CM_0^4}\left[\int_0^1 (u^2+v^2)dx\right]^3 + 2M_0^2. \tag{2.7}$$

It follows from (2.5) and (2.6) that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v^2)dx \le d\left\{-C_2\left[\int_0^1 (u^2+v^2)dx\right]^3 + 2M_0^2 + \frac{1}{d}(a_1+a_2)\int_0^1 (u^2+v^2)dx\right\}. \tag{2.8}$$

This means that there exist positive constants τ_1 and M_1 depending on d_i, a_i, $b_{ij}(i, j = 1, 2)$, such that

$$\int_0^1 u^2 dx, \int_0^1 v^2 dx \le M_1, t \ge \tau_1. \tag{2.9}$$

When $d \ge 1$, M_1 is independent of d because the zero point of the right-hand side in (2.11) can be estimated by positive constants independent of d.

(ii) $t \ge 0$. Repeating estimates in (i) by (2.8)', we can obtain that there exists a positive constant M_1' depending on d_i, a_i, $b_{ij}(i, j = 1, 2)$ and the L^1, L^2-norm of u_0, v_0, such that

$$\int_0^1 u^2 dx, \int_0^1 v^2 dx \le M_1', \quad t \ge 0. \tag{2.9a}$$

When $d \ge 1$, M_1' is independent of d.

Step 3, estimate $|u_x|_2$, $|v_x|_2$.
Introducing the following scaling

$$\tilde{u} = \frac{u}{d_1}, \tilde{v} = \frac{v}{d_1}, \tilde{t} = d_1 t. \tag{2.10}$$

Denoting $\xi = d_2/d_1$, and using u, v, t instead of $\tilde{u}, \tilde{v}, \tilde{t}$, respectively, then system (1.2) reduces to

$$\begin{cases} u_t = P_{xx} + f(u,\ v), & 0 < x < 1,\ t > 0, \\ v_t = Q_{xx} + g(u,\ v), & 0 < x < 1,\ t > 0, \\ u_x(x,\ t) = v_x(x,\ t) = 0, & x = 0, 1,\ t > 0, \\ u(x,\ 0) = \tilde{u}_0(x),\ v(x,\ 0) = \tilde{v}_0(x), & 0 < x < 1, \end{cases} \tag{2.11}$$

where $P = u + \alpha_{11}u^2 + \alpha_{12}uv$, $Q = \xi v + \alpha_{21}uv + \alpha_{22}v^2$, $g(u,v) = a_2 d_1^{-1} v - b_{21}uv - b_{22}v^2 - e_2 d_1 uv^2$, $g(u,v) = a_2 d_1^{-1} v - b_{21}uv - b_{22}v^2 - e_2 d_1 uv^2$.
We still proceed in the following two cases.

(i) $t \ge \tau_1^* = d_1 \tau_1$. It is clear that

$$\begin{aligned} &\int_0^1 u dx, \int_0^1 v dx \le M_0 d_1^{-1}, \\ &\int_0^1 u^2 dx, \int_0^1 v^2 dx \le M_1 d_1^{-2}, \\ &|P|_1,\ |Q|_1 \le A_1 K_1 d_1^{-1}, \end{aligned} \tag{2.12}$$

where $K_1 = (1 + \xi) + M_1 d_1^{-1}$ and $A_1 = \max\{M_0, \alpha_{11} + \alpha_{12}, \alpha_{21} + \alpha_{22}\}$.

Multiplying the first two equations in (2.11) by P_t, Q_t, integrating them over the domain (0, 1), respectively, and then adding up the two integration equalities, we have

$$\begin{aligned} \frac{1}{2}\bar{y}'(t) = &- \int_0^1 u_t^2 dx - \xi \int_0^1 v_t^2 dx - \int_0^1 q(u_t,\ v_t) dx \\ &+ \int_0^1 [(1 + 2\alpha_{11}u + \alpha_{12}v)u_t f + \alpha_{12}uv_t f] dx + \int_0^1 [(\xi + \alpha_{21}u + 2\alpha_{22}v)v_t g + \alpha_{21}vu_t g] dx \end{aligned}$$

where $\bar{y}(t) = \int_0^1 (P_x^2 + Q_x^2)dx$. It is not hard verify by (1.9) that there exists a positive constant C_3 depending only on $\alpha_{ij}(i, j = 1, 2)$, such that

$$q(u_t,\ v_t) \geq C_3(u+v)(u_t^2 + v_t^2).$$

Thus,

$$\begin{aligned}\frac{1}{2}\bar{y}'(t) \leq &-\int_0^1 u_t^2 dx - \xi\int_0^1 v_t^2 dx - C_3\int_0^1 (u+v)(u_t^2+v_t^2)dx \\ &+\int_0^1 [(1+2\alpha_{11}u+\alpha_{12}v)u_t f + \alpha_{12}uv_t f]dx + \int_0^1 [(\xi+\alpha_{21}u+2\alpha_{22}v)v_t g + \alpha_{21}vu_t g]dx.\end{aligned} \tag{2.13}$$

