Grace *et al. Advances in Difference Equations* 2012, **2012**:67
http://www.advancesindifferenceequations.com/content/2012/1/67

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Oscillation of higher order nonlinear dynamic equations on time scales

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Abstract

Some new criteria for the oscillation of nth order nonlinear dynamic equations of the form

$$x^{\Delta^n}(t) + q(t)\left(x^{\sigma}(\xi(t))\right)^{\lambda} = 0$$

are established in delay $\xi(t) \le t$ and non-delay $\xi(t) = t$ cases, where $n \ge 2$ is a positive integer, λ is the ratio of positive odd integers. Many of the results are new for the corresponding higher order difference equations and differential equations are as special cases.

Mathematics Subject Classification (2011): 34C10; 34C15.

Keywords: oscillation, neutral, time scale, higher order

1. Introduction

Consider the nth order nonlinear delay dynamic equation

$$x^{\Delta^n}(t) + q(t)\left(x^{\sigma}(\xi(t))\right)^{\lambda} = 0 \tag{1.1}$$

on an arbitrary time-scale $\mathbb{T} \subseteq \mathbb{R}$ with $\sup \mathbb{T} = \infty$ and $0 \in \mathbb{T}$, where $n \ge 2$ is a positive integer, λ is the ratio of positive odd integers, $q : \mathbb{T} \to \mathbb{R}^+ = (0, \infty)$ and $\xi : \mathbb{T} \to \mathbb{T}$ are real-valued rd-continuous functions, $\xi(t) \le t$, $\xi^{\Delta}(t) \ge 0$, and $\lim_{t\to\infty}\xi(t) = \infty$. Throughout the article by $t \ge s$ for t, $s \in \mathbb{T}$ we shall mean $t \in [s, \infty) \cap \mathbb{T} := [s, \infty)_{\mathbb{T}}$. For the forward jump operator σ, we use the usual notation $x^{\sigma} = x \circ \sigma$.

We recall that a solution x of Equation (1.1) is said to be nonoscillatory if there exists a $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for all $t \ge t_0$; otherwise, it is said to be oscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Recently, there has been an increasing interest in studying the oscillatory behavior of first-and second-order dynamic equations on time-scales, see [1-7]. However, there are very few results regarding the oscillation of higher order equations. Therefore, the purpose of this article is to obtain new criteria for the oscillation of Equation (1.1). This topic is fairly new for dynamic equations on time scales. For a general background on time scale calculus, we may refer to [8,9].

The article is organized as follows: In Section 2, some preliminary lemmas and notations are given, while Section 3 is devoted to the study of Equation (1.1) via comparison with a set of second-order dynamic equations whose oscillatory character is known and have been investigated extensively in the literature. In Section 4, we establish new oscillation criteria for Equation (1.1) when $\xi(t) = t$ for linear, sublinear, and superlinear cases. Further results are presented in Section 5 when there is a special restriction on the function q. We should note that many of our results of this article are new for the corresponding higher order nonlinear differential and difference equations. In fact, the obtained results extend, unify and correlate many of the existing results in the literature.

2. Preliminaries

We shall employ the following lemmas. The first lemma is the well-known Kiguradze's lemma.

Lemma 2.1. *Let* $x \in C_{rd}^{m}([t_0, \infty), \mathbb{R}^+)$. If $x^{\Delta^m}(t)$ *is of constant sign on* $[t_0, \infty)_{\mathbb{T}}$ *and not identically zero on* $[t_1, \infty)_{\mathbb{T}}$ *for any* $t_1 \geq t_0$, *then there exist a* $t_x \geq t_0$ *and an integer* ℓ, $0 \leq \ell \leq m$ *with* $m + \ell$ *even for* $x^{\Delta^m}(t) \geq 0$, *or* $m + \ell$ *odd for* $x^{\Delta^m}(t) \leq 0$ *such that*

$$\ell > 0 \text{ implies } x^{\Delta^k}(t) > 0 \quad \text{for } t \geq t_x, \quad k \in \{1, 2, \ldots, \ell - 1\} \tag{2.1}$$

and

$$\ell \leq m - 1 \text{ implies } (-1)^{\ell+k} x^{\Delta^k}(t) > 0 \text{ for } t \geq t_x,\ k \in \{\ell, \ell + 1, \ldots, m - 1\}. \tag{2.2}$$

Lemma 2.2. *If the inequality*

$$x^{\Delta\Delta} + Q(t)x^{\lambda} \leq 0, \tag{2.3}$$

where Q *is a positive real-valued, rd-continuous function on* $\mathbb{T}$, *has an eventually positive solution, then the equation*

$$x^{\Delta\Delta} + Q(t)x^{\lambda} = 0 \tag{2.4}$$

also has an eventually positive solution.

Proof. Let $x(t)$ be an eventually positive solution of inequality (2.3). It is easy to see that $x^{\Delta}(t) > 0$ eventually. Let t_0 be sufficiently large so that $x(t) > 0$ and $y(t) =: x^{\Delta}(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Then in view of

$$x(t) = x(t_0) + \int_{t_0}^{t} y(s)\,\Delta s,$$

(2.3) becomes

$$y^{\Delta}(t) + Q(t)\left(x(t_0) + \int_{t_0}^{t} y(s)\,\Delta s\right)^{\lambda} \leq 0, t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.5}$$

Integrating (2.5) from t to $u \geq t \geq t_0$ and letting $u \to \infty$, we have

$$y(t) \geq F\big(t, y(t)\big), \quad t \in [t_0, \infty)_{\mathbb{T}},$$

where

$$F\left(t,y\right) := \int_{t}^{\infty} Q\left(v\right)\left(x\left(t_0\right) + \int_{t_0}^{v} y\left(s\right)\Delta s\right)^{\lambda}\Delta v.$$

Next, we define a sequence of successive approximations $\{z_j\left(t\right)\}$ as follows:

$$z_0\left(t\right) = y\left(t\right)$$

$$z_{j+1}\left(t\right) = F\left(t, z_j\left(t\right)\right), \quad j = 0, 1, 2, \ldots.$$

It is easy to show that

$$0 < z_j\left(t\right) \le y\left(t\right) \text{ and } z_{j+1}\left(t\right) \le z_j\left(t\right), \quad j = 0, 1, 2, \ldots.$$

Thus the sequence $\{z_j(t)\}$ is nonincreasing and bounded for each $t \geq t_0$. This means we may define $z(t) = \lim_{j\to\infty} z_j(t) \geq 0$. Since $0 \leq z(t) \leq z_j(t) \leq y(t)$ for all $j \geq 0$, we find that

$$\int_{t_0}^{t} z_j\left(s\right)\Delta s \le \int_{t_0}^{t} y\left(s\right)\Delta s.$$

By the Lebesgue dominated convergence theorem on time scales, one can easily obtain

$$z\left(t\right) = F\left(t, z\left(t\right)\right).$$

Therefore,

$$z^{\Delta}\left(t\right) = -Q\left(t\right) m^{\lambda}\left(t\right), \tag{2.6}$$

where

$$m\left(t\right) = x\left(t_0\right) + \int_{t_0}^{t} z\left(s\right)\Delta s.$$

