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Initial value problem for fractional evolution equations

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Abstract

This article is concerned with the existence of mild solutions to the initial value problem for a class of semilinear evolution equations with fractional order. New existence theorems are obtained by means of fixed point theorem for condensing maps. The results extend some related results in this direction.
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1 Introduction

This article deal with the existence of mild solutions to the initial value problem (IVP) for a class of semilinear evolution equations with fractional order of the form

$$\begin{cases} D^{\beta}u(t) + Au(t) = f(t, u(t)), & t \in J, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where D^{β} is the standard Caputo's derivative of order $0 < \beta < 1$, $J = [0, 1]$, $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear closed densely defined operator, $-A$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ of operators on $\mathbb{X}$, $f : J \times \mathbb{X} \to \mathbb{X}$ is continuous and u_0 is an element of the Banach space $\mathbb{X}$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. (see [1-5]). There has been a significant development in the study of fractional differential equations and inclusions in recent years; see the monograph of Kilbas [6], Lakshmikantham [7], Podlubny [4], and the survey by Agarwal [8]. For some recent contributions on fractional differential equations, see [9-15] and the references therein.

Very recently some basic theory for the initial value problems of fractional differential equations involving Riemann-Liouville differential operator of order $0 < q \leq 1$ has been discussed by Lakshmikantham and Vatsala [16-18].

Among the previous research, only a few be concerned with evolution equations of fractional order under noncompactness conditions. for some recent and deeper results on fractional differential equations under noncompactness conditions, see [19,20]. In this article, we prove the existence of mild solutions for the IVP (1.1) under

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noncompactness measure condition of nonlinear term *f*. For the details of the definition and properties of the measure of noncompactness, see [21].

The rest of this article is organized as follows: In Section 2, we recall briefly some basic definitions, lemmas and preliminary facts which are used throughout this article. The existence theorems of mild solutions for the IVP (1.1) and their proofs are arranged in Section 3.

2 Preliminaries

In this section, we introduce preliminary facts which are used in what follows.

Let $\left(\mathbb{B}(\mathbb{X}), \|\cdot\|_{\mathbb{B}(\mathbb{X})}\right)$ be the Banach space of all linear bounded operators on $\mathbb{X}$. Throughout this article, let $-A$ be the infinitesimal generators of a C_0-semigroup $T(t)(t \geq 0)$ of bounded linear operators on $\mathbb{X}$. Clearly

$$M := \sup_{t\in J} \|T(t)\|_{\mathbb{B}(\mathbb{X})} < \infty. \tag{2.1}$$

Let P be a cone in $\mathbb{X}$ which define a partial ordering in $\mathbb{X}$ by $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, we write $x < y$.

P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of $\mathbb{X}$.

Denote by $C(J, \mathbb{X})$ the Banach space of all continuous functions $x : J \to \mathbb{X}$ with norm $\|x\|_C = \sup_{t\in J} \|x(t)\|$. Set $P_C := \{x \in C(J, \mathbb{X}) : x(t) \geq \theta \text{ for } t \in J\}$, then P_C is a cone in space $C(J, \mathbb{X})$, and so, $C(J, \mathbb{X})$ is partially ordered by P_C : $u \leq v$ if and only if $v - u \in P_C$, i.e., $u(t) \leq v(t)$ for $t \in J$.

Now let Φ_β be the Mainardi function:

$$\Phi_\beta(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}. \tag{2.2}$$

then

$$\Phi_\beta(t) \geq 0 \text{ for all } t > 0; \tag{2.3}$$

$$\int_0^\infty \Phi_\beta(t)dt = 1; \tag{2.4}$$

$$\int_0^\infty t^\eta \Phi_\beta(t)dt = \frac{\Gamma(1+\eta)}{\Gamma(1+\beta\eta)}, \quad \eta \in [0, 1]. \tag{2.5}$$

For the details we refer to [20-22].