Using Young inequality, Hölder inequality, and (2.12), we can obtain the following estimates

$$\begin{aligned}
&\int_0^1 u^3 dx \leq \left(\int_0^1 u^7 dx\right)^{1/5}\left(\int_0^1 u^2 dx\right)^{4/5} \leq M_1^{4/5} d_1^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5},\\
&\int_0^1 u^4 dx \leq \left(\int_0^1 u^7 dx\right)^{2/5}\left(\int_0^1 u^2 dx\right)^{3/5} \leq M_1^{3/5} d_1^{-6/5}\left(\int_0^1 u^7 dx)\right)^{2/5},\\
&\int_0^1 u^5 dx \leq \left(\int_0^1 u^7 dx\right)^{3/5}\left(\int_0^1 u^2 dx\right)^{2/5} \leq M_1^{2/5} d_1^{-4/5}\left(\int_0^1 u^7 dx\right)^{3/5},\\
&\int_0^1 u^6 dx \leq \left(\int_0^1 u^7 dx\right)^{4/5}\left(\int_0^1 u^2 dx\right)^{1/5} \leq M_1^{1/5} d_1^{-2/5}\left(\int_0^1 u^7 dx\right)^{4/5},\\
&\int_0^1 u^2 v dx \leq \left(\int_0^1 u^7 dx\right)^{1/5}\left(\int_0^1 u^2 dx\right)^{3/10}\left(\int_0^1 v^2 dx\right)^{1/2} \leq M_1^{4/5} d_1^{-8/5}\left(\int_0^1 u^7 dx\right)^{1/5},\\
&\int_0^1 u^3 v dx \leq \left(\int_0^1 u^7 dx\right)^{2/5}\left(\int_0^1 u^2 dx\right)^{1/10}\left(\int_0^1 v^2 dx\right)^{1/2} \leq M_1^{3/5} d_1^{-6/5}\left(\int_0^1 u^7 dx\right)^{2/5},\\
&\int_0^1 u^4 v dx \leq \frac{4}{5}\int_0^1 u^5 dx + \frac{1}{5}\int_0^1 v^5 dx \leq \frac{4}{5}M_1^{2/5} d_1^{-4/5}\left[\left(\int_0^1 u^7 dx\right)^{3/5} + \left(\int_0^1 v^7 dx\right)^{3/5}\right],\\
&\int_0^1 u^2 v^2 dx \leq \frac{1}{2}\left(\int_0^1 u^4 dx + \int_0^1 v^4 dx\right) \leq \frac{1}{2}M_1^{3/5} d_1^{-4/5}\left[\left(\int_0^1 u^7 dx\right)^{2/5} + \left(\int_0^1 v^7 dx\right)^{2/5}\right],\\
&\int_0^1 u^3 v^2 dx \leq \frac{3}{5}\int_0^1 u^5 dx + \frac{2}{5}\int_0^1 v^5 dx \leq \frac{3}{5}M_1^{2/5} d_1^{-4/5}\left[\left(\int_0^1 u^7 dx\right)^{3/5} + \left(\int_0^1 v^7 dx\right)^{3/5}\right],\\
&\int_0^1 u^4 v^2 dx \leq \frac{2}{3}\int_0^1 u^6 dx + \frac{1}{3}\int_0^1 v^6 dx \leq \frac{2}{3}M_1^{1/5} d_1^{-4/5}\left[\left(\int_0^1 u^7 dx\right)^{4/5} + \left(\int_0^1 v^7 dx\right)^{4/5}\right],\\
&\int_0^1 u^4 v^3 dx \leq \frac{4}{7}\int_0^1 u^7 dx + \frac{3}{7}\int_0^1 v^7 dx \leq \frac{4}{7}\left(\int_0^1 u^7 dx + \int_0^1 v^7 dx\right),\\
&\int_0^1 u^5 v^2 dx \leq \frac{5}{7}\int_0^1 u^7 dx + \frac{2}{7}\int_0^1 v^7 dx \leq \frac{5}{7}\left(\int_0^1 u^7 dx + \int_0^1 v^7 dx\right).
\end{aligned} \tag{2.14}$$

Applying the above estimates and Gagliardo-Nirenberg-type inequalities to the terms on the right-hand side of (2.13), we have

$$
\begin{aligned}
-\int_0^1 u_t^2 dx &\le -\frac{1}{2}\int_0^1 P_{xx}^2 dx + \int_0^1 f^2 dx,\\
-\xi\int_0^1 v_t^2 dx &\le -\frac{\xi}{2}\int_0^1 Q_{xx}^2 dx + \xi\int_0^1 g^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 f^2 dx \le{}& \int_0^1 (a_1^2 d_1^{-2} u^2 + b_{11}^2 u^4 + b_{12}^2 u^2 v^2 + e_1^2 d_1^2 u^4 v^2 + 2b_{11} b_{12} u^3 v + 2b_{11} e_1 d_1 u^4 v + 2b_{12} e_1 d_1 u^3 v^2) dx\\
&\le a_1^2 M_1 d_1^{-4} + \left(b_{11}^2 + \frac{1}{2}b_{12}^2 + 2b_{11}b_{12}\right) M_1^{3/5} d_1^{-6/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{2/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{2/5}\right]\\
&+ \left(\frac{8}{5}b_{11} + \frac{6}{5}b_{12}\right) e_1 M_1^{2/5} d_1^{1/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{3/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{3/5}\right]\\
&+ \frac{2}{3}e_1^2 M_1^{1/5} d_1^{8/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{4/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{4/5}\right],
\end{aligned}
$$

