Bouzaffour *Advances in Difference Equations* 2012, **2012**:87
http://www.advancesindifferenceequations.com/content/2012/1/87

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

On the q-translation associated with the Askey-Wilson operator

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Abstract

In this article, we solve the open problem 24.5.6 given in the study of Ismail, which consists of extending the action of q-translation operators introduced by Ismail to some measurable functions by means of basic Fourier theory. Also, we prove that the q-exponential function is the only solution of the q-analogue of the Cauchy functional equation. As application we give an inversion formula for the q-Gauss Weierstrass transform.

2000 Mathematics Subject Classification: 33D45; 33D60.

Keywords: basic orthogonal polynomials and functions, basic hypergeometric integrals

Introduction

The concept of the q-translation operators E_q^y introduced by Ismail [1] was defined in polynomials through their action on the continuous q-Hermite polynomials $H_m(x \mid q)$ as follows

$$E_q^y H_n(x|q) = \sum_{m=0}^{n} \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} H_m(x|q) g_{n-m}(y) q^{(m^2-n^2)/4}, \tag{1}$$

where

$$g_n(cos(\theta)) = q^{n^2/4}(1+e^{2i\theta})e^{-in\theta}(-q^{2-n}e^{2i\theta};q^2)_{n-1}.$$

In others words

$$\frac{q^{n^2/4}}{(q;q)_n} E_q^y H_n(x|q) = \sum_{0\le m,j,m+2j\le n} \frac{q^{j+(m^2+(n-m-2j)^2)/4}}{(q^2;q^2)_j} \frac{H_m(x|q)H_{n-m-2j}(y|q)}{(q;q)_m (q;q)_{n-m-2j}},$$

where the polynomials $H_n(x \mid q)$ are defined by (see [2,3])

$$H_n(\cos\theta|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} e^{i(n-2k)\theta}. \tag{2}$$

It was shown in [1] that the q-translation operators E_q^y commute with the Askey-Wilson operator $\mathcal{D}_q$ on the space of all polynomials and by the use of the following expansion [3]

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$$(q\alpha^2;q^2)_\infty \mathcal{E}_q(x;\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(q;q)_n} q^{n^2/4} H_n(x|q). \tag{3}$$

In ([1], (2.21)), the author proved the following product formula for the q-exponential function

$$E_q^y \mathcal{E}_q(x;\alpha) = \mathcal{E}_q(x;\alpha)\mathcal{E}_q(y;\alpha). \tag{4}$$

Furthermore, if y and z are two complex variables, then we have [1]

$$E_q^y E_q^z = E_q^z E_q^y,$$
$$E_q^y f(x) = E_q^x f(y).$$

In [3] problem 24.5.6, Ismail proposed the extension of the action of E_q^y to measurable functions and proving that the only measurable functional solution of the q-analogue of the Cauchy functional equation

$$E_q^y f(x) = f(x)f(y)$$

is the q-exponential function. $\mathcal{E}_q(x;\alpha)$. The purpose of this paper is to define a new q-translation operator T_q^x related to Askey-Wilson operator acting in some measurable functions by means of the basic Fourier series. We show that the new q-Translation coincides with E_q^y on the set of continuous q-Hermite polynomials. In the same context, we establish many properties satisfied by the q-translation operator and generalizing the classical ones.

In the first section, we recall some results of basic Fourier series given in [4]. In Section "Preliminaries", we define and study the q-translation operator T_q^x. Also we solve the following problem

$$\begin{cases} \mathcal{D}_{q,x} u(x,\ y) = \mathcal{D}_{q,y} u(x,\ y) \\ u(x,\ 0) = f(x),\ f \in H_\varepsilon. \end{cases} \tag{5}$$

As a consequence of (5) we solve the basic analogue of the Cauchy functional equation

$$f(x \oplus y) := T_q^x f(y) = f(x)f(y), \tag{6}$$

where the function f is in the same subspace of $L^2(w\ (x)\ dx)$. In addition, we prove the q-translation invariance of the measure $w\ (x)\ dx$ over $(-1, 1)$. Some q-analogous of the Gauss Weierstrass transforms are studied in Section "q-Gauss Weierstrass transform".