We set

$$\mathbb{S}_\beta(t) = \int_0^\infty \Phi_\beta(r)T\left(rt^\beta\right) dr, \tag{2.6}$$

$$\mathbb{P}_\beta(t) = \int_0^\infty \beta r\Phi_\beta(r)T\left(rt^\beta\right)dr. \tag{2.7}$$

Then we have the following result.

Lemma 2.1 [23,24]. Let $\mathbb{S}_\beta$ and $\mathbb{P}_\beta$ be the operators defined, respectively, by (2.6) and (2.7). Then

(*i*) $\|\mathbb{S}_\beta(t)u\| \le M\|u\|; \|\mathbb{P}_\beta(t)u\| \le M\frac{\beta}{\Gamma(\beta+1)}\|u\|$ for all $u \in \mathbb{X}$ and $t \ge 0$.

(*ii*) The operators $\mathbb{S}_\beta(t)(t \ge 0)$ and $\mathbb{P}_\beta(t)(t \ge 0)$ are strongly continuous.

Definition 2.2. A C_0-semigroup $R(t)(t \ge 0)$ in $\mathbb{X}$ is said to be positive, if order inequality $R(t)x \ge \theta$ holds for each $x \ge \theta$, $x \in \mathbb{X}$ and $t \ge 0$.

Remark 2.3. According to (2.6), (2.7) and Definition 2.2, if $T(t)(t \ge 0)$ is positive, then $\mathbb{S}_\beta(t)$ and $\mathbb{P}_\beta(t)$ are also positive.

Definition 2.4 [25,26]. Let $\mathbb{S}_\beta$ and $\mathbb{P}_\beta$ be operators defined, respectively, by (2.6) and (2.7).

Then a continuous function $u : J \to \mathbb{X}$ satisfying for any $t \in [0, 1]$ the equation

$$u(t) = \mathbb{S}_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)f(s,u(s))ds, \tag{2.8}$$

is called a mild solution of the problem (1.1).

Lemma 2.5. Let $T(t)(t \ge 0)$ is positive, $\mathbb{S}_\beta$ and $\mathbb{P}_\beta$ be the operators defined, respectively, by (2.6) and (2.7), $v, w \in C(J, \mathbb{X})$, $f \in C(J \times \mathbb{X}, \mathbb{X})$ and

$$(1)\ v(t) \le \mathbb{S}_\beta(t)v(0) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)f(s,v(s))ds,$$

$$(2)\ w(t) \ge \mathbb{S}_\beta(t)w(0) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)f(s,w(s))ds, \quad 0 \le t \le 1,, 0 \le t \le 1,$$

one of the foregoing inequalities being strict. Suppose further that $f(t, x)$ is nondecreasing in x for each t and

$$v(0) < w(0). \tag{2.9}$$

Then we have

$$v(t) < w(t), \quad 0 \le t \le 1. \tag{2.10}$$

Proof. Suppose that the conclusion (2.10) is not true. Then, because of the continuity of the functions involved and (2.9), it follows that there exists a t_1 such that $0 < t_1 \le 1$ and

$$v(t_1) = w(t_1), \quad v(t) < w(t), \quad 0 < t < t_1. \tag{2.11}$$

Since $v(0) < w(0)$ and $\mathbb{S}_\beta(t)$ is positive, so

$$\mathbb{S}_\beta(t)v(0) \le \mathbb{S}_\beta(t)w(0), \quad 0 \le t \le t_1. \tag{2.12}$$

Similarly, using the nondecreasing nature of f and (2.11), we obtain

$$\mathbb{P}_\beta(t_1-s)f(s,v(s)) \le \mathbb{P}_\beta(t_1-s)f(s,w(s)), \quad 0 \le s \le t_1. \tag{2.13}$$

Without loss of generality, let us suppose that the inequality (2) is strict, according to (2.12) and (2.13) we get

$$\begin{aligned} w(t_1) &> \mathbb{S}_\beta(t_1)w(0) + \int_0^{t_1} (t_1 - s)^{\beta-1}\mathbb{P}_\beta(t_1 - s)f(s, w(s))ds \\ &\geq \mathbb{S}_\beta(t_1)v(0) + \int_0^{t_1} (t_1 - s)^{\beta-1}\mathbb{P}_\beta(t_1 - s)f(s, v(s))ds \\ &\geq v(t_1), \end{aligned}$$

which is a contradiction in view of (2.11). Hence the conclusion (2.10) is valid and the proof is complete.