$$
\begin{aligned}
\xi\int_0^1 g^2 dx \le{}& \xi\int_0^1 (a_2^2 d_1^{-2} v^2 + b_{21}^2 u^2 v^2 + b_{22}^2 v^4 + e_2^2 d_1^2 u^2 v^4 + 2b_{21} b_{22} u v^3 + 2b_{21} e_2 d_1 u^2 v^3 + 2b_{22} e_2 d_1 u v^4) dx\\
&\le \xi a_2^2 M_1 d_1^{-4} + \xi\left(b_{22}^2 + \frac{1}{2}b_{21}^2 + 2b_{21}b_{22}\right) M_1^{3/5} d_1^{-6/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{2/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{2/5}\right]\\
&+ 2\xi\left(\frac{3}{5}b_{21} + \frac{4}{5}b_{22}\right) e_2 M_1^{2/5} d_1^{1/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{3/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{3/5}\right]\\
&+ \frac{2}{3}\xi e_2^2 M_1^{1/5} d_1^{8/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{4/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{4/5}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_0^1 u_t^2 dx - \xi\int_0^1 v_t^2 dx \le{}& -\frac{1}{2}\int_0^1 P_{xx}^2 dx - \frac{\xi}{2}\int_0^1 Q_{xx}^2 dx + \left(a_1^2 + \xi a_2^2\right) M_1 d_1^{-4}\\
&+ \left[\left(b_{11}^2 + \frac{1}{2}b_{12}^2 + 2b_{11}b_{12}\right) + \xi\left(b_{22}^2 + \frac{1}{2}b_{21}^2 + 2b_{21}b_{22}\right)\right] M_1^{3/5} d_1^{-6/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{2/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{2/5}\right]\\
&+ \left[\left(\frac{8}{5}b_{11} + \frac{6}{5}b_{12}\right) e_1 + 2\xi\left(\frac{3}{5}b_{21} + \frac{4}{5}b_{22}\right) e_2\right] M_1^{2/5} d_1^{1/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{3/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{3/5}\right]\\
&+ \frac{2}{3}(e_1^2 + \xi e_2^2) M_1^{1/5} d_1^{8/5}\left[\left(\int_0^1 u^7 \mathrm{d}x\right)^{4/5} + \left(\int_0^1 v^7 \mathrm{d}x\right)^{4/5}\right].
\end{aligned}
\tag{2.15}
$$

Similarly, we can obtain

$$\begin{aligned}\int_0^1 u_t f dx &\le \int_0^1 u_t(a_1 d_1^{-1} u + b_{11} u^2 + b_{12} uv + e_1 d_1 u^2 v) dx \\ &\le \frac{a_1^2 d_1^{-2}}{2\varepsilon} \int_0^1 u dx + \frac{\varepsilon}{2} \int_0^1 u u_t^2 dx + \frac{b_{11}^2}{2\varepsilon} \int_0^1 u^3 dx + \frac{\varepsilon}{2} \int_0^1 u u_t^2 dx \\ &+ \frac{b_{12}^2}{2\varepsilon} \int_0^1 u v^2 dx + \frac{\varepsilon}{2} \int_0^1 u u_t^2 dx + \frac{(e_1 d_1)^2}{2\varepsilon} \int_0^1 u^3 v^2 dx + \frac{\varepsilon}{2} \int_0^1 u u_t^2 dx \\ &\le \frac{a_1^2}{2\varepsilon} M_0 d_1^{-3} + \frac{b_{11}^2 + b_{12}^2}{2\varepsilon} M_1^{4/5} d_1^{-8/5} \left[\left(\int_0^1 u^7 \mathrm{d}x \right)^{1/5} + \left(\int_0^1 v^7 \mathrm{d}x \right)^{1/5} \right] \\ &+ \frac{3e_1^2}{10\varepsilon} M_1^{2/5} d_1^{6/5} \left[\left(\int_0^1 u^7 \mathrm{d}x \right)^{3/5} + \left(\int_0^1 v^7 \mathrm{d}x \right)^{3/5} \right] + 2\varepsilon \int_0^1 u u_t^2 dx,\end{aligned}$$