Preliminaries

Let $0 < q < 1$ and $a \in \mathbb{C}$, the q-shift factorial is defined by (see [2])

$$(a;q)_0 = 1,$$
$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \ldots, \infty.$$

Given a function $f(x)$ with $x = \cos\theta$, $f(x)$ can be viewed as a function of $e^{i\theta}$.

Let

$$\breve{f}(e^{i\theta}) := f(\cos\theta).$$

The Askey-Wilson-divided difference operator $\mathcal{D}_q$ is defined by

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{\frac{1}{2}}e^{i\theta}) - \breve{f}(q^{-\frac{1}{2}}e^{i\theta})}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})i\sin\theta}. \tag{7}$$

The q-exponential function is given by [5]

$$\mathcal{E}_q(\cos\theta,\ \cos\phi;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}\sum_{n=0}^{\infty}(-e^{i(\theta+\phi)}q^{(1-n)/2},\ -e^{i(\theta-\phi)}q^{(1-n)/2};q)_n \times \frac{(\alpha e^{-i\phi})^n}{(q;q)_n}q^{n^2/4}.$$

The q-exponential function $\mathcal{E}_q(\cos\theta, \cos\phi;\alpha)$ is a solution of the q-difference equation of first-order [3]

$$\mathcal{D}_q\mathcal{E}_q(\cos\theta,\ \cos\phi;\alpha) = \frac{2\alpha q^{1/4}}{1-q}\mathcal{E}_q(\cos\theta,\ \cos\phi;\alpha).$$

Put

$$\mathcal{E}_q(x;\alpha) = \mathcal{E}_q(x,\ 0;\alpha),$$

then, we have (see [3])

$$\mathcal{E}_q(x,\ y;\alpha) = \mathcal{E}_q(x;\alpha)\mathcal{E}_q(y;\alpha),$$

and

$$\lim_{q\to 1}\mathcal{E}_q(x;(1-q)\alpha) = \exp(2\alpha x).$$

Ismail and Zhang [5,3] defined the q-cosine and q-sine functions through their q-exponential function as in the standard way, i.e.,

$$\mathcal{E}_q(x;i\omega) = C_q(x;\omega) + iS_q(x;\omega),$$

and used transformation formulas to continue them analytically to entire functions in the variable ω. Bustoz and Suslov [4] have established the following orthogonality relations

$$\int_{-1}^{1}\mathcal{E}_q(x;i\omega_n)\overline{\mathcal{E}_q(x;i\omega_m)}w(x)dx = 2k(\omega_n)\delta_{n,m}, \tag{8}$$

where

$$w(\cos\theta) = \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta};q)_\infty},$$
$$k(\omega) = \pi\frac{(q^{1/2};q)_\infty^2}{(q;q)_\infty^2}\frac{(-\omega^2;q^2)_\infty}{(-q\omega^2;q^2)_\infty}\sum_{k=0}^{\infty}\frac{q^{k/2}}{1+\omega^2q^k},$$

$\omega_0 = 0$, $\omega_1 < \omega_2 < \ldots$, are zeros of the q-sine function $S_q((q^{1/4} + q^{-1/4})/2); \omega)$ and for $n = 1, 2, \ldots, \omega_{-n} = -\omega_n$.

From [4], we have the following asymptotic estimates as $n \to \infty$

$$\omega_n \sim q^{1/4-n}, \text{ and } k(\omega_n) \sim 2\pi \frac{(-q; q^2)_\infty}{(-q^{1/2}; q)_\infty^2}, \tag{9}$$

and for $0 < \varepsilon < 1/2$ and $|x| \leq 1 < (q^{\varepsilon} + q^{-\varepsilon})/2$, we have

$$|\mathcal{E}_q(x; i\omega_n)| < \frac{(-q^{1/4-\varepsilon}|\omega_n|; q^{1/2})_\infty}{(q, q^{2\varepsilon}, q^{1-2\varepsilon}; q)_\infty (-q\omega_n^2; q^2)_\infty} \sim C q^{-2\varepsilon n}, \tag{10}$$

where $C = 1/(-q^{1/2}, q; q)_\infty (q^{\varepsilon}, q^{1/2-\varepsilon}; q^{1/2})_\infty$.