Then, $m(t) > 0$ and $m^{\Delta}(t) = z(t)$. Equation (2.6) then gives

$$m^{\Delta\Delta}\left(t\right) + Q\left(t\right) m^{\lambda}\left(t\right) = 0.$$

Hence, Equation (2.4) has a positive solution $m(t)$. This completes the proof. □

Lemma 2.3 ([4]). *Suppose* $|x|^{\Delta}$ *is of one sign on* $[t_0, \infty)_{\mathbb{T}}$ *and* $\alpha > 0$, $\alpha \neq 1$. *Then*

$$\frac{|x|^{\Delta}}{\left(|x^{\sigma}|\right)^{\alpha}} \le \frac{\left(|x|^{1-\alpha}\right)^{\Delta}}{(1-\alpha)} \le \frac{|x|^{\Delta}}{\left(|x|^{\alpha}\right)}, \quad t \ge t_0. \tag{2.7}$$

It will be convenient to employ the Taylor monomials (see [[8], Sect. 1.6]) $n \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, which are defined recursively as follows:

$$h_0\left(t,s\right) = g_0\left(t,s\right) = 1,$$

$$h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\,\Delta\tau, \quad g_{n+1}(t,s) = \int_s^t g_n(\sigma(\tau),s)\,\Delta\tau, \quad t,s \in \mathbb{T}, n \in \mathbb{N}_0.$$

It is clear that $h_1(t, s) = g_1(t, s) = t - s$ for any time-scales, but simple formulas in general do not hold for $n \geq 2$. It is also known that

$$h_n(t,s) = (-1)^n g_n(s,t).$$

3. Comparison criteria for delay dynamic equations

In this section, we shall consider the equation

$$x^{\Delta^n}(t) + q(t)\,x^{\lambda}(\xi(t)) = 0. \tag{3.1}$$

For $t_0 \in \mathbb{T}$ and $\ell \in \{1, 2, ..., n - 1\}$, we define

$$q_\ell(t,t_0) = \int_t^\infty \tau^{-\lambda} Q_\ell(\tau,t,t_0)\Delta\tau, \quad t \in [t_0,\infty)_{\mathbb{T}},$$

where

$$Q_\ell(\tau,t,t_0) = g_{n-\ell-2}(\sigma(\tau),t)\,R_\ell^\lambda(\tau,t_0)\,q(\tau), \quad \tau \geq t.$$

with

$$R_\ell(\tau,t_0) = \begin{cases} \int_{t_0}^{\xi(\tau)} s h_{\ell-2}(\xi(\tau),\sigma(s)\,\Delta s, & \ell \geq 2 \\ \xi(\tau), & \ell = 1. \end{cases}$$

Theorem 3.1. Let $t_0 \in \mathbb{T}$. *Suppose that for every* $\ell \in \{1, 2, ..., n - 1\}$,

$$\int^\infty Q_\ell(\tau,t_0,t_0)\,\Delta\tau = \infty. \tag{3.2}$$

Then, Equation (3.1) *is oscillatory if*

(i) for n even, the equation

$$y^{\Delta\Delta} + q_\ell(t,t_0)\,y^\lambda = 0, \tag{3.3}$$

for all $\ell \in \{1, 3, ..., n - 1\}$ *is oscillatory;*

(ii) for n odd, the Equation (3.3) *for all* $\ell \in \{2, 4, ..., n - 1\}$ *is oscillatory, and*

$$\limsup_{t\to\infty} \int_{\xi(t)}^t h_{n-1}^\lambda(\xi(s),\xi(t))\,q(s)\,\Delta s > \begin{cases} 0 \text{ when } 0 < \lambda < 1 \\ 1 \text{ when } \lambda = 1. \end{cases} \tag{3.4}$$

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (3.1). Without loss of generality, we may assume that $x(t) > 0$ and $x(\xi(t)) > 0$ for $t \geq t_0$, since otherwise the substitution $w = -x$ transforms Equation (3.1) into an equation of the same form subject to the assumptions of the theorem.

By Lemma 2.1, there exist a $t_1 \geq t_0$ and an integer $\ell \in \{0, 1, ..., n\}$ with $n + \ell$ odd such that (2.1) and (2.2) hold for all $t \geq t_1$. We see that

$$x^{\Delta^{\ell-1}}(t) > 0,\ x^{\Delta^{\ell}}(t) > 0,\ x^{\Delta^{\ell+1}}(t) < 0 \quad \text{for } t \geq t_1,$$

and by Taylor's formula

$$\begin{aligned} x(t) &= \sum_{k=0}^{l-2} x^{\Delta^k}(t_1)\, h_k(t, t_1) + \int_{t_1}^{t} h_{\ell-2}(t, \sigma(\tau)) x^{\Delta^{\ell-1}}(\tau)\,\Delta\tau \\ &\geq \int_{t_1}^{t} h_{\ell-2}(t, \sigma(\tau))\, x^{\Delta^{\ell-1}}(\tau)\,\Delta\tau \quad \text{for } \ell > 1. \end{aligned} \tag{3.5}$$

We claim that

$$\frac{x^{\Delta^{\ell-1}(t)}}{t} \text{ is strictly decreasing for } t \geq t_1 \text{ and } \ell > 0. \tag{3.6}$$

To prove it, set $X(t) = x^{\Delta^{\ell-1}}(t) - t x^{\Delta^{\ell}}(t)$. Because

$$\left(\frac{x^{\Delta^{\ell-1}}}{t}\right)^{\Delta} = \frac{t x^{\Delta^{\ell}} - x^{\Delta^{\ell-1}}}{t\sigma(t)} = -\frac{X(t)}{t\sigma(t)},$$

it suffices to show that $X(t)$ is strictly positive. Suppose on the contrary that $X(t) < 0$. Then $x^{\Delta^{\ell-1}}\big/t$ is strictly increasing and hence

$$x^{\Delta^{\ell-1}}(t) \geq ct \quad \text{for } t \geq t_1, \tag{3.7}$$

where $c = x^{\Delta^{\ell-1}}(t_1)\big/t_1 > 0$. Using (3.7) in (3.5), we have

$$x(\xi(t)) \geq c \int_{t_1}^{\xi(t)} \tau h_{\ell-2}(\xi(t), \sigma(\tau))\,\Delta\tau. \tag{3.8}$$

Let $\ell = 1$, then (3.7) gives $x(\xi(t)) \geq c\xi(t)$ for $t \geq t_1$ by increasing the size of t_1 if necessary. Thus, we obtain

$$x(\xi(t)) \geq cR_{\ell}(t, t_1) \quad \text{for } t \geq t_1 \text{ and } \ell > 0. \tag{3.9}$$

On the other hand, by Taylor's formula we may write that

$$\begin{aligned} x^{\Delta^{l+1}}(t) &= \sum_{k=0}^{n-\ell-2} x^{\Delta^{\ell+k+1}}(s)\, h_k(t, s) + \int_{t}^{s} h_{n-\ell-2}(t, \sigma(\tau)) \left(-x^{\Delta^n}(\tau)\right)\Delta\tau \\ &= \sum_{k=0}^{n-\ell-2} x^{\Delta^{\ell+k+1}}(s)\, (-1)^k g_k(s, t) + \int_{t}^{s} (-1)^{n-\ell-2} g_{n-\ell-2}(\sigma(\tau), t) \left(-x^{\Delta^n}(\tau)\right)\Delta\tau \\ &\leq -\int_{t}^{\infty} g_{n-\ell-2}(\sigma(\tau), t)\, q(\tau)\, x^{\lambda}(\xi(\tau))\,\Delta\tau. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$-x^{\Delta^{\ell+1}}(t_1) \geq c^{\lambda} \int_{t_1}^{\infty} g_{n-\ell-2}(\sigma(\tau), t_1) q(\tau) R_{\ell}^{\lambda}(\tau, t_1) \Delta\tau, \tag{3.11}$$

which contradicts (3.2), and hence completes the proof of the claim.