$$\begin{aligned}2\alpha_{11} \int_0^1 u u_t f dx &\le 2\alpha_{11} \int_0^1 u u_t(a_1 d_1^{-1} u + b_{11} u^2 + b_{12} uv + e_1 d_1 u^2 v) dx \\ &\le \frac{\alpha_{11}^2 a_1^2 d_1^{-2}}{\varepsilon} \int_0^1 u^3 dx + \varepsilon \int_0^1 u u_t^2 dx + \frac{\alpha_{11}^2 b_{11}^2}{\varepsilon} \int_0^1 u^5 dx + \varepsilon \int_0^1 u u_t^2 dx \\ &+ \frac{\alpha_{11}^2 b_{12}^2}{\varepsilon} \int_0^1 u^3 v^2 dx + \varepsilon \int_0^1 u u_t^2 dx + \frac{\alpha_{11}^2 e_1^2 d_1^2}{\varepsilon} \int_0^1 u^5 v^2 dx + \varepsilon \int_0^1 u u_t^2 dx \\ &\le \frac{\alpha_{11}^2 a_1^2}{\varepsilon} M_1^{4/5} d_1^{-18/5} \left(\int_0^1 u^7 dx \right)^{1/5} + \frac{\alpha_{11}^2 (b_{11}^2 + b_{12}^2)}{\varepsilon} M_1^{2/5} d_1^{-4/5} \left[\left(\int_0^1 u^7 \mathrm{d}x \right)^{3/5} + \left(\int_0^1 v^7 \mathrm{d}x \right)^{3/5} \right] \\ &+ \frac{5\alpha_{11}^2 e_1^2}{7\varepsilon} d_1^2 \left(\int_0^1 u^7 \mathrm{d}x + \int_0^1 v^7 \mathrm{d}x \right) + 2\varepsilon \int_0^1 u u_t^2 dx,\end{aligned}$$

$$\begin{aligned}\alpha_{12} \int_0^1 v u_t f dx &\le \alpha_{12} \int_0^1 v u_t(a_1 d_1^{-1} u + b_{11} u^2 + b_{12} uv + e_1 d_1 u^2 v) dx \\ &\le \frac{\alpha_{12}^2 a_1^2 d_1^{-2}}{2\varepsilon} \int_0^1 u v^2 dx + \frac{\varepsilon}{2} \int_0^1 u u_t^2 dx + \frac{\alpha_{12}^2 b_{11}^2}{2\varepsilon} \int_0^1 u^3 v^2 dx + \frac{\varepsilon}{2} \int_0^1 u u_t^2 dx \\ &+ \frac{\alpha_{12}^2 b_{12}^2}{2\varepsilon} \int_0^1 u v^4 dx + \frac{\varepsilon}{2} \int_0^1 u u_t^2 dx + \frac{\alpha_{12}^2 e_1^2 d_1^2}{2\varepsilon} \int_0^1 u^3 v^4 dx + \frac{\varepsilon}{2} \int_0^1 u u_t^2 dx \\ &\le \frac{\alpha_{12}^2 a_1^2}{2\varepsilon} M_1^{4/5} d_1^{-18/5} \left(\int_0^1 v^7 dx \right)^{1/5} + \frac{\alpha_{12}^2 (b_{11}^2 + b_{12}^2)}{2\varepsilon} M_1^{2/5} d_1^{-4/5} \left[\left(\int_0^1 u^7 \mathrm{d}x \right)^{3/5} + \left(\int_0^1 v^7 \mathrm{d}x \right)^{3/5} \right] \\ &+ \frac{2\alpha_{12}^2 e_1^2}{7\varepsilon} d_1^2 \left(\int_0^1 u^7 \mathrm{d}x + \int_0^1 v^7 \mathrm{d}x \right) + 2\varepsilon \int_0^1 u u_t^2 dx,\end{aligned}$$