q-Translation

We define the q-Fourier transform $\mathcal{F}_q$ as

$$\mathcal{F}_q(f)(n) = \int_{-1}^{1} \mathcal{E}_q(x; -i\omega_n) f(x) w(x) dx, \; n \in \mathbb{Z}.$$

Put

$$l^2(k(\omega_n)) = \{(z_n)_{n\in\mathbb{Z}}; \sum_{n=-\infty}^{\infty} \frac{1}{2k(\omega_n)} |z_n|^2 < \infty\}.$$

Theorem 1. *The transform $\mathcal{F}_q$ is an isomorphism from $L^2((-1, 1), w(x)dx)$ into $l^2(k(\omega_n))$ and its inverse is given by*

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2k(\omega_n)} \mathcal{F}_q(f)(n) \mathcal{E}_q(x; i\omega_n).$$

Proof. The result follows from the fact that the family $\{\mathcal{E}_q(x; i\omega_n)\}_{n=-\infty}^{\infty}$ is complete and orthogonal in $L^2((-1, 1), w(x)dx)$ (see [4], [6]). □

Next, we use the q-Fourier series to define the q-translation operators T_q^x. Let denote by H_ε, $(\varepsilon > 0)$, the space of functions in $L^2((-1, 1), w(x)dx)$ such that

$$\sum_{n=-\infty}^{\infty} \frac{q^{-4\varepsilon|n|}}{2k(\omega_n)} |\mathcal{F}(f)(n)|^2 < \infty.$$

Definition 1. Let $0 < \varepsilon < 1/2$ and f in H_ε, we put

$$\begin{aligned} T_q^y f(x) &= \sum_{n=-\infty}^{\infty} \frac{1}{2k(\omega_n)} \mathcal{F}_q(f)(n) \mathcal{E}_q(x; i\omega_n) \mathcal{E}_q(y; i\omega_n), \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2k(\omega_n)} \mathcal{F}_q(f)(n) \mathcal{E}_q(x, \; y; i\omega_n). \end{aligned}$$

The operators T_q^y, are called q–translation operators associated to the Askey-Wilson operator.

Remark 1. (1) The q-translation operators are characterized by the formula

$$\mathcal{F}_q(T_q^y f)(n) = \mathcal{E}_q(y; i\omega_n)\mathcal{F}_q(f)(n), n \in \mathbb{Z}.$$

(2) The Askey-Wilson operator introduced in (7) can be defined on the space $H_{1/2}$ via

$$\mathcal{D}_q f(x) = \frac{q^{1/4}}{1-q} i \sum_{n=-\infty}^{\infty} \frac{1}{k(\omega_n)} \omega_n \mathcal{F}_q(f)(n)\mathcal{E}_q(x; i\omega_n).$$

Proposition 2. *For* $0 < \varepsilon < 1/2$, *we have*

(1) $T_q^0 = id$

(2) $T_q^y \mathcal{E}_q(x; i\omega_n) = \mathcal{E}_q(x; i\omega_n)\mathcal{E}_q(y; i\omega_n)$

(3) $T_q^y f(x) = T_q^x f(y)$

(4) $\mathcal{D}_q T_q^y f = T_q^y \mathcal{D}_q f, f \in H_1$,

where id denotes the identity operator.

Proof. The properties (1)-(3) are evident. To prove property (4), let

$$f(x) = \sum_{n=-\infty}^{\infty} \mathcal{F}_q(f)(n)\mathcal{E}_q(x; i\omega_n) \in H_1.$$

From (9) and (10) we have

$$\begin{aligned} \mathcal{D}_q T_q^y f &= \sum_{n=-\infty}^{\infty} \frac{2q^{1/4}}{1-q} i\omega_m \mathcal{F}_q(f)(n)\mathcal{E}_q(x; i\omega_n)\mathcal{E}_q(x; i\omega_n) \\ &= T_q^y \mathcal{D}_q f. \end{aligned}$$