Let $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness of the bounded set. For any $B \subset C(J, \mathbb{X})$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset \mathbb{X}$. If B is bounded in $C(J, \mathbb{X})$, then $B(t)$ is bounded in $\mathbb{X}$, and $\alpha(B(t)) \leq \alpha(B)$.

Lemma 2.6 [27]. Let $D \subset \mathbb{X}$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \leq 2\alpha(D_0)$.

Lemma 2.7 [28]. Let $H \subset C(J, \mathbb{X})$ is bounded and equicontinuous. Then

$$\alpha(H) = \alpha(H(J)) = \max_{t\in J} \alpha(H(t)).$$

Lemma 2.8 [28]. Let H be a countable set of strongly measurable function $x : J \to \mathbb{X}$ such that there exists an $g \in L(J, [0, +\infty))$ such that $\|x(t)\| \leq g(t)$ *a.e.* $t \in J$ for all $x \in H$. Then $\alpha(H(t)) \in L(J,[0,+\infty))$ and

$$\alpha\left(\left\{\int_J x(t)dt : x \in H\right\}\right) \leq 2\int_J \alpha(H(t))dt.$$

Lemma 2.9 [29]. Suppose $b \geq 0$, $q > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq \infty$), and suppose $x(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$x(t) \leq a(t) + b\int_0^t (t - s)^{q-1}x(s)ds$$

on this interval; then

$$x(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(q))^n}{\Gamma(nq)}(t - s)^{nq-1}a(s)\right]ds, \quad 0 \leq t < T.$$

Lemma 2.10 [21]. Let $\mathbb{X}$ be a Banach space and Ω is a bounded convex closed set in $\mathbb{X}$, $Q : \Omega \to \Omega$ be condensing. Then Q has a fixed point in Ω.

3 Main Results

Theorem 3.1. Let $\mathbb{X}$ be an ordered Banach space, whose positive cone P is normal, $f \in C(J \times \mathbb{X}, \mathbb{X})$. Suppose that the following conditions are satisfied:

(H1) $T(t)(t \geq 0)$ is equicontinuous, i.e., $T(t)$ is continuous in the uniform operator topology for $t > 0$.

(H2) There exists a constant $L > 0$, $\frac{4ML}{\Gamma(\beta+1)} < 1$ such that

$$\alpha(f(t,D)) \leq L\alpha(D), \quad \text{for any } t \in J \text{ and } D \subset B_r,$$

where $B_r = \{u \in C(J,\mathbb{X})\} : \|u\|_C \leq r\}$ ..

(H3) There exists a function $\mu(t) \in L^{\infty}(J,\mathbb{R}^+)$ such that

$$\|f(t,u)\| \leq \mu(t), \quad \text{for all } t \in J \text{ and } u \in \mathbb{X}.$$

Then the IVP (1.1) has a mild solution in $C(J,\mathbb{X})$.

Proof. Let

$$\Omega = \{u \in C(J,\mathbb{X}) : \|u\|_C \leq R\},$$

where

$$R > M\left(\|u_0\| + \frac{\|\mu\|_{L^{\infty}(J,\mathbb{R}^+)}}{\Gamma(\beta+1)}\right).$$

Define the operator $Q : \Omega \to C(J,\mathbb{X})$ by

$$(Qu)(t) - \mathbb{S}_{\beta}(t)u_0 + \int_0^t (t-s)^{\beta-1}\mathbb{P}_{\beta}(t-s)f(s,u(s))ds. \tag{3.1}$$

It is obvious that the mild solution of the IVP (1.1) is equivalent to the fixed point of Q. Then we proceed in two steps.

Step 1. $Q : \Omega \to \Omega$.