$$
\begin{aligned}
\alpha_{12}\int_0^1 uv_t f dx &\le \alpha_{12}\int_0^1 uv_t(a_1 d_1^{-1}u + b_{11}u^2 + b_{12}uv + e_1 d_1 u^2 v)dx \\
&\le \frac{\alpha_{12}^2 a_1^2 d_1^{-2}}{2\varepsilon}\int_0^1 u^3 dx + \frac{\varepsilon}{2}\int_0^1 uv_t^2 dx + \frac{\alpha_{12}^2 b_{11}^2}{2\varepsilon}\int_0^1 u^5 dx + \frac{\varepsilon}{2}\int_0^1 uv_t^2 dx \\
&\quad + \frac{\alpha_{12}^2 b_{12}^2}{2\varepsilon}\int_0^1 u^3 v^2 dx + \frac{\varepsilon}{2}\int_0^1 uv_t^2 dx + \frac{\alpha_{12}^2 e_1^2 d_1^2}{2\varepsilon}\int_0^1 u^5 v^2 dx + \frac{\varepsilon}{2}\int_0^1 uv_t^2 dx \\
&\le \frac{\alpha_{12}^2 a_1^2}{2\varepsilon} M_1^{4/5} d_1^{-18/5}\left(\int_0^1 u^7 dx\right)^{1/5} + \frac{\alpha_{12}^2 (b_{11}^2 + b_{12}^2)}{2\varepsilon} M_1^{2/5} d_1^{-4/5}\left[\left(\int_0^1 u^7 dx\right)^{3/5} + \left(\int_0^1 v^7 dx\right)^{3/5}\right] \\
&\quad + \frac{5\alpha_{12}^2 e_1^2}{14\varepsilon} d_1^2\left(\int_0^1 u^7 dx + \int_0^1 v^7 dx\right) + 2\varepsilon\int_0^1 uv_t^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
\xi\int_0^1 v_t g dx &\le \xi\int_0^1 v_t(a_2 d_1^{-1}v + b_{21}uv + b_{22}v^2 + e_2 d_1 uv^2)dx \\
&\le \frac{\xi^2 a_2^2 d_1^{-2}}{2\varepsilon}\int_0^1 v dx + \frac{\varepsilon}{2}\int_0^1 vv_t^2 dx + \frac{\xi^2 b_{21}^2}{2\varepsilon}\int_0^1 u^2 v dx + \frac{\varepsilon}{2}\int_0^1 vv_t^2 dx \\
&\quad + \frac{\xi^2 b_{22}^2}{2\varepsilon}\int_0^1 v^3 dx + \frac{\varepsilon}{2}\int_0^1 vv_t^2 dx + \frac{\xi^2 e_2^2 d_1^2}{2\varepsilon}\int_0^1 u^2 v^3 dx + \frac{\varepsilon}{2}\int_0^1 vv_t^2 dx \\
&\le \frac{\xi^2 a_2^2}{2\varepsilon} M_0 d_1^{-3} + \frac{\xi^2 (b_{21}^2 + b_{22}^2)}{2\varepsilon} M_1^{4/5} d_1^{-8/5}\left[\left(\int_0^1 u^7 dx\right)^{1/5} + \left(\int_0^1 v^7 dx\right)^{1/5}\right] \\
&\quad + \frac{3\xi^2 e_2^2}{10\varepsilon} M_1^{2/5} d_1^{6/5}\left[\left(\int_0^1 u^7 dx\right)^{3/5} + \left(\int_0^1 v^7 dx\right)^{3/5}\right] + 2\varepsilon\int_0^1 vv_t^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{21}\int_0^1 uv_t g dx &\le \alpha_{21}\int_0^1 uv_t(a_2 d_1^{-1}v + b_{21}uv + b_{22}v^2 + e_2 d_1 uv^2)dx \\
&\le \frac{\alpha_{21}^2 a_2^2 d_1^{-2}}{2\varepsilon}\int_0^1 u^2 v dx + \frac{\varepsilon}{2}\int_0^1 vv_t^2 dx + \frac{\alpha_{21}^2 b_{21}^2}{2\varepsilon}\int_0^1 u^4 v dx + \frac{\varepsilon}{2}\int_0^1 vv_t^2 dx \\
&\quad + \frac{\alpha_{21}^2 b_{22}^2}{2\varepsilon}\int_0^1 u^2 v^3 dx + \frac{\varepsilon}{2}\int_0^1 vv_t^2 dx + \frac{\alpha_{21}^2 e_2^2 d_1^2}{2\varepsilon}\int_0^1 u^4 v^3 dx + \frac{\varepsilon}{2}\int_0^1 vv_t^2 dx \\
&\le \frac{\alpha_{21}^2 a_2^2}{2\varepsilon} M_1^{4/5} d_1^{-18/5}\left(\int_0^1 u^7 dx\right)^{1/5} + \frac{\alpha_{21}^2 (b_{21}^2 + b_{22}^2)}{2\varepsilon} M_1^{2/5} d_1^{-4/5}\left[\left(\int_0^1 u^7 dx\right)^{3/5} + \left(\int_0^1 v^7 dx\right)^{3/5}\right] \\
&\quad + \frac{5\alpha_{21}^2 e_2^2}{14\varepsilon} d_1^2\left(\int_0^1 u^7 dx + \int_0^1 v^7 dx\right) + 2\varepsilon\int_0^1 vv_t^2 dx,
\end{aligned}
$$

$$\begin{aligned}2\alpha_{22}\int_0^1 vv_t g dx \leq& 2\alpha_{22}\int_0^1 vv_t(a_2d_1^{-1}v+b_{21}uv+b_{22}v^2+e_2d_1uv^2)dx\\ \leq& \frac{\alpha_{22}^2a_2^2d_1^{-2}}{\varepsilon}\int_0^1 v^3dx+\varepsilon\int_0^1 vv_t^2dx+\frac{\alpha_{22}^2b_{21}^2}{\varepsilon}\int_0^1 u^2v^3dx+\varepsilon\int_0^1 vv_t^2dx\\ &+\frac{\alpha_{22}^2b_{22}^2}{\varepsilon}\int_0^1 v^5dx+\varepsilon\int_0^1 vv_t^2dx+\frac{\alpha_{22}^2e_2^2d_1^2}{\varepsilon}\int_0^1 u^2v^5dx+\varepsilon\int_0^1 vv_t^2dx\\ \leq& \frac{\alpha_{22}^2a_2^2}{\varepsilon}M_1^{4/5}d_1^{-18/5}\left(\int_0^1 v^7dx\right)^{1/5}+\frac{\alpha_{22}^2(b_{21}^2+b_{22}^2)}{\varepsilon}M_1^{2/5}d_1^{-4/5}\left[\left(\int_0^1 u^7dx\right)^{3/5}+\left(\int_0^1 v^7dx\right)^{3/5}\right]\\ &+\frac{5\alpha_{22}^2e_2^2}{7\varepsilon}d_1^2\left(\int_0^1 u^7dx+\int_0^1 v^7dx\right)+2\varepsilon\int_0^1 vv_t^2dx,\end{aligned}$$