□

Theorem 3. *For* $0 < \varepsilon < 1/2$ *and* f *in* H_ε, *the function*

$$u(x,\ y) = T_q^y f(x),$$

is the unique solution of the system

$$\begin{cases} \mathcal{D}_{q,x} u(x,\ y) = \mathcal{D}_{q,y} u(x,\ y), \\ \quad u(x,\ 0) = f(x). \end{cases} \tag{11}$$

Proof. It is clear that the function $T_q^y f(x)$ is a solution of the system (11). Applying the q-Fourier transform to each member of the system (11), we obtain for $n \in \mathbb{Z}$,

$$\begin{cases} \mathcal{D}_{q,y}\mathcal{F}_q\ (u\ (.;\ y))(n) = \dfrac{2iq^{1/4}}{1-q}\omega_n \mathcal{F}_q(u\ (.;\ y)), \\ \quad \mathcal{F}_q\ (u\ (.;\ 0))(n) = \mathcal{F}_q(f)(n). \end{cases}$$

Hence

$$\mathcal{F}_q\ (u\ (.;\ y))(n) = \mathcal{F}_q(f)(n)\mathcal{E}_q(y; i\omega_n).$$

So that

$$\begin{aligned} u(x,\ y) &= \sum_{n=-\infty}^{\infty} \mathcal{F}_q(f)(n)\mathcal{E}_q(y; i\omega_n)\mathcal{E}_q(x; i\omega_n) \\ &= T_q^y f(x). \end{aligned}$$

□

Proposition 4. *Let* $0 < \varepsilon < 1/4$, *we have*

$$\mathcal{E}_q\ (x;\ it) \in H_\varepsilon.$$

Proof. From the integral (3.13) in [7], we get

$$\begin{aligned} \mathcal{F}\ (\ \mathcal{E}_q\ (.;\ it))\ (n) &= \int_{-1}^{1} \mathcal{E}_q(\ x;\ it)\ \mathcal{E}_q(x; -i\omega_n)w(x)dx \\ &= \mathrm{Sinc}_q(t, n), \end{aligned}$$

where

$$\mathrm{Sinc}_q(t,\ n) = \frac{(q^{1/2}; q)_{1/2}(-q\omega_n^2; q^2)_\infty \mathrm{Im}((it, -i\omega_n; q^{1/2})_\infty)}{(-qt^2; q^2)_\infty(\omega_n + t)\frac{\partial}{\partial t}\mathrm{Im}((it, -i\omega_n; q^{1/2})_\infty)|_{t=-\omega_n}}.$$

By (9), we have the asymptotic estimates as $n \to \infty$

$$\begin{aligned} (-q\omega_n^2; q^2)_\infty &\sim (-q^{1/2-2n}; q^2)_\infty \\ &\sim (-)^n q^{-n^2+1/2n}(-q^{1/2}; q^2)_n(-q^{3/2}; q^2)_\infty, \end{aligned}$$

and similarly

$$\mathrm{Im}(it,\ -i\omega_n; q^{1/2})_\infty \sim (-1)^n q^{-n^2}\mathrm{Im}(it,\ iq^{1/4}; q^{1/2})_\infty.$$

Then from (9) and relation ((3.15), [7]), we obtain as $n \to \infty$

$$\mathrm{Sinc}_q(t, n) \sim \frac{(q, q^2)_\infty^3 \mathrm{Im}(it, iq^{1/4}; q^{1/2})_\infty}{2q^{1/4}(q, q)_\infty^2(q^{-3/2}, q^{1/2}, q; q^2)_\infty} q^{n/2}, \tag{12}$$

and

$$\mathrm{Sinc}_q(t,\ -n) \sim \frac{(q, q^2)_\infty^3 \mathrm{Im}(-it, iq^{1/4}; q^{1/2})_\infty}{2q^{1/4}(q, q)_\infty^2(q^{-3/2}, q^{1/2}, q, q^2)_\infty} q^{n/2}. \tag{13}$$

So that

$$\frac{q^{-4|n|}}{2k(\omega_n)}|\mathcal{F}(\mathcal{E}_q(.; it))(n)|^2 \sim Cq^{(1-4\varepsilon)|n|}.$$

Then the following series

$$\sum_{n=-\infty}^{\infty} \frac{q^{-4\varepsilon|n|}}{2k(\omega_n)}|\mathcal{F}(\mathcal{E}_q(.; it))(n)|^2,$$

converges iff $\varepsilon < 1/4$.