Now in view of (3.6) it follows from (3.5) that

$$x(t) \geq \frac{x^{\Delta^{\ell-1}}(t)}{t} \int_{t_1}^{t} \tau h_{\ell-2}(t, \sigma(\tau)) \Delta\tau, \quad t \geq t_1. \tag{3.12}$$

Replacing t by $\xi(t)$ in (3.12) and using (3.6), we have

$$x(\xi(t)) \geq x^{\Delta^{\ell-1}}(t) \int_{t_1}^{\xi(t)} \frac{\tau}{t} h_{\ell-2}(\xi(t), \sigma(\tau)) \Delta\tau, \quad \ell > 1 \tag{3.13}$$

for all $t \geq t_2$ for some $t_2 \geq t_1$.

If $\ell = 1$, then we may write that

$$x(\xi(t)) = \frac{x^{\Delta^{\ell-1}}(\xi(t))}{\xi(t)} \xi(t) \geq \frac{x^{\Delta^{\ell-1}}(t)}{t} \xi(t), \quad t \geq t_2. \tag{3.14}$$

Thus, from (3.13) and (3.14) for all $t \geq t_2$,

$$x(\xi(t)) \geq \frac{x^{\Delta^{\ell-1}}(t)}{t} R_{\ell}(t, t_1), \quad \ell > 0. \tag{3.15}$$

Substituting (3.15) into (3.10) gives

$$-x^{\Delta^{\ell+1}}(t) \geq \left(x^{\Delta^{\ell-1}}(t)\right)^{\lambda} \int_{t}^{\infty} \tau^{-\lambda} g_{n-\ell-2}(\sigma(\tau), t) R_{\ell}^{\lambda}(\tau, t_1) q(\tau) \Delta\tau, t \geq t_2. \tag{3.16}$$

Set $w(t) = x^{\Delta^{\ell-1}}(t)$ in (3.16), then $w(t) > 0$ satisfies

$$w^{\Delta\Delta} + q_{\ell}(t, t_1) w^{\lambda} \leq 0, \quad t \geq t_2.$$

By Lemma 2.2, the equation

$$w^{\Delta\Delta} + q_{\ell}(t, t_1) w^{\lambda} = 0$$

has a nonoscillatory solution. But this is impossible by the hypothesis.

Finally, we let $\ell = 0$. This is the case, when n is odd. By applying Taylor's formula and using (2.2) with $\ell = 0$, we can easily find

$$x(u) \geq h_{n-1}(u, v) x^{\Delta^{n-1}}(v) \tag{3.17}$$

for $v \geq u \geq t_1$, which implies that

$$x(\xi(s)) \geq h_{n-1}(\xi(s), \xi(t)) x^{\Delta^{n-1}}(\xi(t)), \quad t > s \geq t_3. \tag{3.18}$$

for some $t_3 \geq t_1$. Integrating equation (3.1) from $\xi(t) \geq t_3$ to $t \geq t$, we get

$$x^{\Delta^{n-1}}(\xi(t)) \geq \int_{\xi(t)}^{t} q(s)\, x^{\lambda}(\xi(s))\Delta s. \tag{3.19}$$

Using (3.18) in (3.19), we have

$$x^{\Delta^{n-1}}(\xi(t)) \geq \left(x^{\Delta^{n-1}}(\xi(t))\right)^{\lambda} \int_{\xi(t)}^{t} h_{n-1}^{\lambda}(\xi(s), \xi(t))\, q(s)\,\Delta s$$

or

$$\left(x^{\Delta^{n-1}}(\xi(t))\right)^{1-\lambda} \geq \int_{\xi(t)}^{t} h_{n-1}^{\lambda}(\xi(s), \xi(t))\, q(s)\,\Delta s$$

Taking the lim sup as $t \to \infty$, we obtain a contradiction to condition (3.4). □

The following immediate result can be extracted from Theorem 3.1.

Corollary 3.1. *Let n be an odd and condition* (3.4) *hold. Then every bounded solution of Equation* (3.1) *is oscillatory.*

Next, we claim that inequality (3.15) can be replaced by

$$x(\xi(t)) \geq \frac{1}{t} h_{\ell}(\xi(t), t_1)\, x^{\Delta^{\ell-1}}(t). \tag{3.20}$$

To prove this, we write that

$$x^{\Delta^{\ell-2}}(t) \geq \int_{t_1}^{t} x^{\Delta^{\ell-1}}(s)\,\Delta s = \int_{t_1}^{t} s\left(\frac{x^{\Delta^{\ell-1}}(s)}{s}\right)\Delta s$$

and hence by (3.6) we find

$$x^{\Delta^{\ell-2}}(t) \geq h_2(t, t_1)\left(\frac{x^{\Delta^{\ell-1}}(t)}{t}\right).$$

Integrating this inequality (ℓ - 2)-times from t_1 to $t \geq t_1$ and using (3.6), we obtain

$$x(t) \geq h_{\ell}(t, t_1)\left(\frac{x^{\Delta^{\ell-1}}(t)}{t}\right).$$

Thus, there exists a $t_2 \geq t_1$ such that

$$x(\xi(t)) \geq h_{\ell}(\xi(t), t_1)\,\frac{x^{\Delta^{\ell-1}}(\xi(t)}{\xi(t))} \geq \frac{1}{t} h_{\ell}(\xi(t), t_1)\, x^{\Delta^{\ell-1}}(t), \quad t \geq t_2.$$

This completes the proof of our claim.

Set

$$Q_{\ell}^{*}(\tau, t, t_0) = g_{n-\ell-2}(\sigma(\tau), t)\, h_{l}^{\lambda}(\xi(\tau), t_0)\, q(\tau), \quad \tau \geq t$$

and

$$q_\ell^*(t,t_0) = \int_t^\infty \tau^{-\lambda} Q_\ell^*(\tau,t,t_0)\,\Delta\tau, \quad t \ge t_0.$$

In view of Theorem 3.1 and inequality (3.20) we may state the following theorem.

