Gupta et al. Advances in Difference Equations 2012, 2012:86
http://www.advancesindifferenceequations.com/content/2012/1/86

Advances in Difference Equations
a SpringerOpen Journal

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Properties of *q*-analogue of Beta operator

Vijay Gupta[1], Honey Sharma[2], Taekyun Kim[3*] and Sang-Hun Lee[4]

* Correspondence: taekyun64@hotmail.com
[3]Department of Mathematics, Kwangwoon University, Seoul 139-701, South Korea
Full list of author information is available at the end of the article

Abstract

In this article, we introduce the *q*-variant of Beta operator. We find the recurrence formula for *m*th-order moments. Here, we establish some direct theorems in terms of modulus of continuity for these operators. We also propose conditions for better approximation. In the end, we also propose the Stancu-type generalization.
Mathematical Subject Classification: 41A25; 41A35.

Keywords: *q*-integers, *q*-beta function, *q*-Beta operators, modulus of continuity, asymptotic formula

1. Introduction

In the last four decades after the integral modification of Bernstein polynomials by Durrmeyer, several new Durrmeyer-type operators were introduced and their approximation properties were discussed. In 2007, Gupta et al. [1]proposed a family of linear positive operators as

$$B_n(f,\ x) = \frac{1}{(n+1)} \sum_{k=1}^{\infty} b_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t)dt, x \in [0,\ \infty), \tag{1.1}$$

where

$$b_{n,k}(t) = \frac{1}{B(k, n+1)} t^{k-1}(1+t)^{-n-k-1},$$

for $f \in C_\gamma\,[0, \infty)$, where $C_\gamma\,[0, \infty)$, $\gamma > 0$ be the class of all continuous functions defined on $[0, \infty)$ satisfying the growth condition $|f(t)| \le Ct^\gamma$, $C > 0$ and $B(k, n + 1)$ is beta function. They [1] established the direct and inverse results for these operators.

In the recent years, q-calculus was used in approximation theory and several new operators were introduced and their approximation properties were discussed (see [2-6], etc.). Motivated by these operators, we now introduce the q-analogue of (1.1). For $f \in C[0,\infty)$ and $0 < q < 1$, we propose the q-Beta operators as

$$B_n^q(f,\ x) = \frac{1}{[n+1]_q} \sum_{k=1}^{\infty} b_{n,k}^q(x) \int_0^{\infty/A} b_{n,k}^q(t) f(t) d_q t, x \in [0,\ \infty), \tag{1.2}$$

where

$$b_{n,k}^q(t) = \frac{q^{(k-1)^2/2}}{B_q(k, n+1)} t^{k-1}(1+t)_q^{-n-k-1},$$

and $B_q(t, s)$ denote the q-Beta function [7] is given by

$$B_q(t,\ s) = K(A,\ t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x,$$

where $K(x,\ t) = \frac{1}{1+x} x^t (1+1/x)_q^t (1+x)_q^{1-t}$. In particular, for any positive integer n, $K(x,\ n) = q^{\frac{n(n-1)}{2}}$, $K(x,0) = 1$, $B_q(t,\ s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}$ and $(a+b)_q^n = \prod_{s=0}^{n-1}(a+q^s b)$, $n \in \mathbb{N}$, a, $b \in \mathbb{R}$.

This article is the extension of the earlier work of [1]. Here, we consider the q variant of the operators discussed in [1] and obtain the recurrence relations for moments. We also obtain some direct results for the q operators, which also include the asymptotic formula. In the end, we establish the conditions for better approximation.

2. Preliminaries

To make the article self-content, here we mention certain basic definitions of q-calculus, details can be found in [8,9] and the other recent articles. For each nonnegative integer k, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are, respectively, defined by

$$[k]_q = \begin{cases} (1-q^k)/(1-q), & q \neq 1 \\ k, & q = 1 \end{cases},$$

and

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1 \\ 1 & k = 0 \end{cases}.$$

For the integers n, k satisfying $n \geq k \geq 0$, the q-binomial coecients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x},\ x \neq 0,$$

and the q-improper integral (see [10]) is given by

$$\int_0^{\infty/A} f(t) d_q t = (1-q) \sum_{j=-\infty}^{\infty} f(q^j/A)(q^j/A),\ A > 0.$$

The q-integral by parts is given by

$$\int_b^a f(t) D_q(g(t)) d_q t = [f(t)g(t)]_a^b - \int_a^b g(qt) D_q(f(t)) d_q t.$$

Lemma 1. *For n, $k \geq 0$, we have*

$$D_q(1+x)_q^{n+k+1} = [n+k+1]_q(1+qx)_q^{n+k}, \tag{2.1}$$

$$D_q\left(\frac{1}{(1+x)_q^{n+k+1}}\right) = -\frac{[n+k+1]_q}{(1+x)_q^{n+k+2}}. \tag{2.2}$$

Proof. Using q-derivative operator, we can write

$$\begin{aligned} D_q(1+x)_q^{n+k+1} &= \frac{1}{(q-1)x}\left(\prod_{j=0}^{n+k}(1+q^{j+1}x) - \prod_{j=0}^{n+k}(1+q^jx)\right) \\ &= \frac{(q^{n+k+1}-1)}{(q-1)}\prod_{j=0}^{n+k-1}(1+q^{j+1}x) \\ &= [n+k+1]_q(1+qx)_q^{n+k}. \end{aligned}$$

Equation (2.2) can be obtained directly by using q-quotient rule as follows:

$$\begin{aligned} D_q\left(\frac{1}{(1+x)_q^{n+k+1}}\right) &= \frac{-[n+k+1]_q(1+qx)_q^{n+k}}{(1+x)_q^{n+k+1}(1+qx)_q^{n+k+1}} \\ &= \frac{-[n+k+1]_q}{(1+x)_q^{n+k+1}(1+q^{n+k+1}x)} \\ &= \frac{-[n+k+1]_q}{(1+x)_q^{n+k+2}}. \end{aligned}$$