$$\begin{aligned}\alpha_{21}\int_0^1 vu_t g dx \leq& \alpha_{21}\int_0^1 vu_t(a_2d_1^{-1}v+b_{21}uv+b_{22}v^2+e_2d_1uv^2)dx\\ \leq& \frac{\alpha_{21}^2a_2^2d_1^{-2}}{2\varepsilon}\int_0^1 v^3dx+\frac{\varepsilon}{2}\int_0^1 vu_t^2dx+\frac{\alpha_{21}^2b_{21}^2}{2\varepsilon}\int_0^1 u^2v^3dx+\frac{\varepsilon}{2}\int_0^1 vu_t^2dx\\ &+\frac{\alpha_{21}^2b_{22}^2}{2\varepsilon}\int_0^1 v^5dx+\frac{\varepsilon}{2}\int_0^1 vu_t^2dx+\frac{\alpha_{21}^2e_2^2d_1^2}{2\varepsilon}\int_0^1 u^2v^5dx+\frac{\varepsilon}{2}\int_0^1 vu_t^2dx\\ \leq& \frac{\alpha_{21}^2a_2^2}{2\varepsilon}M_1^{4/5}d_1^{-18/5}\left(\int_0^1 v^7dx\right)^{1/5}+\frac{\alpha_{21}^2(b_{21}^2+b_{22}^2)}{2\varepsilon}M_1^{2/5}d_1^{-4/5}\left[\left(\int_0^1 u^7dx\right)^{3/5}+\left(\int_0^1 v^7dx\right)^{3/5}\right]\\ &+\frac{5\alpha_{21}^2e_2^2}{14\varepsilon}d_1^2\left(\int_0^1 u^7dx+\int_0^1 v^7dx\right)+2\varepsilon\int_0^1 vv_t^2dx.\end{aligned}$$

By the above inequalities and the condition (1.9), we have

$$\begin{aligned}&\int_0^1[(1+2\alpha_{11}u+\alpha_{12}v)u_tf+\alpha_{12}uv_tf]dx+\int_0^1[(\xi+\alpha_{21}u+2\alpha_{22}v)v_tg+\alpha_{21}vu_tg]dx\\ &\leq\bar{\lambda}\varepsilon\int_0^1(u+v)(u_t^2+v_t^2)dx+\frac{C_4}{\varepsilon}(1+\xi^2)M_0d_1^{-3}+\frac{C_5}{\varepsilon}(1+\xi^2+d_1^{-2})M_1^{4/5}d_1^{-8/5}\left[\int_0^1(u^7+v^7)dx\right]^{1/5}\\ &\quad+\frac{C_5}{\varepsilon}(1+\xi^2+d_1^{-2})M_1^{2/5}d_1^{6/5}\left[\int_0^1(u^7+v^7)dx\right]^{3/5}+\frac{C_6}{\varepsilon}d_1^2\int_0^1(u^7+v^7)dx,\end{aligned}\tag{2.16}$$

where $\bar{\lambda}$ is a constant. Choose a small enough positive number $\varepsilon(\alpha_{ij}, a_i, b_{ij}, e_i)(i, j = 1, 2)$, such that $\bar{\lambda}\varepsilon < C_3$.

Substituting inequalities (2.15) and (2.16) into (2.13), one can obtain

$$\frac{1}{2}\bar{y}'(t)\leq-\frac{1}{2}\int_0^1 P_{xx}^2dx-\frac{\xi}{2}\int_0^1 Q_{xx}^2dx+B_1K_2d_1^{-3}+B_2d_1^2Y+B_3K_3d_1^{6/5}Y^{3/5}+B_4K_4d_1^{-8/5}Y^{1/5},\tag{2.17}$$

where $Y=\int_0^1(u^7+v^7)dx$, $K_2=(1+\xi^2)M_0$, $K_3=\left(1+\xi^2+d_1^{-2}\right)M_1^{2/5}$, $K_4=\left(1+\xi^2+d_1^{-2}\right)M_1^{4/5}$.

Clearly,

$$P \geq \alpha_{11} u^2, \quad Q \geq \alpha_{22} v^2.$$

It follows from (2.12) and (2.3) to functions P, Q that

$$\begin{aligned} Y &\leq B_5 \int_0^1 (P^{7/2} + Q^{7/2}) dx \leq B_6 K_1^{11/6} d_1^{-11/6} \bar{y}^{5/6} + B_6 K_1^{7/2} d_1^{-7/2}, \\ Y^{1/5} &\leq B_7 K_1^{11/30} d_1^{-11/30} \bar{y}^{1/6} + B_7 K_1^{7/10} d_1^{-7/10}, \\ Y^{3/5} &\leq B_8 K_1^{11/10} d_1^{-11/10} \bar{y}^{1/2} + B_8 K_1^{21/10} d_1^{-21/10}. \end{aligned} \tag{2.18}$$