This show for $0 < \varepsilon < 1/4$ we have

$$\mathcal{E}_q\,(x;\;it) \in H_\varepsilon,$$

and

□

$$\mathcal{E}_q(x;it) = \sum_{n=-\infty}^{\infty} \mathrm{Sinc}_q(t,n)\mathcal{E}_q(x;i\omega_n). \tag{14}$$

Proposition 5. *For* $t \neq -iq^{-1/2-n}$, $n = 0, \pm 1, \pm 2, \ldots$ *we have*

$$T_q^y \mathcal{E}_q(x;it) = \mathcal{E}_q(x;it)\mathcal{E}_q(y;it).$$

Proof. From Proposition 2, we see that the following two functions

$$(x,y) \to \mathcal{E}_q(x;it)\mathcal{E}_q(y;it),$$

and

$$(x,y) \to T_q^y \mathcal{E}_q(x;it),$$

are solutions of the system

$$\begin{cases} \mathcal{D}_{q,x}u(x,y) = \mathcal{D}_{q,y}u(x,y), \\ u(x,0) = \mathcal{E}_q(x;\alpha). \end{cases}$$

The result follows by Theorem 3. □

In the following proposition, we find the invariance of the measure $w(x)dx$ over $(-1, 1)$ by the q-translation operators.

Proposition 6. *Let* $0 < \varepsilon < 1/2$ *and* $f \in H_\varepsilon$. *Then*

$$\int_{-1}^{1} T_q^y f(x)w(x)dx = \int_{-1}^{1} f(x)w(x)dx.$$

Proof. We have

$$\int_{-1}^{1} T_q^y f(x)w(x)dx = \int_{-1}^{1} \sum_{n=-\infty}^{\infty} \frac{1}{2k(\omega_n)} \mathcal{F}(f)(n)\mathcal{E}_q(x;i\omega_n)\mathcal{E}_q(y;i\omega_n)w(x)dx.$$

Then, we get after interchangement of integral and sum

$$\int_{-1}^{1} T_q^y f(x)w(x)dx = \mathcal{F}(f)(0) = \int_{-1}^{1} f(x)w(x)dx. \tag{15}$$

To justify the interchangement of integral and summation, we put

$$\sum_{n=-\infty}^{\infty} \int_{-1}^{1} |\frac{1}{2k(\omega_n)} \mathcal{F}(f)(n)\mathcal{E}_q(x;i\omega_n)\mathcal{E}_q(y;i\omega_n)|w(x)dx = \sum_{-\infty}^{\infty} c_n. \tag{16}$$

Let $0 < \eta < 1/4$ and $\delta > 0$ such that $2\eta + \delta < 1/2$, by (9) and (10), we have

$$\begin{aligned} c_n &\sim \frac{1}{2k(\omega_n)} |\mathcal{F}(f)(n)| q^{-4\eta|n|}, \\ &\sim \frac{1}{2k(\omega_n)} |\mathcal{F}(f)(n)| q^{-2(2\eta+\delta)|n|} q^{2\delta|n|}. \end{aligned}$$

The convergence of the series $\sum_{-\infty}^{\infty} c_n$ follows from the Cauchy inequality and the fact that $f \in H_\varepsilon$.

□

In the following proposition, we show that the q-translation are self-adjoint operators.

Proposition 7. *Let f and* $g \in H_\varepsilon$ *where* $0 < \varepsilon < 1/2$. *Then*

$$\int_{-1}^{1} T_q^x f(-y) g(y) w(y) dy = \int_{-1}^{1} f(y) T_q^x g(-y) w(y) dy.$$

Ismail [1] proved that the q-exponential function $\mathcal{E}_q(x;\alpha)$ is the only solution of the functional equation

$$f(x \oplus y) = f(x) f(y), \tag{17}$$

where $f(x)$ has the expansion $f(x) = \sum_{n=0}^{\infty} \frac{f_n}{(q;q)_n} g_n(x)$, which converges uniformly on compact subsets of a domain Ω.