□

Remark 1. By using (2.1) and $D_q x^{k-1} = [k-1]_q x^{k-2}$, we get

$$\begin{aligned} D_q\left(\frac{x^{k-1}}{(1+x)_q^{n+k+1}}\right) &= \frac{(1+x)_q^{n+k+1}[k-1]_q x^{k-2} - x^{k-1}[n+k+1]_q(1+qx)_q^{n+k}}{(1+x)_q^{n+k+1}(1+qx)_q^{n+k+1}} \\ &= \frac{[k-1]_q x^{k-2}}{(1+qx)_q^{n+k+1}} - \frac{[n+k+1]_q x^{k-1}}{(1+q^{n+k+1}x)(1+x)_q^{n+k+1}} \\ &= \frac{x^{k-1}}{x(1+x)(1+qx)_q^{n+k+1}}[(1+x)[k-1]_q - x[n+k+1]_q] \\ &= \frac{x^{k-1}}{x(1+x)(1+qx)_q^{n+k+1}}[[k-1]_q - q^{k-1}[n+2]_q x]. \end{aligned}$$

Hence, we obtain

$$x(1+x)D_q\left(\frac{x^{k-1}}{(1+x)_q^{n+k+1}}\right) = \frac{x^{k-1}}{(1+qx)_q^{n+k+1}}([k-1]_q - q^{k-1}[n+2]_q x). \tag{2.3}$$

Lemma 2. *We have following equalities*

$$qx(1+x)D_q(b_{n,k}^q(x)) = [n+2]_q b_{n,k}^q(qx)\left(\frac{[k-1]_q}{q^{k-2}[n+2]_q} - qx\right), \tag{2.4}$$

$$\frac{t}{q}\left(1+\frac{t}{q}\right)D_q\left(b_{n,k}^q\left(\frac{t}{q}\right)\right) = \frac{[n+2]_q}{q^2}b_{n,k}^q(t)\left(\frac{[k-1]_q}{q^{k-2}[n+2]_q} - t\right). \tag{2.5}$$

Proof. Above equalities can be obtained by direct computations using definition of operator and (2.3). □

Theorem 1. *If mth* ($m > 0$, $m \in \mathbb{N}$)*-order moment of operator (1.2) is defined as*

$$B_{n,m}^q(x) := B_n^q(t^m, x) = \frac{1}{[n+1]_q} \sum_{k=1}^{\infty} b_{n,k}^q(x) \int_0^{\infty/A} b_{n,k}^q(t) t^m d_q t, x \in [0, \infty),$$

then $B_{n,0}^q(x) = 1$ *and for* $n > m$, *we have following recurrence relation,*

$$([n+2]_q - [m+2]_q) B_{n,m+1}^q(qx) = q([m+1]_q + x[n+2]_q) B_{n,m}^q(qx) + qx(1+x) D_q(B_{n,m}^q(x)). \tag{2.6}$$

Proof. By using (2.4), we have

$$\begin{aligned} qx(1+x)D_q(B_{n,m}^q(x)) &= \frac{1}{[n+1]_q} \sum_{k=1}^{\infty} qx(1+x) D_q b_{n,k}^q(x) \int_0^{\infty/A} b_{n,k}^q(t) t^m d_q t \\ &= \frac{[n+2]_q}{[n+1]_q} \sum_{k=1}^{\infty} b_{n,k}^q(qx) \int_0^{\infty/A} \left(\frac{[k-1]_q}{q^{k-2}[n+2]_q} - t \right) b_{n,k}^q(t) t^m d_q t \\ &+ \frac{[n+2]_q}{[n+1]_q} \sum_{k=1}^{\infty} b_{n,k}^q(qx) \int_0^{\infty/A} b_{n,k}^q(t) t^{m+1} d_q t \\ &- qx \frac{[n+2]_q}{[n+1]_q} \sum_{k=1}^{\infty} b_{n,k}^q(qx) \int_0^{\infty/A} b_{n,k}^q(t) t^m d_q t \\ &= I + [n+2]_q B_{n,m+1}^q(qx) - qx[n+2]_q B_{n,m}^q(qx), \end{aligned}$$

by (2.5) and q-integration by parts, we get

$$\begin{aligned} I &= \frac{1}{[n+1]_q} \sum_{k=1}^{\infty} b_{n,k}^q(qx) \int_0^{\infty/A} (qt^{m+1} + t^{m+2}) D_q b_{n,k}^q(t/q) d_q t \\ &= -\frac{1}{[n+1]_q} \sum_{k=1}^{\infty} b_{n,k}^q(qx) \int_0^{\infty/A} (q[m+1]_q t^m + [m+2]_q t^{m+1}) b_{n,k}^q(t) d_q t \\ &= -[m+1]_q q B_{n,m}^q(qx) - [m+2]_q B_{n,m+1}^q(qx), \end{aligned}$$

by combining above two equations, we can write

$$\begin{aligned} qx(1+x)D_q(B_{n,m}^q(x)) &= -[m+1]_q q B_{n,m}^q(qx) - [m+2]_q B_{n,m+1}^q(qx) \\ &+ [n+2]_q B_{n,m+1}^q(qx) - qx[n+2]_q B_{n,m}^q(qx) \\ &= ([n+2]_q - [m+2]_q) B_{n,m+1}^q(qx) - q([m+1]_q + x[n+2]_q) B_{n,m}^q(qx). \end{aligned}$$

Hence, the result follows. □

Corollary 2. *We have*

$$B_{n,1}^q(x) = \frac{(q + x[n+2]_q)}{([n+2]_q - [2]_q)}, \tag{2.7}$$

$$B_{n,2}^q(x) = \frac{(q[2]_q + x[n+2]_q)(q + x[n+2]_q)}{([n+2]_q - [2]_q)([n+2]_q - [3]_q)} + \frac{x(q+x)[n+2]_q}{q([n+2]_q - [2]_q)([n+2]_q - [3]_q)} \tag{2.8}$$

Corollary 3. *If we denote central moments by* $\phi_{n,m}^q(x) = B_n^q((t-x)^m, x)$, $m = 1,2$, *then we have*

$$\phi_{n,1}^q(x) = \frac{(q + x[2]_q)}{([n+2]_q - [2]_q)}, \tag{2.9}$$

$$\phi_{n,2}^q(x) = \frac{x^2\left(\left(\frac{1+q^3}{q}\right)[n+2]_q + [2]_q[3]_q\right) + x((q^2+1)[n+2]_q + 2q[3]_q) + q^2[2]_q}{([n+2]_q - [2]_q)([n+2]_q - [3]_q)} \tag{2.10}$$

Remark 2. As a special case when $q \to 1^-$, we have

$$B_{n,1}(x) = \frac{(2+n)x + 1}{n},$$
$$B_{n,2}(x) = \frac{(n+3)(n+2)}{n(n-1)}x^2 + 4\frac{(n+2)}{n(n-1)}x + \frac{2}{n(n-1)}.$$

The first two central moments for $q \to 1^-$ are

$$B_n((t-x), x) = \frac{2x+1}{n},$$
$$B_n((t-x)^2, x) = \frac{x(1+x)(2n+10)+2}{n(n-1)}.$$

which are the moments obtained by Gupta et al. [1] for Beta operator.