Moreover, one can obtain by (2.4) and (2.12)

$$-\frac{1}{2} \int_0^1 P_{xx}^2 dx - \frac{\xi}{2} \int_0^1 Q_{xx}^2 dx \leq -B_9 \min\{1, \ \xi\} K_1^{-4/3} d_1^{4/3} \bar{y}^{5/3} + (1+\xi) K_1^2 d_1^{-2}. \tag{2.19}$$

Combining (2.16), (2.18), and (2.19), we have

$$\begin{aligned} \frac{1}{2} \bar{y}'(t) \leq &- A_1 \min\{1, \ \xi\} K_1^{-4/3} d_1^{4/3} \bar{y}^{5/3} \\ &+ A_2 \left[(1+\xi) K_1^2 d_1^{-2} + K_2 d_1^{-3} + K_1^{7/2} d_1^{-3/2} + K_1^{21/10} K_3 d_1^{-9/10} + K_1^{7/10} K_4 d_1^{-23/10} \right] \\ &+ A_3 K_1^{11/6} d_1^{1/6} \bar{y}^{5/6} + A_4 K_1^{11/10} K_3 d_1^{1/10} \bar{y}^{1/2} + A_5 K_1^{11/30} K_4 d_1^{-59/30} \bar{y}^{1/6}. \end{aligned} \tag{2.20}$$

Multiplying inequality (2.20) by d_1^2, we have

$$\begin{aligned} \frac{1}{2} y'(t) \leq &- A_1 \min\{1, \ \xi\} K_1^{-4/3} y^{5/3} \\ &+ A_2 \left[(1+\xi) K_1^2 + K_2 d_1^{-1} + K_1^{7/2} d_1^{1/2} + K_1^{21/10} K_3 d_1^{11/10} + K_1^{7/10} K_4 d_1^{-3/10} \right] \\ &+ A_3 K_1^{11/6} d_1^{1/2} y^{5/6} + A_4 K_1^{11/10} K_3 d_1^{11/10} y^{1/2} + A_5 K_1^{11/30} K_4 d_1^{-3/10} y^{1/6}, \end{aligned} \tag{2.21}$$

where $y = \int_0^1 \left[(d_1 P_x)^2 + (d_1 Q_x)^2 \right] dx$. The inequality (2.21) implies that there exist $\tilde{\tau}_2 > 0$ and positive constant $\widetilde{M_2}$ depending on d_i, α_{ij}, a_i, b_{ij}, $e_i (i, j = 1, 2)$, such that

$$\int_0^1 (d_1 P_x)^2 dx, \ \int_0^1 (d_1 Q_x)^2 dx \leq \widetilde{M_2}, t \geq \tilde{\tau}_2. \tag{2.22}$$

In the case that $d_1, d_2 \geq 1$, $\xi \in [\underline{d}, \bar{d}]$, the coefficients of inequality (2.20) can be estimated by some constants which depend on $\underline{d}, \bar{d}$, but do not depend on d_1, d_2. So $\widetilde{M_2}$ depends on α_{ij}, a_i, b_{ij}, $e_i (i, j = 1, 2)$, $\underline{d}$ and $\bar{d}$, but it is irrelevant to d_1, d_2, when d_1, $d_2 \geq 1$ and $\xi \in [\underline{d}, \bar{d}]$. Since

$$\begin{pmatrix} P_x \\ Q_x \end{pmatrix} = \begin{pmatrix} P_u & P_v \\ Q_u & Q_v \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix},$$

we can transform the formulations of u_x, v_x into fraction representations, then distribute the denominators of the absolute value of the fractions to the numerators item and enlarge the term concerning with u_x, v_x to obtain

$$|d_1u_x| + |d_1v_x| \ \le L(|\,d_1P_x| + |d_1Q_x|), \quad 0 < x < 1, t > 0, \tag{2.23}$$

where L is a constant depending only on ξ, $\alpha_{ij}(i, j = 1, 2)$. After scaling back and contacting estimates (2.22) and (2.23), there exist positive constant M_2 depending on d_i, α_{ij}, a_i, b_{ij}, $e_i(i, j = 1, 2)$ and $\tau_2 > 0$, such that

$$\int_0^1 u_x^2 dx, \int_0^1 v_x^2 dx \le M_2, \quad t \ge \tau_2. \tag{2.24}$$

When d_1, $d_2 \ge 1$ and $\xi \in [\underline{d}, \bar{d}]$, M_2 is independent of d_1, d_2.

(ii) $t \ge 0$. Modifying the dependency of the coefficients in inequalities (2.12)-(2.14), namely replacing M_0, M_1 with M'_0, M'_1, there exists a positive constant M'_2 depending on d_i, α_{ij}, a_i, b_{ij}, $e_i(i, j = 1, 2)$ and the W_2^1-norm of u_0, v_0, such that

$$\int_0^1 u_x^2 dx, \int_0^1 v_x^2 dx \le M'_2, \quad t \ge 0. \tag{2.24a}$$

Furthermore, in the case that d_1, $d_2 \ge 1$, $\xi \in [\underline{d}, \bar{d}]$, M'_2, depends on $\underline{d}, \bar{d}$, but does not depend on d_1, d_2.