Theorem 3.2. *In Theorem 3.1, let q_ℓ and Q_ℓ be replaced by q_ℓ^* and Q_ℓ^*, respectively. Then the conclusions of Theorem 3.1 hold.*

Let $\mathbb{T} = \mathbb{R}$, i.e., the continuous case. Here Equation (3.1) becomes

$$x^{(n)}(t) + q(t)x^\lambda(\xi(t)) = 0 \tag{3.21}$$

and the functions q_ℓ^* and Q_ℓ^* take the form

$$q_\ell^c(t,t_0) = \int_t^\infty \tau^{-\lambda} Q_\ell^c(\tau,t,t_0)\,d\tau$$

and

$$Q_\ell^c(\tau,t,t_0) = \frac{(\xi(\tau)-t_0)^{\lambda\ell}}{(\ell!)^\lambda}\frac{(\tau-t)^{n-\ell-2}}{(n-\ell-2)!}q(\tau).$$

From Theorem 3.2 we have the following theorem.

Theorem 3.3. *Let $t_0 \in \mathbb{T}$. Suppose that for $\ell \in \{1, 2, ..., n-1\}$,*

$$\int^\infty Q_\ell^c(\tau,t_0,t_0)\,d\tau = \infty. \tag{3.22}$$

Then, Equation (3.21) is oscillatory if

(i) for n even, the equation

$$y'' + q_\ell^c(t,t_0)y = 0, \tag{3.23}$$

for all $\ell \in \{1, 3, ..., n-1\}$ is oscillatory;

(ii) for n odd, the Equation (3.23) for all $\ell \in \{2, 4, ..., n-1\}$ is oscillatory and

$$\limsup_{t\to\infty} \int_{\xi(t)}^t \left(\frac{(\xi(s)-\xi(t))^{n-1}}{(n-1)!}\right)^\lambda q(s)\,\Delta s > \begin{cases} 0 \text{ when } 0 < \lambda < 1 \\ 1 \text{ when } \lambda = 1. \end{cases} \tag{3.24}$$

Next, we let $\mathbb{T} = \mathbb{Z}$, i.e., the discrete case. Then, Equation (3.1) reads as

$$\Delta^n x(m) + q(m)x^\lambda(\xi(m) = 0 \tag{3.25}$$

and the functions q_ℓ^* and Q_ℓ^* become

$$q_\ell^d(m,m_0) = \sum_{j=m}^\infty j^{-\lambda} Q_\ell^d(j,m,m_0)$$

and

$$Q_\ell^d(j, m, m_0) = \frac{\left[\left(\xi(j) - m_0\right)^{(\ell)}\right]^\lambda}{(\ell!)^\lambda} \frac{\left(j - m + n - \ell - 2\right)^{(n-\ell-2)}}{(n - \ell - 2)!} q(j),$$

where $t^{(m)} = t(t - 1)(t - 2) \ldots (t - m + 1)$ is the usual factorial function.

Theorem 3.4. *Let* $m_0 \in \mathbb{Z}$. *Suppose that for* $\ell \in \{1, 2, \ldots, n - 1\}$

$$\sum_{j=m_0}^{\infty} Q_\ell^d(j, m_0, m_0) = \infty. \tag{3.26}$$

Then, Equation (3.25) *is oscillatory if*

(i) for n even, the second-order difference equation

$$\Delta^2 y(m) + q_\ell^d(m, m_0) y^\lambda(m) = 0, \tag{3.27}$$

for all $\ell \in \{1, 3, \ldots, n - 1\}$ *is oscillatory;*

(ii) for n odd, the Equation (3.27) *for all* $\ell \in \{2, 4, \ldots, n - 1\}$ *is oscillatory and*

$$\limsup_{m\to\infty} \sum_{j=\xi(m)}^{m} \left(\frac{\left(\xi(j) - \xi(m)\right)^{(n-1)}}{(n-1)!}\right)^\lambda q(j) > \begin{cases} 0 \text{ when } 0 < \lambda < 1 \\ 1 \text{ when } \lambda = 1. \end{cases} \tag{3.28}$$

Remark 1. *The oscillation of Equation* (3.1) *is obtained via a comparison with a set of second-order dynamic equations whose oscillatory behavior has been studied extensively in the literature. In fact, there are many sufficient conditions for the oscillation of Equation* (3.3) *which can be employed rather easily.*

4. Even order dynamic equations without delay

In this section, we present new oscillation criteria for (3.1) when n is even. That is, we consider

$$x^{\Delta^{2n}} + q(t)\left(x^\sigma\right)^\lambda = 0. \tag{4.1}$$

For $t \in \mathbb{T}$, we define

$$\hat{Q}_\ell(t) = \int_t^\infty \int_{s_{2n-\ell-1}}^\infty \cdots \int_{s_1}^\infty q(s)\Delta s \Delta s_1 \ldots \Delta s_{2n-\ell-1}, \ \ell \in \{1, 3, \ldots, 2n-1\}. \tag{4.2}$$

Theorem 4.1. *Let* $\lambda > 1$ *and* $t_0 \in \mathbb{T}$. *If for every integer* $\ell \in \{1, 3, \ldots, 2n - 1\}$,

$$\int_{t_0}^\infty h_{\ell-1}(s, t_0)\hat{Q}_\ell(s)\Delta s = \infty, \tag{4.3}$$

then Equation (4.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (4.1), say, $x(t) > 0$ for $t \geq t_0$. From Equation (4.1), we see that $x^{\Delta^{2n}}(t) \leq 0$ for $t \geq t_0$, where $x^{\Delta^{2n}}(t)$ is not identically zero for all large t. Using Lemma 2.1 there exist a $t_1 \geq t_0$ and an integer $\ell \in \{1, 3, \ldots, 2n - 1\}$ such that (2.1) and (2.2) hold for all $t \geq t_1$. From (2.1), we see that $x^{\Delta^\ell}(t) > 0$ and decreasing on $[t_1, \infty)_\mathbb{T}$. Now,

$$x^{\Delta^{\ell-1}}(s) - x^{\Delta^{\ell-1}}(t_1) = \int_{t_1}^{s} x^{\Delta^{\ell}}(\tau)\,\Delta\tau \geq h_1(s,t_1)\,x^{\Delta^{\ell}}(s),$$

or

$$x^{\Delta^{\ell-1}}(s) \geq h_1(s,t_1)\,x^{\Delta^{\ell}}(s), \quad s \geq t_1. \tag{4.4}$$

Integrating (4.4) (ℓ - 2)-times from t_1 to $s \geq t_1$, we have

$$x^{\Delta}(s) \geq h_{\ell-1}(s,t_1)\,x^{\Delta^{\ell}}(s), \quad s \geq t_1. \tag{4.5}$$

Next, we integrate Equation (4.1) from $s_1 \geq t_1$ to $v \geq s_1$ and let $v \to \infty$ to get

$$x^{\Delta^{2n-1}}(s_1) \geq \int_{s_1}^{\infty} q(\tau)\,x^{\lambda}(\sigma(\tau))\,\Delta\tau \geq \left(\int_{s_1}^{\infty} q(\tau)\,\Delta\tau\right) x^{\lambda}(\sigma(s_1)).$$

Integrating this inequality from $s_2 \geq t_1$ to $v \geq s_2$ and then letting $v \to \infty$ and using (2.2), we get

$$-x^{\Delta^{2n-2}}(s_2) \geq \left(\int_{s_2}^{\infty}\int_{s_1}^{\infty} q(\tau)\Delta\tau\Delta s_1\right) x^{\lambda}(\sigma(s_2)).$$