For $f \in L^2((-1,1), w(x)dx)$, we put

$$\widehat{f}(t) = \int_{-1}^{1} f(x) \mathcal{E}_q(x; -it) w(x) dx.$$

Proposition 8. *Let* $f \in L^2((-1,1)\ w(x)dx)$, *then the function*

$$F(t) = (-qt^2; q^2)_\infty \widehat{f}(t),$$

is an entire function such that

$$\lim_{r\to\infty} \frac{\ln(M(r,F))}{\ln^2 r} \le \frac{1}{\ln q - 1}.$$

Furthermore, the function F is of order 0 *and has infinitely many zeros.*

Proof. The function f is in $L^2((-1,1)\ w(x)dx)$, then

$$\int_{-1}^{1} |f(x)| w(x) dx < \infty.$$

From (2) we have the following estimate

$$|H_n(x|q)| \le H_n(1|q), \ for\ |x| < 1,$$

and by (3) we can write for all $t \in \mathbb{C}$

$$|F(t)| \le \int_{-1}^{1} |f(x)| w(x) dx (-q|t|^2; q^2)_\infty \mathcal{E}_q(1; |t|).$$

On the other hand

$$(-q|t|^2; q^2)_\infty \mathcal{E}_q(1; |t|) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(q;q)_n} q^{n^2/4} H_n(1|q).$$

The result follows by a similar proof as in Lemma 14.1.4 and Corollary 14.1.5 in [3].

□

Proposition 9. *We have*

$$\widehat{T_q^y f}(t) = \mathcal{E}_q(y; it)\widehat{f}(t).$$

Proof. It is easy to see that the function $w(x)$ is even and the q-exponential $\mathcal{E}_q(x; it)$ satisfies

$$\mathcal{E}_q(-x;\ it) = \mathcal{E}_q(x; -it).$$

Then from Propositions 4 and 7, we get

$$\begin{aligned}\widehat{T_q^y f}(t) &= \int_{-1}^{1} T_q^y f(x)\mathcal{E}_q(-x; it)w(x)dx\\ &= \int_{-1}^{1} f(x)T_q^y\mathcal{E}_q(-x; it)w(x)dx\\ &= \mathcal{E}_q(y; it)\int_{-1}^{1} f(x)\mathcal{E}_q(-x; it)w(x)dx\\ &= \mathcal{E}_q(y; it)\widehat{f}(t).\end{aligned}$$

□

In the following Proposition, we show that the q-translation operator T_q^x coincides with the q-Translation E_q^x defined by Ismail on the set of q-Hermite polynomials (2).

Proposition 10. *For* $n = 0, 1, 2, \ldots,$ *we have*

$$T_q^x H_n(y|q) = E_q^x H_n(y|q). \tag{18}$$

Proof. By Proposition 5, we get

$$(-qt^2; q^2)_\infty T_q^x\mathcal{E}(y; it) = (-qt^2; q^2)_\infty\mathcal{E}(x; it)\mathcal{E}(y; it).$$

Then the formula ([3, 14.6.7]) and (3) lead to

$$\sum_{n=0}^{\infty}\frac{(it)^n q^{n^2/4}}{(q;q)_n}T_q^y H_n(x|q) = \sum_{n=0}^{\infty}\frac{g_n(y)}{(q;q)_n}(it)^n\sum_{m=0}^{\infty}\frac{(it)^m q^{m^2/4}}{(q;q)_m}H_m(x|q).$$

Hence,

$$T_q^y H_n(x|q) = \sum_{m=0}^{n}\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}H_m(x|q)g_{n-m}(y)q^{(m^2-n^2)/4}.$$

□

Theorem 11. *Let f be a function in* $L^2((-1, 1), w(x)dx)$ *satisfying the following functional equation*

$$T_q^y f(x) = f(x)f(y),$$

and we denote by Z_f *the set of zeros of*

$$F(t) = (-qt^2; q^2)_\infty\widehat{f}(t).$$

Then f is a function of two variables x and t equal to the q-exponential function $\mathcal{E}_q(x; it)$, *for* $|x| < 1$ *and* $t \in \mathbb{C} - Z_f$