3. Ordinary approximation

Let $C_B\,[0, \infty)$ be the space of all real valued continuous bounded function f on $[0, \infty)$ endowed with the norm $||f|| = \sup\{|f(x)| : x \in [0, \infty)\}$. Further let us consider the following K-functional:

$$K_2(f, \delta) = \inf_{g \in W^2}\{||f - g|| + \delta||g''||\},$$

where $\delta > 0$ and

$$W^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}.$$

From [11], there exist an absolute constant $C > 0$ such that

$$K_2(f, \delta) \le C\omega_2(f, \delta^{1/2}), \tag{3.1}$$

where

$$\omega_2(f, \delta^{1/2}) = \sup_{0<h\le\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h) - 2f(x+h) + f(x)|$$

is the second-order modulus of smoothness of $f \in C_B\,[0, \infty)$. By

$$\omega(f, \delta) = \sup_{0<h\le\delta}\sup_{x\in[0,\infty)}|f(x+h) - f(x)|$$

we denote the usual modulus of continuity of $f \in C_B\,[0, \infty)$.

Theorem 4. *Let* $0 < q < 1$, *we have*

$$|B_n^q(f, x) - f(x)| \le C\omega_2\left(f, \sqrt{\phi_{n,2}^q(x) + {\phi_{n,1}^q}^2(x)}\right) + \omega(f, \phi_{n,1}^q(x))$$

for every $x \in [0, \infty)$ and $f \in C_B[0, \infty)$, where C is a positive constant.

Proof. We consider modified operators $\overline{B_n^q}$ defined by

$$\overline{B}_n{}^q(f, x) = B_n^q(f, x) - f\left(\frac{q + [n+2]_q x}{[n+2]_q - [2]_q}\right) + f(x), \tag{3.2}$$

$x \in [0, \infty)$. The operators $\overline{B}_n^q$ preserve the linear functions:

$$\overline{B}_n^q(t - x, x) = 0 \tag{3.3}$$

Let $g \in W^2$ and $t \in [0, \infty)$. Using Taylor's expansion, we have

$$g(t) = g(x) - g'(x)(t - x) + \int_x^t (t - u)g''(u)du,$$

and (3.3), we have

$$\bar{B}_n^q(g, x) = g(x) + \overline{B}_n^q\left(\int_x^t (t - u)g''(u)du, x\right).$$

Therefore, from (3.2), we have

$$\begin{aligned} |\bar{B}_n^q(g, x) - g(x)| &\le \left|B_n^q\left(\int_x^t (t - u)g''(u)du, x\right)\right| + \left|\int_x^{\frac{q+[n+2]_q x}{[n+2]_q-[2]_q}} \left(\frac{q + [n+2]_q x}{[n+2]_q - [2]_q} - u\right)g''(u)du\right| \\ &\le B_n^q\left(\int_x^t |t - u||g''(u)|du, x\right) + \int_x^{\frac{q+[n+2]_q x}{[n+2]_q-[2]_q}} \left|\frac{q + [n+2]_q x}{[n+2]_q - [2]_q} - u\right| |g''(u)|du \\ &\le \left[B_n^q((t - x)^2, x) + \left(\frac{q + [2]_q x}{[n+2]_q - [2]_q}\right)\right] ||g''||. \end{aligned} \tag{3.4}$$

From Corollary 3, we get

$$|\overline{B}_n^q(g, x) - g(x)| \le \left(\phi_{n,2}^q(x) + \phi_{n,1}^{q2}(x)\right) ||g''||. \tag{3.5}$$

By (3.2) and Corollary 1, we have

$$|\overline{B}_n^q(f, x)| \le |B_n^q(f, x)| + 2||f|| \le ||f||B_n^q(1, x) + 2||f|| \le 3||f||. \tag{3.6}$$

Now using (3.2), (3.5), and (3.6), we obtain

$$\begin{aligned} |B_{n,q}(f, x) - f(x)| &\le |\overline{B}_n^q(f - g, x) - (f - g)(x)| + |\overline{B}_n^q(g, x) - g(x)| + \left|f\left(\frac{q + [n+2]_q x}{[n+2]_q - [2]_q}\right) - f(x)\right| \\ &\le 4||f - g|| + (\phi_{n,2}^q(x) + \phi_{n,1}^{q2}(x))||g''|| + \left|f\left(\frac{q + [n+2]_q x}{[n+2]_q - [2]_q}\right) - f(x)\right|. \end{aligned}$$

Thus taking infimum on the right-hand side over all $g \in W^2$, we get

$$|B_n^q(f, x) - f(x)| \le 4K_2\left(f, \phi_{n,2}^q(x) + \phi_{n,1}^{q2}(x)\right) + \omega(f, \phi_{n,1}^q(x)).$$

In the view of (3.1), we get

$$|B_n^q(f,x)-f(x)|\leq C\omega_2\left(f,\sqrt{\phi_{n2}^q(x)+\phi_{n1}^{q2}(x)}\right)+\omega(f,\phi_{n,1}^q(x)).$$

This completes the proof of the theorem. □

4. Weighted approximation

Here, we give weighted approximation theorem for the operator $B_n^q(f,x)$. Similar type of results are given in [3].

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on the interval $[0, \infty)$ satisfying the condition

$$|f(x)|\leq M_f(1+x^2),$$

where M_f is a constant depending on f. $B_{x^2}[0,\infty)$ is a normed space with the norm

$$\|f\|_{x^2}=\sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2},\quad f\in B_{x^2}[0,\infty).$$

$C_{x^2}[0,\infty)$ denotes the subspace of all continuous functions in $B_{x^2}[0,\infty)$ and $C_{x^2}^*[0,\infty)$ denotes the subspace of all functions $f\in C_{x^2}[0,\infty)$ with $\lim_{x\to\infty}\frac{f(x)}{1+x^2}=k$.