Continuing this process, one can easily find

$$x^{\Delta^{\ell}}(s) \geq \left(\int_{s}^{\infty}\int_{s_{2n-\ell-1}}^{\infty}\cdots\int_{s_1}^{\infty} q(\tau)\Delta\tau\Delta s_1\ldots\Delta s_{2n-\ell-1}\right) x^{\lambda}(\sigma(s)),$$

or

$$x^{\Delta^{\ell}}(s) \geq \hat{Q}_{\ell}(s)\,x^{\lambda}(\sigma(s)), \quad s \geq t_1. \tag{4.6}$$

From (4.5) and (4.6), we find

$$x^{-\lambda}(\sigma(s))\,x^{\Delta}(s) \geq h_{\ell-1}(s,t_1)\,\hat{Q}_{\ell}(s), \quad s \geq t_1,$$

and hence

$$\int_{t_1}^{t} x^{-\lambda}(\sigma(s))\,x^{\Delta}(s)\,\Delta s \geq \int_{t_1}^{t} h_{\ell-1}(s,t_1)\,\hat{Q}_{\ell}(s)\,\Delta s.$$

By employing the first inequality in Lemma 2.3, we get

$$\int_{t_1}^{t} \frac{\left(x^{1-\lambda}(s)\right)^{\Delta}}{1-\lambda}\Delta s \geq \int_{t_1}^{t} h_{\ell-1}(s,t_1)\,\hat{Q}_{\ell}(s)\,\Delta s,$$

and so

$$\int_{t_1}^{\infty} h_{\ell-1}(s,t_1)\,\hat{Q}_{\ell}(s)\,\Delta s \leq \frac{x^{1-\lambda}(t_1)}{\lambda-1} < \infty.$$

But this contradicts condition (4.3). The proof is complete. □

Theorem 4.2. *Let* $\lambda > 1$ *and* $t_0 \in \mathbb{T}$. *If for every integer* $\ell \in \{1, 3, ..., 2n - 1\}$,

$$\int_{t_0}^{\infty} h_{\ell-1}(s, t_0) \left(\int_{s}^{\infty} g_{2n-\ell-1}(\sigma(\tau), s) q(\tau) \Delta\tau \right) \Delta s = \infty, \tag{4.7}$$

then Equation (4.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (1.1), say, $x(t) > 0$ for $t \geq t_0$. By Taylor's formula, we see that

$$x^{\Delta^{\ell}}(s) \geq -\int_{s}^{\infty} g_{2n-\ell-1}(\sigma(\tau), s) x^{\Delta^{2n}}(\tau) \Delta\tau, \quad s \geq t_1. \tag{4.8}$$

Using Equation (4.1) in (4.8), we get

$$\begin{aligned} x^{\Delta^{\ell}}(s) &\geq \int_{s}^{\infty} g_{2n-\ell-1}(\sigma(\tau), s) q(\tau) x^{\lambda}(\sigma(\tau)) \Delta\tau \\ &\geq \left(\int_{s}^{\infty} g_{2n-\ell-1}(\sigma(\tau), s) q(\tau) \Delta\tau \right) x^{\lambda}(\sigma(s)), \quad s \geq t_1. \end{aligned} \tag{4.9}$$

Combining (4.8) with (4.9), we find

$$x^{\Delta}(s) \geq h_{\ell-1}(s, t_1) \left(\int_{s}^{\infty} g_{2n-\ell-1}(\sigma(\tau), s) q(\tau) \Delta\tau \right) x^{\lambda}(\sigma(s)), \quad s \geq t_1.$$

Dividing both sides by $x^{\lambda}(\sigma(s))$ and integrating from t_1 to $t \geq t_1$, we have

$$\int_{t_1}^{t} x^{-\lambda}(\sigma(s)) x^{\Delta}(s) \Delta s \geq \int_{t_1}^{t} h_{\ell-1}(s, t_1) \int_{s}^{\infty} g_{2n-\ell-1}(\sigma(\tau), s) q(\tau) \Delta\tau \Delta s.$$

The rest of the proof is similar to that of Theorem 4.1 and hence it is omitted. This completes the proof. □

Next, we apply Theorems 4.1 and 4.2 to obtain oscillation criteria for Equation (4.1) when $\lambda \leq 1$.

Theorem 4.3. *Let* $\lambda \leq 1$ *and* $t_0 \in \mathbb{T}$. *Assume that there exists a positive constant* α *such that* $\alpha + \lambda > 1$. *If for every* $\ell \in \{1, 3, ..., 2n - 1\}$, *condition* (4.3) *or* (4.7) *holds with* $q(t)$ *replaced by* $cq(t) h_{\ell}^{-\alpha}(t, 0)$, *where c is any positive constant, then Equation* (4.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (4.1) and assume that there exists a $t_0 > 0$ such that $x(t) > 0$ for $t \geq t_0$ and (2.1) and (2.2) hold for $t \geq t_0$. From (2.1) and the decreasing nature of $x^{\Delta^{\ell}}(t)$, there exists a constant $c_1 > 0$ such that $x^{\Delta^{\ell}}(t) \leq c_1$ for $t \geq t_0$. Integrating this inequality ℓ -times from t_0 to t, we have

$$x(t) \leq c h_{\ell}(t, 0), \quad t \geq t_0, \tag{4.10}$$

where c is a positive constant. Now, from Equation (4.1), we have

$$\begin{aligned} 0 &= x^{\Delta^{2n}}(t) + q(t)\,x^{-\alpha}(\sigma(t))\,x^{\lambda+\alpha}(\sigma(t)) \\ &\geq x^{\Delta^{2n}}(t) + c^{-\alpha} q(t)\,h_{\ell}^{-\alpha}(\sigma(t),0)\,x^{\lambda+\alpha}(\sigma(t)), \quad t \geq t_0. \end{aligned} \tag{4.11}$$

By applying Theorems 4.1 and 4.2 with inequality (3.20), we arrive at the desired conclusion. This completes the proof. □

Theorem 4.4. *Let* $\lambda < 1$ *and* $t_0 \in \mathbb{T}$. *If for every* $\ell \in \{1, 3, ..., 2n - 1\}$,

$$\int_{t_0}^{\infty} q(t)\left(\int_{t_0}^{t} h_{\ell-1}(t,\sigma(u))\,g_{2n-\ell-1}(t,u)\,\Delta u\right)^{\lambda} \Delta t = \infty, \tag{4.12}$$

then Equation (4.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (4.1), say, $x(t) > 0$ for $t \geq t_0$. As in the proof of Theorem 4.1, we see that (2.1) and (2.2) hold for $t \geq t_1 \geq t_0$. It is easy to see that

$$x(t) \geq \int_{t_1}^{t} h_{\ell-1}(t,\sigma(u))\,x^{\Delta^{\ell}}(u)\,\Delta u$$

and

$$x^{\Delta^{\ell}}(u) \geq g_{2n-\ell-1}(t,u)\,x^{\Delta^{2n-1}}(t), \quad t \geq u \geq t_1.$$

Therefore,

$$x(t) \geq \left(\int_{t_1}^{t} h_{\ell-1}(t,\sigma(u))\,g_{2n-\ell-1}(t,u)\,\Delta u\right) x^{\Delta^{2n-1}}(t) \text{ for } t \geq t_1.$$