Proof. Let f be a function in L^2 $((-1, 1), w(x)dx)$ satisfying

$$T_q^y f(x) = f(x)f(y).$$

By Proposition 9, we have

$$\mathcal{E}_q(x; it)\widehat{f}(t) = f(x)\widehat{f}(t).$$

Then for all complex numbers t such that $\widehat{f}(t) \neq 0$, we have

$$f(x) := f(x, t) = \mathcal{E}_q(x; it),$$

and f is a function of two variables x and t. □

q-Gauss Weierstrass transform

We conclude this study by an application of the q-translation operators. We consider the q-analogue of the Gauss Weierstrass transform by (see [3]).

$$F_W(f)(y) = \frac{(q;q)_\infty}{2\pi} \int_0^\pi T_q^y f(x) W(x) dx, \tag{19}$$

where

$$W(x) = (e^{2i\theta}, e^{-2i\theta}; q)_\infty.$$

In [1], the author proved that (19) can be inverted by the Askey-Wilson operator

$$f(y) = \left(\frac{1}{4} q^{1/2} (1-q)^2 \mathcal{D}_q^2; q^2\right)_\infty F_W(f)(y). \tag{20}$$

where f is a polynomial.

In the following theorem we prove that the inversion formula (20) is still valid in the space

$$H_\infty = \bigcap_{\varepsilon > 0} H_\varepsilon.$$

Theorem 12. *The q-Gauss Weierstrass transform has the inversion formula*

$$f(y) = \left(\frac{1}{4} q^{1/2} (1-q)^2 \mathcal{D}_q^2; q^2\right)_\infty F_W(y), \ f \in H_\infty.$$

Proof. Let $f \in H_\infty$, then by the formula

$$\frac{1}{(qt^2; q^2)_\infty} = \frac{(q;q)_\infty}{2\pi} \int_0^\pi \mathcal{E}_q(x; t) W(x) dx,$$

we have

$$\begin{aligned} F_W(y) &= \frac{(q;q)_\infty}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2k(\omega_n)} \mathcal{F}_q(f)(n) \mathcal{E}_q(y; i\omega_n) \int_{-1}^{1} \mathcal{E}_q(x; i\omega_n) W(x) dx \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2k(\omega_n)} \frac{1}{(-q\omega_n^2; q^2)_\infty} \mathcal{F}_q(f)(n) \mathcal{E}_q(y; i\omega_n). \end{aligned}$$

So that

$$\begin{aligned}&\left(\frac{1}{4}q^{1/2}(1-q)^2\mathcal{D}_q^2;q^2\right)_\infty F_W(x)\\&=\sum_{n=-\infty}^{\infty}\frac{1}{2k(\omega_n)}\mathcal{F}_q(f)(n)\mathcal{E}_q(y;i\omega_n)\\&=f(y)\end{aligned}$$

□

Another q-analogue of Gauss Weierstrass transform can be defined by

$$F_\gamma(y)=\frac{(q,\gamma^2;q)_\infty}{2\pi(\gamma,q\gamma^2;q)_\infty}\int_{-1}^{1}T_q^y f(x)w(x|\gamma,\ q)dx,$$

where

$$w(\cos\theta|\gamma,\ q)=\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}.$$

In a similar way as in Theorem 12, we can prove the following inversion formula for the transform F_γ.

Theorem 13. *The transform F_γ has the inversion formula*

$$f(y)=\varphi_\gamma\left(\frac{1}{4}q^{1/2}(1-q)^2\mathcal{D}_q^2\right)F_\gamma(y),\ f\in H_\infty,$$

where

$$\varphi_\gamma(\alpha)=\frac{1}{{}_0\phi_1(-;q\gamma;q;q\gamma\alpha^2)}.$$

Acknowledgements
The author thank the referees for her careful reading of the manuscript and for helpful suggestions. This research was supported by the NPST Program of King Saud University, project number 10-MAT1293-02.

Competing interests
The authors declare that they have no competing interests.

Received: 27 December 2011 Accepted: 25 June 2012 Published: 25 June 2012

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doi:10.1186/1687-1847-2012-87
Cite this article as: Bouzaffour: On the q-translation associated with the Askey-Wilson operator. *Advances in Difference Equations* 2012 **2012**:87.