Theorem 5. *Let $q = q_n \in (0, 1)$ such that $q_n \to 1$ as $n \to \infty$, then for each $f\in C_{x^2}^*[0,\infty)$, we have*

$$lim_{n\to\infty}\left\|B_n^{q_n}(f,x)-f(x)\right\|_{x^2}=0. \tag{4.1}$$

Proof. By the Korovkin's theorem (see [12]), $B_n^{q_n}(f,x)$ converges to f uniformly as $n\to\infty$ for $f\in C_{x^2}^*[0,\infty)$ if it satisfies $B_n^{q_n}(t^i;x)\to x^i$ for $i = 0, 1, 2$ uniformly as $n \to \infty$.

As, $B_n^{q_n}(1,x)=1$,

$$||B_n^{q_n}(1,x)-1||_{x^2}=0. \tag{4.2}$$

By Corollary 2, for $n > 1$,

$$\begin{aligned}||B_n^{q_n}(t,x)-x||_{x^2}&=\sup_{x\in[0,\infty)}\frac{|B_n^{q_n}(t,x)-x|}{1+x^2}\\&\leq\frac{q_n}{([n+2]_{q_n}-[2]_{q_n})}+\frac{[n+2]_{q_n}}{([n+2]_{q_n}-[2]_{q_n})}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\&\leq\frac{q_n}{([n+2]_{q_n}-[2]_{q_n})}+\frac{[n+2]_{q_n}}{([n+2]_{q_n}-[2]_{q_n})},\end{aligned}$$

as $n \to \infty$, we get

$$||B_n^{q_n}(t,x)-f(x)||_{x^2}\to 0. \tag{4.3}$$

Similarly for $n > 1$, we have

$$\begin{aligned}\left\|B_n^{q_n}(t^2,x)-x^2\right\|_{x^2} &= \sup_{x\in[0,\infty)}\frac{|B_n^{q_n}(t^2,x)-x^2|}{1+x^2}\\ &\le \frac{[n+2]_{q_n}^2+[n+2]_{q_n}/q_n}{([n+2]_{q_n}-[2]_{q_n})([n+2]_{q_n}-[3]_{q_n})}\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\ &+\frac{(1+q_n+q_n[2]_{q_n}}{([n+2]_{q_n}-[2]_{q_n})([n+2]_{q_n}-[3]_{q_n})}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{q_n^2[2]_{q_n}}{([n+2]_{q_n}-[2]_{q_n})([n+2]_{q_n})}\\ &\le \frac{[n+2]_{q_n}^2+[n+2]_{q_n}/q_n}{([n+2]_{q_n}-[2]_{q_n})([n+2]_{q_n}-[3]_{q_n})}+\frac{(1+q_n+q_n[2]_{q_n}}{([n+2]_{q_n}-[2]_{q_n})([n+2]_{q_n}-[3]_{q_n})}\\ &+\frac{q_n^2[2]_{q_n}}{([n+2]_{q_n}-[2]_{q_n})([n+2]_{q_n}-[3]_{q_n})},\end{aligned}$$

as $n \to \infty$, we get

$$||B_n^{q_n}(t^2,x)-x^2||_{x^2} \to 0. \tag{4.4}$$

By (4.2), (4.3), (4.4), and Korovkin's theorem, we get the desired result. □

Theorem 6. *Let $q = q_n \in (0, 1)$ such that $q_n \to 1$ as $n \to \infty$, then for each $f \in C_{x^2}[0,\infty)$ and $\alpha > 0$, we have*

$$lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|B_n^{q_n}(f,x)-f(x)|}{(1+x^2)^{\alpha+1}}=0. \tag{4.5}$$

Proof. For any fixed $x_0 > 0$, we have

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|B_n^{q_n}(f,x)-f(x)|}{(1+x^2)^{\alpha+1}} &\le \sup_{x\le x_0}\frac{|B_n^{q_n}(f,x)-f(x)|}{(1+x^2)^{\alpha+1}}+\sup_{x\ge x_0}\frac{|B_n^{q_n}(f,x)-f(x)|}{(1+x^2)^{\alpha+1}}\\ &\le \quad ||B_n^{q_n}(f,x)-f(x)||_{C[0,x_0]}+||f||\sup_{x\ge x_0}\frac{|B_n^{q_n}(1+t^2,x)|}{(1+x^2)^{\alpha+1}}\\ &+ \quad \sup_{x\ge x_0}\frac{|f(x)|}{(1+x^2)^{\alpha+1}}.\end{aligned}$$

By Theorem 5 and Corollary 2 first two terms of above inequality tends to 0 as $n \to \infty$. Last term of inequality can be made small enough for large $x_0 > 0$. This completes the proof. □

5. Central moments and asymptotic formula

In this section, we observe that it is not possible to estimate recurrence formula $B_n^q((t-x)^m,x)$ in q calculus, there may be some techniques, but at the moment it can be considered as an open problem. Here we establish the recurrence relation for the central moments and obtain asymptotic formula.

Lemma 3. *If we de ne the central moments as*

$$T_{n,m}(x)=B_n^q((t-x)_q^m,x)=\frac{1}{[n+1]_q}\sum_{k=1}^{\infty}b_{n,k}^q(x)\int_0^{\infty/A}b_{n,k}^q(t)(t-x)_q^m d_qt, x\in[0,\infty),$$

then

$$\begin{aligned}T_{n,0}(x)&=1,\\ T_{n,1}(x)&=\frac{(q+x[2]_q)}{([n+2]_q-[2]_q)},\\ T_{n,2}(x)&=\frac{(q[2]_q+x[n+2]_q)(q+x[n+2]_q)}{([n+2]_q-[2]_q)([n+2]_q-[3]_q)}+\frac{x(q+x)[n+2]_q}{q([n+2]_q-[2]_q)([n+2]_q-[3]_q)}\\ &-\frac{(qx+x)(q+x[n+2]_q)}{([n+2]_q-[2]_q)}+qx^2.\end{aligned}$$

and for n >m, we have the following recurrence relation:

$$\begin{aligned}\left([n+2]_q-[m+2]_q\right)T_{n,m+1}(qx) &= qx(1+x)\left[D_qT_{n,m}(x)+[m]_qT_{n,m-1}(qx)\right]\\ &+\left([3]_qq^mx+q-x\right)[m+1]_qT_{n,m}(qx)\\ &+\left[[2]_qq^mx\left([3]_qq^mx+q-x\right)-[3]_qq^{2m+1}x^2-qx\right][m]_qT_{n,m-1}(qx)\\ &+\left[q^mx\left\{[2]_qq^mx\left([3]_qq^mx+q-x\right)-[3]_qq^{2m+1}x^2-qx\right\}+q^{2m+1}x^2\left\{q^2x-[3]_qq^mx-q+x\right\}\right][m-1]_qT_{n,m-2}(qx)\\ &+x(1-q^{m+1})[n+2]_qT_{n,m}(qx)+qx(1-q^{m-1})[n+2]_qT_{n,m}(qx)-qx^2(1-q^{m-1})(1-q^m)[n+2]_qT_{n,m-1}(qx).\end{aligned}$$

proof. Using the identity

$$qx(1+x)D_q[b^q_{n,k}(x)]=\left(\frac{[k-1]_q}{q^{k-2}[n+2]_q}-qx\right)[n+2]_qb^q_{n,k}(qx)$$

and q derivatives of product rule, we have

$$\begin{aligned}qx(1+x)D_q[T_{n,m}] &= \frac{1}{[n+1]_q}\sum_{k=1}^{\infty}qx(1+x)D_q(b^q_{n,k}(x))\int_0^{\infty/A}b^q_{n,k}(t)(t-x)^m_q\,d_qt\\ &-\frac{[m]_q}{[n+1]_q}\sum_{k=1}^{\infty}qx(1+x)b^q_{n,k}(qx)\int_0^{\infty/A}b^q_{n,k}(t)(t-qx)^{m-1}_q\,d_qt\end{aligned}$$

Thus

$$\begin{aligned}E &:= qx(1+x)\left[D_q(T_{n,m}(x))+[m]_qT_{n,m-1}(qx)\right]\\ &= \frac{1}{[n+1]_q}\sum_{k=1}^{\infty}qx(1+x)D_q(b^q_{n,k}(x))\int_0^{\infty/A}b^q_{n,k}(t)(t-x)^m_q\,d_qt\\ &= \frac{[n+2]_q}{[n+1]_q}\sum_{k=1}^{\infty}b^q_{n,k}(qx)\left(\frac{[k-1]_q}{q^{k-2}[n+2]_q}-qx\right)\int_0^{\infty/A}b^q_{n,k}(t)(t-x)^m_q\,d_qt\\ &= \frac{[n+2]_q}{[n+1]_q}\sum_{k=1}^{\infty}b^q_{n,k}(qx)\int_0^{\infty/A}\left(\frac{[k-1]_q}{q^{k-2}[n+2]_q}-t+t-q^mx-qx+q^mx\right)b^q_{n,k}(t)(t-x)^m_q\,d_qt\\ &= \frac{1}{[n+1]_q}\sum_{k=1}^{\infty}b^q_{n,k}(qx)\int_0^{\infty/A}q^2\left[\frac{t}{q}\left(1+\frac{t}{q}\right)\right]D_q\left[b^q_{n,k}\left(\frac{t}{q}\right)\right](t-x)^m_q\,d_qt\\ &+\frac{[n+2]_q}{[n+1]_q}\sum_{k=1}^{\infty}b^q_{n,k}(qx)\int_0^{\infty/A}b^q_{n,k}(t)(t-x)^{m+1}_q\,d_qt\\ &+\frac{[n+2]_q}{[n+1]_q}qx(q^{m-1}-1)\sum_{k=1}^{\infty}b^q_{n,k}(qx)\int_0^{\infty/A}b^q_{n,k}(t)(t-x)^m_q\,d_qt\\ &= \frac{1}{[n+1]_q}\sum_{k=1}^{\infty}b^q_{n,k}(qx)\int_0^{\infty/A}(tq+t^2)D_q\left[b^q_{n,k}\left(\frac{t}{q}\right)\right](t-x)^m_q\,d_qt\\ &+\frac{[n+2]_q}{[n+1]_q}\sum_{k=1}^{\infty}b^q_{n,k}(qx)\int_0^{\infty/A}b^q_{n,k}(t)(t-x)^{m+1}_q\,d_qt\\ &+\frac{[n+2]_q}{[n+1]_q}qx(q^{m-1}-1)\sum_{k=1}^{\infty}b^q_{n,k}(qx)\int_0^{\infty/A}b^q_{n,k}(t)(t-x)^m_q\,d_qt.\end{aligned}$$

Using the identities

$$(t-q^m x)(t-q^{m+1}x)=t^2-[2]_q q^m xt+q^{2m+1}x^2$$

and

$$(t-q^m x)(t-q^{m+1}x)(t-q^{m+2}x)=t^3-[3]_q q^m xt^2+[3]_q q^{2m+1}x^2 t-q^{3m+3}x^3$$

we obtain the following identity after simple computation

$$\begin{aligned}(qt+t^2)(t-x)_q^m &= (qt+t^2)(t-x)(t-qx)_q^{m-1}=\left[t^3+(q-x)t^2-qxt\right](t-qx)_q^{m-1}\\ &= (t-qx)_q^{m+2}+\left([3]_q q^m x+q-x\right)(t-qx)_q^{m+1}\\ &+\left[[2]_q q^m x\left\{[3]_q q^m x+q-x\right\}-[3]_q q^{2m+1}x^2-qx)\right](t-qx)_q^m\\ &+\left[q^m x\left\{[2]_q q^m x\left([3]_q q^m x+q-x\right)-[3]_q q^{2m+1}x^2-qx\right\}+q^{2m+1}x^2\left\{q^2x-[3]_q q^m x-q+x\right\}\right](t-qx)_q^{m-1}.\end{aligned}$$