Using this inequality in Equation (4.1), we get

$$\begin{aligned} -\left(x^{\Delta^{2n-1}}(t)\right)^{\Delta} &= q(t)\,x^{\lambda}(\sigma(t)) \geq q(t)\,x^{\lambda}(t) \\ &\geq q(t)\left(\int_{t_1}^{t} h_{\ell-1}(t,\sigma(u))\,g_{2n-\ell-1}(t,u)\,\Delta u\right)^{\lambda}\left(x^{\Delta^{2n-1}}(t)\right)^{\lambda}, \quad t \geq t_1. \end{aligned}$$

Set $w(t) = x^{\Delta^{2n-1}}(t)$, then

$$-w^{\lambda}(t)\,w^{\Delta}(t) \geq q(t)\left(\int_{t_1}^{t} h_{\ell-1}(t,\sigma(u))\,g_{2n-\ell-1}(t,u)\,\Delta u\right)^{\lambda}, \quad t \geq t_1.$$

Finally, in view of a chain rule, we integrate the last inequality from t_1 to t to get

$$\infty > \frac{w^{1-\lambda}(t_1)}{1-\lambda} \geq \int_{t_1}^{t} q(s)\left(\int_{t_1}^{s} h_{\ell-1}(s,\sigma(u))\,g_{2n-\ell-1}(s,u)\,\Delta u\right)^{\lambda} \Delta s,$$

a contradiction with condition (4.12). □

As an example, we shall reformulate some of the above results for the case $\mathbb{T} = \mathbb{Z}$, i.e., the discrete case. The Equation (4.1) takes the form

$$\Delta^{2n} x(m) + q(t) x^{\lambda}(m+1) = 0 \tag{4.13}$$

and establish new criteria for the oscillation of Equation (4.13).

We let

$$\hat{Q}_{\ell}^{d}(m) = \sum_{s_{2n-\ell-1}=t}^{\infty} \cdots \sum_{s_1=s_2}^{\infty} \sum_{u=s_1}^{\infty} q(u), \quad \ell \in \{1, 3, \ldots, 2n - -1\}, \quad m \geq m_0.$$

Theorem 4.5. *Let* $\lambda > 1$ *and* $m_0 \in \mathbb{Z}$. *If for every* $\ell \in \{1, 3, ..., 2n - 1\}$,

$$\sum_{s=m_0}^{\infty} s^{(\ell-1)} \hat{Q}_{\ell}^{d}(s) = \infty, \tag{4.14}$$

then Equation (4.13) *is oscillatory.*

Theorem 4.6. *Let* $\lambda > 1$ *and* $m_0 \in \mathbb{Z}$. *If for every* $\ell \in \{1, 3, ..., 2n - 1\}$,

$$\sum_{s=m_0}^{\infty} s^{(\ell-1)} \sum_{\tau=s}^{\infty} (\tau - s + 1)^{(2n-\ell-1)} q(\tau) = \infty, \tag{4.15}$$

then Equation (4.13) *is oscillatory.*

Theorem 4.7. *Let* $\lambda < 1$ *and* $m_0 \in \mathbb{Z}$. *If*

$$\sum_{s=m_0}^{\infty} \left(s^{\lambda}\right)^{(2n-1)} q(s) = \infty, \tag{4.16}$$

then Equation (4.13) *is oscillatory.*

Theorem 4.8. *Let* $\lambda \leq 1$ *and* $m_0 \in \mathbb{Z}$. *Assume that there exists a positive constant* α *such that* $\alpha + \lambda > 1$. *If for every* $\ell \in \{1, 3, ..., 2n - 1\}$ *condition* (4.14) *or* (4.15) *holds with* $q(t)$ *be replaced by c* $q(t)(t)^{(\ell)}/\ell!)^{-\alpha}$, *where c is any positive constant, then Equation* (4.13) *is oscillatory.*

Remark 2. *For Equation* (4.1) *of odd order, one may obtain results for the oscillatory and asymptotic behavior, while for complete oscillation, we may consider Equation* (1.1) *and employ the technique given in Theorem 3.1. The details are left to the reader.*

5. Further oscillation criteria

In this section, we consider

$$x^{\Delta^n} + q(t)\left(x^{\sigma}\right)^{\lambda} = 0, \tag{5.1}$$

subject to the condition

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \int_{u}^{\infty} q(s)\, \Delta s \Delta u \Delta v = \infty. \tag{5.2}$$

Note that if $x(t)$, $t \geq t_0$ is a positive solution of Equation (5.1), then by Lemma 2.1, Equations (2.1), and (2.2) hold for $t \geq t_1$. Here, we claim that $\ell = n - 1$. Otherwise, we find $x^{\Delta^{n-1}}(t) > 0$, $x^{\Delta^{n-2}}(t) < 0$ and $x^{\Delta^{n-3}}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Integrating Equation

(5.1) from $t \geq t_1$ to $u \geq t$ and letting $u \to \infty$, we have

$$x^{\Delta^{n-1}}(t) \geq \int_t^{\infty} q(s)\, x^{\lambda}(\sigma(s))\, \Delta s. \tag{5.3}$$

Since x is increasing on $[t_1, \infty)_{\mathbb{T}}$, there exists a constant $c > 0$ such that

$$x(t) \geq c, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{5.4}$$

Using (5.4) in (5.3), we get

$$-x^{\Delta^{n-1}}(t) \leq -c^{\lambda} \int_t^{\infty} q(s)\, \Delta s.$$

Integrating this inequality twice, once from $v \geq t$ to $w \geq v$ and letting $w \to \infty$ and then from t_1 to $t \geq t_1$, we have

$$x^{\Delta^{n-3}}(t) \leq -c^{\lambda} \int_{t_1}^{t} \int_{v}^{\infty} \int_{s}^{\infty} q(u)\, \Delta u \Delta s \Delta v \to \infty \quad \text{as } t \to \infty,$$

which contradicts (5.2). Thus, we must have $\ell = n - 1$, i.e.,

Thus, we have

$$x^{\Delta^{n-2}}(t) = x^{\Delta^{n-2}}(t_1) + \int_{t_1}^{t} x^{\Delta^{n-1}}(s)\, \Delta s \geq h_1(t, t_1)\, x^{\Delta^{n-1}}(t), \quad t \geq t_1.$$

Integrating this inequality $(n - 2)$-times from t_1 to t, we obtain

$$x(t) \geq h_{n-1}(t, t_1)\, x^{\Delta^{n-1}}(t), \quad t \geq t_1. \tag{5.5}$$

Now, by making use of earlier results in [6], we obtain the following interesting theorems.

Theorem 5.1. *Let condition* (5.2) *hold. If there exists a positive nondecreasing, differentiable function* $\eta \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ *such that for any* $t_1 \geq t_0$,

$$\limsup_{t \to \infty} \int_{t_1}^{t} \left[\eta(s)\, q(s) - \eta^{\Delta}(s) \frac{A(s, t_0)}{h_{n-1}(s, t_0)} \right] \Delta s = \infty, \tag{5.6}$$

where

$$A(t, t_0) = \begin{cases} c_1, c_1 \text{ is any positive constant,} & \text{when } \lambda > 1 \\ 1, & \text{when } \lambda = 1 \\ c_2 h_{n-1}^{1-\lambda}(t, t_0),\ c_2 \text{ is any positive constant,} & \text{when } \lambda < 1, \end{cases}$$

then Equation (5.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (5.1), say, $x(t) > 0$ for $t \geq t_1 \geq t_0$.