In view of (2.1), (H3) and Lemma 2.1, we have for $u \in \Omega$ and $t \in J$,

$$\begin{aligned}
\|(Qu)(t)\| &\leq \|\mathbb{S}_{\beta}(t)u_0\| + \left\|\int_0^t (t-s)^{\beta-1}\mathbb{P}_{\beta}(t-s)f(s,u(s))ds\right\| \\
&\leq M\|u_0\| + \frac{M\beta}{\Gamma(\beta+1)}\int_0^t (t-s)^{\beta-1}\mu(s)ds \\
&\leq M\left(\|u_0\| + \frac{\|\mu\|_{L^{\infty}(J,\mathbb{R}+)}}{\Gamma(\beta+1)}\right) \\
&\leq R,
\end{aligned}$$

that is

$$\|Qu\|_C \leq R.$$

So Q: $\Omega \to \Omega$.

Step 2. $Q : \Omega \to \Omega$ is condensing.

First, by using analog argument performed in [30], one can prove $Q(\Omega)$ is equicontinuous, we omit it here.

For any $B \subset \Omega$, by Lemma 2.6, there exists a countable set $B_1 = \{u_n\} \subset B$, such that

$$\alpha(Q(B)) \leq 2\alpha(Q(B_1)). \tag{3.2}$$

Since $Q(B_1) \subset Q(\Omega)$ is equicontinuous, in view of Lemma 2.7

$$\alpha\left(Q(B_1)\right) = \max_{t\in J} \alpha\left(Q(B_1)(t)\right). \tag{3.3}$$

For $t \in J$, according to Lemma 2.1, Lemma 2.8 and (H2), we have

$$\begin{aligned}
\alpha\left(Q(B_1)(t)\right) &= \alpha\left(\left\{\mathbb{S}_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)f(s,u_n(s))ds\right\}\right) \\
&\le \alpha\left(\left\{\int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)f(s,u_n(s))ds\right\}\right) \\
&\le 2\int_0^t (t-s)^{\beta-1}\alpha\left(\left\{\mathbb{P}_\beta(t-s)f(s,u_n(s))\right\}\right)ds \\
&\le \frac{2M\beta}{\Gamma(\beta+1)}\int_0^t (t-s)^{\beta-1}\alpha\left(f(s,B_1(s))\right)ds \\
&\le \frac{2ML\beta}{\Gamma(\beta+1)}\alpha(B)\int_0^t (t-s)^{\beta-1}ds \\
&\le \frac{2ML}{\Gamma(\beta+1)}\alpha(B).
\end{aligned}$$

So, we can conclude that

$$\alpha\left(Q(B_1)(t)\right) \le \frac{2ML}{\Gamma(\beta+1)}\alpha(B). \tag{3.4}$$

Thus, a combination of (3.2), (3.3), and (3.4) gives that

$$\alpha(Q(B)) \le \frac{4ML}{\Gamma(\beta+1)}\alpha(B). \tag{3.5}$$

From (H2), $Q : \Omega \to \Omega$ is condensing.

Finally, Lemma 2.10 guarantees that Q has a fixed point in Ω.

Now we discuss the existence of minimal and maximal mild solutions for IVP (1.1).

Theorem 3.2. Let $\mathbb{X}$ be an ordered Banach space, whose positive cone P is normal with normal constant N, $T(t)x > \theta$ holds for each $x > \theta$, $x \in \mathbb{X}$ and $t \ge 0$, $f \in C(J \times \mathbb{X}, \mathbb{X})$. If conditions (H1)-(H3) and the following condition are satisfied:

(H4) $t \in J$, $u_1 \le u_2$ implies $f(t,u_1) \le f(t,u_2)$.

Then the IVP (1.1) has minimal and maximal mild solutions in $C(J, \mathbb{X})$.