Using the above identity and q integral by parts

$$\int_a^b u(t)D_q(v(t))d_qt=[u(t)v(t)]_a^b-\int_a^b v(qt)D_q[u(t)]d_qt,$$

we have

$$\begin{aligned}E=&-[m+2]_qT_{n,m+1}(qx)-\left([3]_q q^m x+q-x\right)[m+1]_qT_{n,m}(qx)\\ &-\left[[2]_q q^m x\left([3]_q q^m x+q-x\right)-[3]_q q^{2m+1}x^2-qx\right][m]_qT_{n,m-1}(qx)\\ &-\left[q^m x\left\{[2]_q q^m x\left([3]_q q^m x+q-x\right)-[3]_q q^{2m+1}x^2-qx\right\}+q^{2m+1}x^2\left\{q^2x-[3]_q q^m x-q+x\right\}\right][m-1]_qT_{n,m-2}(qx)\\ &+\frac{[n+2]_q}{[n+1]_q}\sum_{k=1}^{\infty}b_{n,k}^q(qx)\int_0^{\infty/A}b_{n,k}^q(t)(t-x)_q^{m+1}d_qt\\ &-\frac{[n+2]_q}{[n+1]_q}qx(1-q^{m-1})\sum_{k=1}^{\infty}b_{n,k}^q(qx)\int_0^{\infty/A}b_{n,k}^q(t)(t-x)_q^{m}d_qt\end{aligned}$$

Finally, using

$$(t-x)_q^{m+1}=(t-x)(t-qx)_q^m=(t-qx)_q^{m+1}-x(1-q^{m+1})(t-qx)_q^m$$

and

$$(t-x)_q^m=(t-x)(t-qx)_q^{m-1}=(t-qx)_q^m-x(1-q^m)(t-qx)_q^{m-1},$$

we get

$$\begin{aligned}E=&-[m+2]_qT_{n,m+1}(qx)-([3]_q q^m x+q-x)[m+1]_qT_{n,m}(qx)\\ &-\left[[2]_q q^m x\left([3]_q q^m x+q-x\right)-[3]_q q^{2m+1}x^2-qx\right][m]_qT_{n,m-1}(qx)\\ &-\left[q^m x\left\{[2]_q q^m x\left([3]_q q^m x+q-x\right)-[3]_q q^{2m+1}x^2+qx\right\}+q^{2m+1}x^2\left\{q^2x-[3]_q q^m x-q+x\right\}\right][m-1]_qT_{n,m-2}(qx)\\ &+[n+2]_qT_{n,m+1}(qx)-x(1-q^{m+1})[n+2]_qT_{n,m}(qx)\\ &-qx(1-q^{m-1})[n+2]_qT_{n,m}(qx)+qx^2(1-q^{m-1})(1-q^m)[n+2]_qT_{n,m-1}(qx).\end{aligned}$$

This completes the proof of recurrence relation.

□

Theorem 7. *Let $f \in C[0,\infty)$ be a bounded function and (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then we have for a point $x \in (0,\infty)$*

$$\lim_{n\to\infty}[n]_{q_n}(B_n^{q_n}(f,x)-f(x))=(2x+1)\lim_{n\to\infty}D_{q_n}f(x)+x(1+x)\lim_{n\to\infty}D_{q_n}^2f(x).$$

Proof. By q-Taylor's formula [7] on f, we have

$$f(t) = f(x) + D_q f(x)(t-x) + \frac{1}{[2]_q} D_q^2 f(x)(t-x)_q^2 + \Phi_q(x;t)(t-x)_q^2$$

for $0 < q < 1$, where

$$\Phi_q(x;t) = \begin{cases} \frac{f(t)-f(x)-D_q f(x)(t-x)-\frac{1}{[2]_q}D_q^2 f(x)(t-x)_q^2}{(t-x)_q^2}, & \text{if } x \neq t \\ 0, & \text{if } x = t. \end{cases} \tag{5.1}$$

We know that for n large enough

$$\lim_{t\to x} \Phi_{q_n}(x;t) = 0. \tag{5.2}$$

That is for any $\varepsilon > 0$, $A > 0$, there exists a $\delta > 0$ such that

$$|\Phi_{q_n}(x;t)| < \varepsilon \tag{5.3}$$

for $|t - x| < \delta$ and n sufficiently large. Using (5.1), we can write

$$B_n^{q_n}(f,x) - f(x) = D_{q_n} f(x) T_{n,1}(x) + \frac{D_{q_n}^2 f(x)}{[2]_{q_n}} T_{n,2}(x) + E_n^{q_n}(x),$$

where

$$E_n^q(x) = \frac{1}{[n+1]_q} \sum_{k=1}^{\infty} b_{n,k}^q(x) \int_0^{\infty/A} b_{n,k}^q(t) \Phi_q(x;t)(t-x)_q^2 d_q t.$$

We can easily see that

$$\lim_{n\to\infty} [n]_{q_n} T_{n,1}(x) = 2x + 1 \quad \text{and} \quad \lim_{n\to\infty} [n]_{q_n} T_{n,2}(x) = 2x(1+x).$$

In order to complete the proof of the theorem, it is suffficient to show that $lim_{n\to\infty}[n]_{q_n} E_n^{q_n}(x) = 0$. We proceed as follows:

Let

$$R_{n,1}^{q_n}(x) = [n]_{q_n} \frac{1}{[n+1]_{q_n}} \sum_{k=1}^{\infty} b_{n,k}^{q_n}(x) \int_0^{\infty/A} b_{n,k}^{q_n}(t) \Phi_{q_n}(x;t)(t-x)_{q_n}^2 \chi_x(t) d_q t$$

and

$$R_{n,2}^{q_n}(x) = [n]_{q_n} \frac{1}{[n+1]_{q_n}} \sum_{k=1}^{\infty} b_{n,k}^{q_n}(x) \int_0^{\infty/A} b_{n,k}^{q_n}(t) \Phi_{q_n}(x;t)(t-x)_{q_n}^2 (1-\chi_x(t)) d_q t,$$

so that

$$[n]_{q_n} E_n^{q_n}(x) = R_{n,1}^{q_n}(x) + R_{n,2}^{q_n}(x),$$

where $\chi_x(t)$ is the characteristic function of the interval $\{t : |t - x| < \delta\}$.