Define

$$w(t) = \eta(t)\frac{x^{\Delta^{n-1}}(t)}{x^{\lambda}(t)}, \quad t \geq t_1. \tag{5.7}$$

It is easy to see that for $t \geq t_1$,

$$\begin{aligned} w^{\Delta} &= \left(\frac{\eta}{x^{\lambda}}\right)^{\Delta}\left(x^{\Delta^{n-1}}\right)^{\sigma} + \left(\frac{\eta}{x^{\lambda}}\right)\left(x^{\Delta^{n}}\right) \\ &= -\eta q\left(\frac{x^{\sigma}}{x}\right)^{\lambda} + \left(x^{\Delta^{n-1}}\right)^{\sigma}\left[\frac{\eta^{\Delta}x^{\lambda} - \eta\left(x^{\lambda}\right)\Delta}{x^{\lambda}(x^{\sigma})^{\lambda}}\right]. \end{aligned} \tag{5.8}$$

By [[8], Theorem 1.90],

$$\left(x^{\lambda}\right)^{\Delta} = \lambda x^{\Delta}\int_0^1 \left[x + \mu h x^{\Delta}\right]^{\lambda-1} dh > 0. \tag{5.9}$$

Using (5.9) in (5.8) we have

$$w^{\Delta}(t) \leq -\eta(t)q(t) + \eta^{\Delta}(t)\frac{\left(x^{\Delta^{n-1}}(t)\right)^{\sigma}}{(x^{\sigma}(t))^{\lambda}} \leq -\eta(t)q(t) + \eta^{\Delta}(t)\frac{x^{\Delta^{n-1}}(t)}{x^{\lambda}(t)}, \quad t \geq t_1,$$

and hence in view of (5.5), we find

$$w^{\Delta}(t) \leq -\eta(t)q(t) + \frac{\eta^{\Delta}(t)}{h_{n-1}(t,t_1)}x^{1-\lambda}(t), \quad t > t_1. \tag{5.10}$$

Let $\lambda > 1$. Since there exist $c > 0$ and $t_2 \geq t_1$ such that $x(t) \geq c$ for all $t \geq t_2$, we have $x^{1-\lambda}(t) \leq c^{1-\lambda} := c_1$ for all $t \geq t_2$. If $\lambda = 1$, then $x^{1-\lambda}(t) = 1$ for all $t \geq t_1$. If $\lambda < 1$, then there exist $b > 0$ and $t_3 \geq t_1$ such that $x^{\Delta^{n-1}}(t) \leq b$ for all $t \geq t_3$, and hence $x^{1-\lambda}(t) \leq c_2 h_{n-1}^{1-\lambda}(t,t_1)$ for all $t \geq t_3$, where $c_2 := b^{1-\lambda}$. Combining all these we see that

$$x^{1-\lambda}(t) \leq A(t,t_1), \quad t \geq t_4 \tag{5.11}$$

for some $t_4 \geq \max\{t_2, t_3\}$. From (5.10) and (5.11),

$$w^{\Delta}(t) \leq -\eta(t)q(t) + \eta^{\Delta}(t)\frac{A(t,t_1)}{h_{n-1}(t,t_1)}, \quad t \geq t_4.$$

Integrating this inequality from t_4 to t, we find

$$\int_{t_4}^{t}\left[\eta(s)q(s) - \eta^{\Delta}(s)\frac{A(s,t_1)}{h_{n-1}(s,t_1)}\right]\Delta s \leq w(t_4).$$

Taking limit superior as $t \to \infty$, we obtain a contradiction to condition (5.6). This completes the proof. □

In the following result, we employ the lemma below, see [10].

Lemma 5.1. *If X and Y are nonnegative and $\alpha > 1$, then*

$$X^{\alpha} - \alpha XY^{\alpha-1} + (\alpha - 1)Y^{\alpha} \geq 0, \tag{5.12}$$

where equality holds if and only if $X = Y$.

Theorem 5.2. *Let condition* (5.2) *hold. If there exists a positive, nondecreasing, differentiable function* $\eta \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ *such that for any* $t_1 \geq t_0$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\eta(s)\, q(s) - \frac{\left(\eta^{\Delta}(s)\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1} \left(h_{n-2}(s,t_0)\, \eta(s)\, B(s,t_0)\right)^{\lambda}} \right] \Delta s = \infty, \tag{5.13}$$

where

$$B(t,t_0) = \begin{cases} c_1, c_1 \text{ is any positive constant}, & \text{when } \lambda > 1 \\ 1, & \text{when } \lambda = 1 \\ c_2 \left(h_{n-1}^{\sigma}(t,t_0)\right)^{\lambda+1}, \ c_2 \text{ is any positive constant}, & \text{when } \lambda < 1, \end{cases} \tag{5.14}$$

then Equation (5.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (5.1), say, $x(t) > 0$ for $t \geq t_0$. Let w be as in (5.7). Then (5.8) and (5.9) hold. We also have

$$w^{\Delta} \leq -\eta q + \frac{\eta^{\Delta}}{\eta} w^{\sigma} - \lambda \eta \left(\frac{x^{\Delta}}{x}\right) (w/\eta)^{\sigma}. \tag{5.15}$$

Using the fact that $\ell = n - 1$ and

$$\frac{x^{\Delta^{n-1}}}{x} \geq \left(\left(\frac{w}{\eta}\right)^{\sigma} \right)^{1/\lambda} \left(x^{\sigma}\right)^{\lambda-1}$$

in (5.15), we obtain

$$w^{\Delta} \leq -\eta q + \frac{\eta^{\Delta}}{\eta} w^{\sigma} - \lambda \eta h_{n-2} \left(\left(\frac{w}{\eta}\right)^{\sigma} \right)^{1+1/\lambda} \left(x^{\sigma}\right)^{\lambda-1}. \tag{5.16}$$

If $\lambda > 1$, then from $x^{\sigma}(t) \geq x^{\sigma}(t_1)$ for $t \geq t_1$, we have $(x^{\sigma}(t))^{\lambda-1} \geq c_1 = (x^{\sigma}(t_1))^{\lambda-1}$. In case $\lambda = 1$, $(x^{\sigma}(t))^{\lambda-1} = 1$ for all $t \geq t_1$. Finally, let $\lambda < 1$. We see that there exist $t_2 \geq t_1$ and $b > 0$ such that $x^{\Delta^{n-1}}(t) \leq b$ for all $t \geq t_2$. It follows that $x(t) \leq b h_{n-1}(t, t_1)$ for all $t \geq t_2$, and hence $\left(x^{\sigma}(t)\right)^{\lambda-1} \geq b^{\lambda-1} \left(h_{n-1}^{\sigma}(t,t_1)\right)^{\lambda-1}$ for all $t \geq t_2$, where $c_2 = b^{\lambda-1}$. Putting all these together, we have

$$\left(x^{\sigma}(t)\right)^{\lambda-1} \geq B(t,t_1), \quad t \geq t_2. \tag{5.17}$$