Summarizing estimates (2.5), (2.9), (2.24), and Sobolev embedding theorem, there exist positive constants M, M' depending only on d_i, α_{ij}, a_i, b_{ij}, $e_i(i, j = 1, 2)$, such that (1.10) and (1.11) hold. In particular, M, M' depend only on α_{ij}, a_i, b_{ij}, $e_i(i, j = 1, 2)$, $\underline{d}$ and $\bar{d}$, but do not depend on d_1, d_2, when d_1, $d_2 \ge 1$ and $\xi \in [\underline{d}, \bar{d}]$. Similarly, according to (2.5)', (2.9)', (2.24)', there exists a positive constant M'' depending on d_i, α_{ij}, a_i, b_{ij}, $e_i(i, j = 1, 2)$ and the initial functions u_0, v_0, such that

$$|u(\cdot,\ t)|_{1,2},\ |v(\cdot,\ t)|_{1,2} \le M'', \quad t \ge 0.$$

Further, in the case that d_1, $d_2 \ge 1$, $\xi \in [\underline{d}, \bar{d}]$, M" depends only on $\underline{d}, \bar{d}$, but do not depend on d_1, d_2. Thus, $T = +\infty$.

This completes the proof of Theorem 1.1.

3 Global stability

In order to obtain the uniform convergence of the solution to system (1.2), we recall the following result which can be found in [21].

Lemma 3.1. Let a and b be positive constants. Assume that ϕ, $\psi \in C^1([a, +\infty))$, $\psi(t) \ge 0$ and ϕ is bounded from below. If $\phi'(t) \le -b\psi(t)$ and $\psi'(t)$ is bounded from above in $[a, +\infty)$, then $\lim_{t\to\infty} \psi(t) = 0$.

Proof of Theorem 1.2. Let (u, v) be a solution for the system (1.8) with initial functions $u_0(x)$, $v_0(x) \ge$ (⊠)0. From the strong maximum principle for parabolic equations, it is not hard to verify that u, $v > 0$ for $t > 0$. Define the function

$$H(u, v) = \int_0^1 \left(u - u^* - u^* \ln \frac{u}{u^*}\right) dx + \int_0^1 \left(v - v^* - v^* \ln \frac{v}{v^*}\right) dx.$$

Then the time derivative of $H(u, v)$ for the system (1.8) satisfies

$$\begin{aligned}\frac{dH}{dt} &= \int_0^1 \frac{u-u^*}{u} u_t dx + \int_0^1 \frac{v-v^*}{v} v_t dx \\ &= -\int_0^1 \Big[\frac{u^*}{u^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 + (\frac{\alpha_{12}u^*}{u} + \frac{\alpha_{21}v^*}{v})u_x v_x + \frac{\alpha v^*}{v^2}(d_2 + \alpha_{21}u + 2\alpha_{22}v)v_x^2\,\Big]dx \\ &\quad -\int_0^1 \Big[(u-u^*)^2(b_{11}+e_1 v) + (b_{12}+b_{21}+e_1u^*+e_2v^*)(u-u^*)(v-v^*) + (v-v^*)^2(b_{22}+e_2u)\Big]dx.\end{aligned} \tag{3.1}$$

The first integrand in the right-hand side of (3.1) is positive definite if

$$4u^*v^*(d_1 + 2\alpha_{11}u + \alpha_{12}v)(d_2 + \alpha_{21}u + 2\alpha_{22}v) > (\alpha_{12}u^*v + \alpha_{21}v^*u)^2, \tag{3.2}$$

and the condition (1.12) implies (3.2). The second integrand in the right-hand side of (3.1) is positive definite if

$$4(b_{11} + e_1v)(b_{22} + e_2u) > (b_{12} + b_{21} + e_1u^* + e_2v^*)^2, \tag{3.3}$$

and the condition (1.13) implies (3.3). Consequently, there exists $\delta > 0$, such that

$$\frac{dH}{dt} \le -\delta \int_0^1 [(u-u^*)^2 + (v-v^*)^2]dx, \quad \frac{dH}{dt} \le 0, \quad (u,\ v) \ne (u^*,\ v^*). \tag{3.4}$$

By the maximum-norm estimate in Theorem 1.1 and some tedious calculations, we can verify $(d/dt)\int_0^1 [(u-u^*)^2 + (v-v^*)^2]dx$ is bounded from above. Then from Lemmas 3.1 and (3.4), we obtain

$$\lim_{t\to\infty} \int_0^1 (u-u^*)^2 dx = \lim_{t\to\infty} \int_0^1 (v-v^*)^2 dx = 0. \tag{3.5}$$

It follows from (3.5) and Gagliardo-Nirenberg-type inequality $|u|_\infty \le C|u|_{1,2}^{1/2}|u|_2^{1/2}$ that (u, v) converges uniformly to (u^*, v^*). By the fact that $H(u, v)$ is decreasing for $t \ge 0$, it is obvious that (u^*, v^*) is globally asymptotically stable. So the proof of Theorem 1.2 is completed.

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Competing interests
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