It follows from (5.1)

$$|R_{n,1}^{q_n}(x)| < \varepsilon 2x(x+1) \quad \text{as } n \to \infty.$$

if $|t-x| \geq \delta$, then $|\Phi_{q_n}(x;t)| \leq \frac{M}{\delta^2}(t-x)^2$, where $M > 0$ is a constant. Since

$$\begin{aligned}(t-x)^2 &= (t-q^2x+q^2x-x)(t-q^3x+q^3x-x)\\ &= (t-q^2x)(t-q^3x)+x(q^3-1)(t-q^2x)+x(q^2-1)(t-q^2x)\\ &\quad + x^2(q^2-1)(q^2-q^3)+x^2(q^2-1)(q^3-1),\end{aligned}$$

we have

$$\begin{aligned}|R^{q_n}_{n,2}(x)| &\leq \frac{M}{\delta^2}[n]_{q_n}\frac{1}{[n+1]_{q_n}}\sum_{k=1}^{\infty} b^{q_n}_{n,k}(x)\int_0^{\infty/A} b^{q_n}_{n,k}(t)(t-x)^4_{q_n}d_qt\\ &+\frac{M}{\delta^2}x(|(q_n^3-1)+(q_n^2-1)|)[n]_{q_n}\frac{1}{[n+1]_{q_n}}\sum_{k=1}^{\infty} b^{q_n}_{n,k}(x)\int_0^{\infty/A} q_n^k b^{q_n}_{n,k}(t)(t-x)^3_{q_n}d_qt\\ &+\frac{M}{\delta^2}x^2(q_n^2-1)^2[n]_{q_n}\frac{1}{[n+1]_{q_n}}\sum_{k=1}^{\infty} b^{q_n}_{n,k}(x)\int_0^{\infty/A} q_n^k b^{q_n}_{n,k}(t)(t-x)^2_{q_n}d_qt\end{aligned}$$

and

$$|R^{q_n}_{n,2}(x)| \leq \frac{M}{\delta^2}\left\{[n]_{q_n}T_{n,4}(x)+x(2-q_n^2-q_n^3)[n]_{q_n}T_{n,3}(x)+x^2(q_n^2-1)^2[n]_{q_n}T_{n,2}(x)\right\}.$$

Using Lemma 3, we have

$$T_{n,4}(x) \leq \frac{C_{1,x}}{[n]^2_{q_n}},\quad T_{n,3}(x) \leq \frac{C_{2,x}}{[n]^2_{q_n}} \text{ and } T_{n,4}(x) \leq \frac{C_{3,x}}{[n]_{q_n}}.$$

Thus, for n suciently large $R^{q_n}_{n,2}(x) \to 0$. This completes the proof of theorem. □

Corollary 8. *Let $f \in C[0,\infty)$ be a bounded function and (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Suppose that the first and second derivative $f'(x)$ and $f''(x)$ exist at a point $x \in (0,\infty)$, we have*

$$\lim_{n\to\infty}[n]_{q_n}(B^{q_n}_n(f,x)-f(x)) = f'(x)(2x+1)+x(1+x)f''(x).$$

6. Better error approximation

King [13] in 2003 proposed a new approach to modify the Bernstein polynomials to improve rate of convergence, by making operator to preserve test functions e_0 and e_1. As the q-Beta operators $B^q_n(f,x)$ reproduce only constant functions, this motivated us to propose the modification of (1.2), so that they reproduce constant as well as linear functions.

Define sequence $\{u_{n,q}(x)\}$ of real valued continues functions on $[0,\infty)$ with $0 \leq u_{n,q}(x) < \infty$, as

$$u_{n,q}(x) = \frac{x([n+2]_q-[2]_q)-q}{[n+2]_q}.$$

We replace x in definition of operator (1.2) with $u_{n,q}(x)$. Therefore, modified operator is given as

$$\widetilde{B}_n^q(f;x) = \frac{1}{[n+1]_q}\sum_{k=1}^{\infty} b_{n,k}^q(u_{n,q}(x)) \int_0^{\infty/A} b_{n,k}^q(t)f(t)d_qt, x \in I \equiv [1/q[n]_q, \infty), \tag{6.1}$$

Remark 3. By simple computation we can write

$$\widetilde{B}_n^q(1;x) = 1, \widetilde{B}_n^q(t;x) = x$$

$$\widetilde{B}_n^q(t^2;x) = x^2\left(\frac{[n+3]_q([n+2]_q-[2]_q)}{q[n+2]_q([n+2]_q-[3]_q)}\right) + x\left(\frac{[n+2]_q-2+q}{[n+2]_q([n+2]_q-[3]_q)}\right) - \frac{q([n+2]_q-1)}{[n+2]_q([n+2]_q-[2]_q)([n+2]_q-[3]_q)}.$$

Therefore,

$$\widetilde{B}_n^q(t-x)^2;x) = x^2\left(\frac{[n+3]_q([n+2]_q-[2]_q)}{q[n+2]_q([n+2]_q-[3]_q)} - 1\right) + x\left(\frac{[n+2]_q-2+q}{[n+2]_q([n+2]_q-[3]_q)}\right) \frac{q([n+2]_q-1)}{[n+2]_q([n+2]_q-[2]_q)([n+2]_q-[3]_q)}.$$

Theorem 9. *Let $f \in C_B(I)$, then for every $x \in I$ and for $n > 1$, $C > 0$, we have*

$$|\widetilde{B}_n^q(f;x) - f(x)| \leq C\omega_2(f,\delta_n), \tag{6.2}$$

where $\delta_n = \sqrt{\widetilde{B}_n^q((t-x)^2;x)}$.

Proof. Let $g \in W_\infty^2$, by Taylor's series

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t g''(s)(t-s)ds$$

therefore, by linearity and Remark 3, we get

$$\begin{aligned}|\widetilde{B}_n^q(g;x) - g(x)| &\leq \widetilde{B}_n^q(t-x,x)\,\|g'\| + \frac{\|g''\|}{2}\widetilde{B}_n^q((t-x)^2;x)\\ &\leq \quad \widetilde{B}_n^q((t-x)^2;x)\,\|g''\|.\end{aligned}$$

Also

$$|\widetilde{B}_n^q(f,x)| \;\leq\; ||f|||\widetilde{B}_n^q(1,x)| \;=\; \|f\|.$$

Therefore,

$$\begin{aligned}|\widetilde{B}_n^q(f;x) - f(x)| &\leq |\widetilde{B}_n^q(f-g;x) - (f-g)(x)| + |\widetilde{B}_n^q(g;x) - g(x)|\\ &\leq 2\,\|f-g\| + \overline{B_n^q}((t-x)^2;x)\,\|g''\|\end{aligned}$$

on choosing $\delta_n = \sqrt{\widetilde{B}_n^q((t-x)^2;x)}$, taking infimum over $g \in W_\infty^2$, we get the desired result.