Proof. Let

$$\theta < \cdots < \varepsilon_n < \varepsilon_{n-1} < \cdots \varepsilon_2 < \varepsilon_1, \quad n = 1, 2, ...,$$

where

$$\|\varepsilon_n\| < \delta \quad \text{and} \quad \lim_{n\to\infty} \|\varepsilon_n\| = 0. \tag{3.6}$$

Set

$$\Omega_1 = \left\{u \in C(J, \mathbb{X}) : \|u\|_C \le R_1\right\},$$

where

$$R_1 > M\left[\|u_0\| + \delta + \frac{\|\mu\|_{L^\infty(J,\mathbb{R}+)} + \delta}{\Gamma(\beta+1)}\right].$$

We consider the following fractional evolution equation

$$\begin{cases} D^\beta u(t) + Au(t) = f(t, u(t)) + \varepsilon_n, \quad t \in J, \\ u(0) = u_0 + \varepsilon_n, \end{cases} \tag{3.7}$$

by Lemma 2.4, if $u(t)$ is a mild solution of IVP (3.7), then

$$u(t) = \mathbb{S}_\beta(t)\,(u_0 + \varepsilon_n) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s, u(s)) + \varepsilon_n\right] ds. \tag{3.8}$$

It follows from (3.6), (3.8), (H3) and Lemma 2.1 that

$$\begin{aligned} \left\|u(t)\right\| &\le M\,(\|u_0\| + \delta) + \frac{M\beta\left(\|\mu\|_{L^\infty(J,\mathbb{R}+)} + \delta\right)}{\Gamma(\beta+1)} \int_0^t (t-s)^{\beta-1} ds \\ &\le M\,(\|u_0\| + \delta) + \frac{M\beta\left(\|\mu\|_{L^\infty(J,\mathbb{R}+)} + \delta\right)}{\Gamma(\beta+1)} \\ &< R_1. \end{aligned}$$

Thus

$$\|u\|_C \le R_1.$$

From the proof of Theorem 3.1, we know that the IVP (3.7) has a mild solution $u(t,\varepsilon_n)$ in Ω_1.

By (3.8), we know that

$$u(t, \varepsilon_n) = \mathbb{S}_\beta(t)u(0, \varepsilon_n) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s, u(s, \varepsilon_n)) + \varepsilon_n\right] ds, \tag{3.9}$$

where

$$u(0, \varepsilon_n) = u_0 + \varepsilon_n.$$

This yields

$$u(0, \varepsilon_n) < u(0, \varepsilon_{n-1}), \quad n = 2, 3, \ldots.$$

$$u(t, \varepsilon_2) = \mathbb{S}_\beta(t)u(0, \varepsilon_2) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s, u(s, \varepsilon_2)) + \varepsilon_2\right] ds,$$

$$\begin{aligned} u(t, \varepsilon_1) &= \mathbb{S}_\beta(t)u(0, \varepsilon_1) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s, u(s, \varepsilon_1)) + \varepsilon_1\right] ds \\ &> \mathbb{S}_\beta(t)u(0, \varepsilon_1) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s, u(s, \varepsilon_1)) + \varepsilon_2\right] ds. \end{aligned}$$

Combining (H4) with Lemma 2.5, we have

$$u(t,\varepsilon_2) < u(t,\varepsilon_1), \quad t \in J.$$

Hence

$$\cdots < u(t,\varepsilon_n) < u(t,\varepsilon_{n-1}) < \cdots < u(t,\varepsilon_2) < u(t,\varepsilon_1). \tag{3.10}$$

Let

$$V(t) = \{u(t,\varepsilon_n) : n = 1,2,\ldots\}, \quad \varphi(t) = \alpha(V(t)), \quad t \in J.$$

From (3.9), (H2), Lemmas 2.1 and 2.8, we have

$$\begin{aligned}
\varphi(t) &= \alpha(V(t)) \\
&= \alpha\left(\left\{\mathbb{S}_\beta(t)u(0,\varepsilon_n) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s,u(s,\varepsilon_n)) + \varepsilon_n\right]ds\right\}\right) \\
&\le \alpha\left(\left\{\int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s,u(s,\varepsilon_n)) + \varepsilon_n\right]ds\right\}\right) \\
&< 2\int_0^t \alpha\left(\left\{(t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s,u(s,\varepsilon_n)) + \varepsilon_n\right]\right\}\right)ds \\
&\le \frac{2M\beta}{\Gamma(\beta+1)}\int_0^t (t-s)^{\beta-1}\alpha(f(s,V(s)))ds \\
&\le \frac{2M\beta}{\Gamma(\beta+1)}\int_0^t (t-s)^{\beta-1}\varphi(s)ds.
\end{aligned}$$