In view of (5.17) and (5.16), we find

$$w^{\Delta}(t) \leq -\eta(t)\, q(t) + \frac{\eta^{\Delta}(t)}{\eta(t)} w^{\sigma}(t) - \lambda \eta(t)\, h_{n-2}(t,t_1)\, B(t,t_1) \left(\left(\frac{w(t)}{\eta(t)}\right)^{\sigma} \right)^{1+1/\lambda}, \quad t \geq t_2. \tag{5.18}$$

Now, setting

$$X = (\lambda \eta h_{n-2} B)^{\lambda/(\lambda+1)} \left(\frac{w}{\eta}\right)^{\sigma} \text{ and } Y = \left(\frac{\lambda}{\lambda+1}\right)^{\lambda} \left(\eta^{\Delta}\right)^{\lambda} \left(\left(\frac{1}{\lambda h_{n-2} B}\right)^{\lambda/(\lambda+1)} \right)^{\lambda}$$

and $\alpha = (\lambda + 1)/\lambda > 1$ in Lemma 5.1, we have

$$\lambda \eta h_{n-2} B \left(\left(\frac{w}{\eta}\right)^{\sigma} \right)^{1+1/\lambda} - \eta^{\Delta} \left(\frac{w}{\eta}\right)^{\sigma} + \frac{\left(\eta^{\Delta}\right)^{\lambda+1}}{(\lambda+1)^{\lambda+1} (\eta h_{n-1} B)^{\lambda}} \geq 0.$$

Therefore, from (5.18)

$$w^{\Delta} \leq -\eta q + \frac{1}{(\lambda+1)^{\lambda+1}} - \frac{\left(\eta^{\Delta}\right)^{\lambda+1}}{(\eta h_{n-1} B)^{\lambda}}, \quad t \geq t_2.$$

Integrating this inequality from t_2 to t results in

$$\int_{t_2}^{t} \left[\eta(s) q(s) - \frac{1}{(\lambda+1)^{\lambda+1}} \frac{\left(\eta^{\Delta}(s)\right)^{\lambda+1}}{(\eta(s) h_{n-2}(s,t_1) B(s,t_1))^{\lambda}} \right] \Delta s \leq w(t_2),$$

which contradicts (5.15). This completes the proof. □

Finally, we present the following result.

Theorem 5.3. *Let condition* (5.2) *hold. If there exists a positive, nondecreasing differentiable function* η *such that for any* $t_1 \geq t_0$,

$$\limsup_{t \to \infty} \int_{t_1}^{t} \left[\eta(s) q(s) - \frac{\left(\eta^{\Delta}(s)\right)^{\lambda}}{4\lambda \eta(s) B(s,t_0) h_{n-2}(s,t_0) \left(h_{n-1}^{\sigma}(s,t_0)\right)^{\lambda}} \right] \Delta s = \infty, \quad (5.19)$$

where $B(t, t_0)$ *is as in* (5.14), *then Equation* (5.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Equation (5.1), say, $x(t) > 0$ for $t \geq t_0$. Proceeding as in the proof of Theorem 5.2, we obtain

$$\begin{aligned} w^{\Delta} &\leq -\eta q + \eta^{\Delta}\left(\frac{w}{\eta}\right)^{\sigma} - \lambda \eta h_{n-2} B \left(\left(\frac{w}{\eta}\right)^{\sigma}\right)^{1+1/\lambda} \\ &= -\eta q + \eta^{\Delta}\left(\frac{w}{\eta}\right)^{\sigma} - \lambda \eta h_{n-2} B \frac{(w^{\sigma})^{1/\lambda - 1}}{(\eta^{\sigma})^{1/\lambda+1}} (w^{\sigma})^2, \end{aligned}$$

where $B = B(t, t_1)$ and $h_{n-2} = h_{n-2}(t, t_1)$. Since

$$w^{1/\lambda - 1}(t) = \lambda^{1/\lambda - 1} \left(\frac{x^{\Delta^{n-1}}(t)}{x(t)} \right)^{1-\lambda} \geq \eta^{1/\lambda - 1}(t) h_{n-1}^{\lambda - 1}(t, t_1),$$

it follows that

$$\begin{aligned} w^{\Delta} &\leq -\eta q + \eta^{\Delta}\left(\frac{w}{\eta}\right)^{\sigma} - \lambda \eta B h_{n-1} \left(h_{n-1}^{\sigma}\right)^{\lambda-1} \left(\frac{w^{\sigma}}{\eta^{\sigma}}\right)^2 \\ &= -\eta q - \left[\left(\lambda \eta B h_{n-2} \left(h_{n-1}^{\sigma}\right)^{\lambda-1}\right)^{1/2} \left(\frac{w}{\eta}\right)^{\sigma} - \frac{\eta^{\Delta}}{2\left(\lambda \eta B h_{n-2}\left(h_{n-1}^{\sigma}\right)^{\lambda-1}\right)^{1/2}} \right]^2 \\ &\quad + \frac{\left(\eta^{\Delta}\right)^2}{4\lambda \eta B h_{n-2}\left(h_{n-1}^{\sigma}\right)^{\lambda-1}} \\ &\leq -\eta q + \frac{\left(\eta^{\Delta}\right)^2}{4\lambda \eta B h_{n-2}\left(h_{n-1}^{\sigma}\right)^{\lambda-1}}, \quad t \geq t_2. \end{aligned}$$

Integrating this inequality from t_2 to t, we have

$$\int_{t_2}^{t}\left[\eta(s)\,q(s)-\frac{\left(\eta^{\Delta}(s)\right)^{2}}{4\lambda\eta(s)\,B(s,t_1)\,h_{n-2}(s,t_1)\left(h_{n-1}^{\sigma}(s,t_1)\right)^{\lambda-1}}\right]\Delta s\leq w(t_2),$$

which contradicts (5.19). This completes the proof. □

Remark 3. *We note that the oscillation criteria given in this article are new for the corresponding difference equations and some of these results are new for the corresponding differential and/or delay differential equations. The results can be extended easily to equations of the form*

$$x^{\Delta^n}(t)+f(t,x(\xi(t)))=0,$$

when $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ *is continuous and f is strongly superlinear or f is strongly sublinear, see* [4].

As examples, we have reformulated some of the obtained results for the time-scales $\mathbb{T}=\mathbb{R}$ (i.e., the continuous case) and $\mathbb{T}=\mathbb{Z}$ (i.e., the discrete case). One may obtain more results by employing other types of time scales such as $\mathbb{T}=h\mathbb{Z}$ with $h>0$, $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, and $\mathbb{T}=\mathbb{N}_0^2$, see [8]. The details are left to the reader.

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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 7 February 2012 Accepted: 23 May 2012 Published: 23 May 2012

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doi:10.1186/1687-1847-2012-67
Cite this article as: Grace *et al.*: **Oscillation of higher order nonlinear dynamic equations on time scales.** *Advances in Difference Equations* 2012 **2012**:67.