□

Theorem 10. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C_{x^2}^*(I)$ we have*

$$\lim_{n\to\infty}\left\|\widetilde{B}_n^{q_n}(f)-f\right\|_{x^2}=0.$$

Proof. To prove theorem (Using the Theorem in [14]), it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|\widetilde{B}_n^{q_n}-(t^\nu,x)-x^\nu\right\|_{x^2}=0,\quad \nu=0,1,2. \tag{6.3}$$

By Remark 3, we have $\widetilde{B}_n^{q_n}(1,x)=1$ and $\widetilde{B}_n^{q_n}(t,x)=x$, the first and second condition of (6.3) is fulfilled for $\nu=0$ and $\nu=1$.

And

$$\begin{aligned}\left\|\widetilde{B}_n^{q_n}(t^2,x)-x^2\right\|_{x^2} \le & \left|\frac{[n+3]_{q_n}([n+2]_{q_n}-[2]_{q_n})}{q_n[n+2]_{q_n}([n+2]_{q_n}-[3]_{q_n})}-1\right|\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\ &+\left(\frac{[n+2]_{q_n}-2+q_n}{[n+2]_{q_n}([n+2]_{q_n}-[3]_{q_n})}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\ &+\frac{\left|q_n([n+2]_{q_n}-1)\right|}{[n+2]_{q_n}([n+2]_{q_n}-[2]_{q_n})([n+2]_{q_n}-[3]_{q_n})},\end{aligned}$$

which implies that as $n\to\infty$

$$\lim_{n\to\infty}\left\|\widetilde{B}_n^{q_n}(t^2,x)-x^2\right\|_{x^2}=0.$$

Thus, the proof is completed. □

7. Stancu approach

In 1968, Stancu introduced Bernstein-Stancu operators in [15] a linear positive operator depending on two non-negative parameters α and β satisfying the condition $0 \le \alpha \le \beta$. Recently, many researcher applied this approach to many operators, for detail see [16,17,2], etc.

For $f\in C[0,\infty)$ and $0<q<1$, we define the q-Beta Stancu operators as

$$\widehat{B}_n^q(f,x)=\frac{1}{[n+1]_q}\sum_{k=1}^{\infty}b_{n,k}^q(x)\int_0^{\infty/A}b_{n,k}^q(t)f\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)d_qt,\ x\in[0,\infty), \tag{7.1}$$

where $0\le\alpha\le\beta$.

In each of following theorems, we assume that $q=q_n$, where q_n is a sequence of real numbers such that $0<q_n<1$ for all n and $\lim_{n\to\infty}q_n=1$.

Theorem 11. *For* $\widehat{B}_n^q(t^s,x)$, $s=0,1,2$ *the following identities hold:*

$$\begin{aligned}&\widehat{B}_n^q(1,x)=1,\\ &\widehat{B}_n^q(t,x)=\frac{[n]_q}{[n]_q+\beta}\left(\frac{(q+x[n+2]_q)}{([n+2]_q-[2]_q)}\right)+\frac{\alpha}{[n]_q+\beta},\\ &\widehat{B}_n^q(t^2,x)=\frac{[n]_q^2}{([n]_q+\beta)^2}\left(\frac{(q[2]_q+x[n+2]_q)(q+x[n+2]_q)}{([n+2]_q-[2]_q)([n+2]_q-[3]_q)}+\frac{x(q+x)[n+2]_q}{q([n+2]_q-[2]_q)([n+2]_q-[3]_q)}\right)\\ &\qquad+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}\left(\frac{(q+x[n+2]_q)}{([n+2]_q-[2]_q)}\right)+\frac{\alpha^2}{([n]_q+\beta)^2}.\end{aligned}$$

Proof of the theorem can be obtained directly by using linearity of operator and Corollary 2.

Corollary 12. *If we denote central moments by* $\Phi_{n,m}^{q}(x) = \widehat{B_n^q}((t-x)^m, x)$, $m = 1, 2$, *then we have*

$$\Phi_{n,1}^{q}(x) = \frac{[n]_q}{[n]_q+\beta}\left(\frac{(q+x[n+2]_q)}{([n+2]_q-[2]_q)}\right) + \frac{\alpha}{[n]_q+\beta} - x,$$

$$\begin{aligned}\Phi_{n,2}^{q}(x) = x^2 &+ \frac{[n]_q^2}{([n]_q+\beta)^2}\left(\frac{(q[2]_q+x[n+2]_q)(q+x[n+2]_q)}{([n+2]_q-[2]_q)([n+2]_q-[3]_q)} + \frac{x(q+x)[n+2]_q}{q([n+2]_q-[2]_q)([n+2]_q-[3]_q)}\right)\\ &+ \frac{2\alpha[n]_q}{([n]_q+\beta)^2}\left(\frac{(q+x[n+2]_q)}{([n+2]_q-[2]_q)}\right) + \frac{\alpha^2}{([n]_q+\beta)^2} - 2x\left(\frac{[n]_q}{[n]_q+\beta}\left(\frac{(q+x[n+2]_q)}{([n+2]_q-[2]_q)}\right) + \frac{\alpha}{[n]_q+\beta}\right).\end{aligned}$$

Theorem 13. *Let* $0 < q < 1$, *we have*

$$\left|\widehat{B_n^q}(f,x) - f(x)\right| \le C\omega_2\left(f, \sqrt{\Phi_{n2}^{q}(x) + \Phi_{n1}^{q2}(x)}\right) + \omega(f, \Phi_{n,1}^{q}(x))$$

for every $\alpha, \beta \ge 0$, $x \in [0, \infty)$ *and* $f \in C_B[0, \infty)$, *where C is a positive constant.*

Proof of theorem is just similar to Theorem 4.

Acknowledgements
The authors would like to express their deep gratitude to the referees for their valuable suggestions and comments.

Author details
[1]School of Applied Science, Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi-110078, India
[2]Department of Mathematics, Dr. B. R. Ambedkar National Institute of Technology, Jalandhar-144011 (Punjab), India
[3]Department of Mathematics, Kwangwoon University, Seoul 139-701, South Korea [4]Division of General Education, Kwangwoon University, Seoul 139-701, South Korea

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 20 April 2012 Accepted: 25 June 2012 Published: 25 June 2012

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doi:10.1186/1687-1847-2012-86
Cite this article as: Gupta *et al.*: Properties of *q*-analogue of Beta operator. *Advances in Difference Equations* 2012 **2012**:86.