This together with Lemma 2.9, we obtain that $\phi(t) \equiv 0$ on J. This means that $V(t)$ is precompact in $\mathbb{X}$. On the other hand, from the proof of Theorem 3.1 we know that $Q(\Omega_1)$ is equicontinuous, consequently, V is also equicontinuous. By Ascoli-Arzela theorem, we can obtain that V is relatively compact in $C(J,\mathbb{X})$, and so, there exists a subsequence of $\{u(t,\varepsilon_n)\}$ which converges uniformly on J to some $u^* \in C(J,\mathbb{X})$. In view of (3.10), we see that $\{u(t,\varepsilon_n)\}$ is non-increasing. Let $\{u(t,\varepsilon_{n_i})\}$ converge to u^*, for $i > j$, we have $u(t,\varepsilon_{n_i}) < u(t,\varepsilon_{n_j})$, which implies that $u^* \le u(t,\varepsilon_n)$. For any $\epsilon > 0$, there exists k such that

$$\|u(t,\varepsilon_{n_k}) - u^*\| < \frac{\epsilon}{N}.$$

Thus, for $n \ge n_k$, we get $u^* \le u(t,\varepsilon_n) \le u(t,\varepsilon_{n_k})$. This, together with the normality of P, yields that

$$\|u(t,\varepsilon_n) - u^*\| \le N\|u(t,\varepsilon_{n_k}) - u^*\| < \epsilon,$$

which implies that $\{u(t,\varepsilon_n)\}$ itself converges to u^* uniformly on J. So, we have

$$f(t,u(t,\varepsilon_n)) + \varepsilon_n \to f(t,u^*(t)), \quad n \to \infty, \quad t \in J. \tag{3.11}$$

On the other hand, we know that

$$(t-s)^{\beta-1}\left\|\mathbb{P}_\beta(t-s)\left[f(s,u(s,\varepsilon_n))+\varepsilon_n\right]\right\| \le \frac{M\beta}{\Gamma(\beta+1)}\left(\|\mu\|_{L^\infty(J,\mathbb{R}_+)}+\delta\right)(t-s)^{\beta-1}. \quad (3.12)$$

It follows from (3.9), (3.11), (3.12) and the Lebesgue dominated convergence theorem that

$$u^* = \mathbb{S}_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)f(s,u^*(s))ds.$$

Consequently, u^* is a mild solution of the IVP (1.1).

Let $u(t)$ be any solution of the IVP (1.1). It is obvious that

$$u(0) = u_0 < u_0 + \varepsilon_n = u(0,\varepsilon_n)$$

$$u(t) < \mathbb{S}_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s,u(s))+\varepsilon_n\right]ds,$$

$$u(t,\varepsilon_n) = \mathbb{S}_\beta(t)(u_0+\varepsilon_n) + \int_0^t (t-s)^{\beta-1}\mathbb{P}_\beta(t-s)\left[f(s,u(s,\varepsilon_n))+\varepsilon_n\right]ds.$$

By (H4) and Lemma 2.5, we deduce that

$$u(t) < u(t,\varepsilon_n).$$

Let $n \to \infty$, we have

$$u(t) \le u^*(t).$$

Thus u^* is a maximal mild solution of the IVP (1.1).

Similar to the above proof, one can prove that the IVP (1.1) has a minimal mild solution in Ω_1, we omit it here.

The proof is complete.

Remark 3.3. In Theorem 3.2, we do not assume $f(t,B_r) = \{f(t,u) : u \in B_r\}$ is relatively compact in $\mathbb{X}$ for any $t \in J$ and $r > 0$, therefore, Theorem 3.2 in this article is the extension of the main result in [15, Theorem 2.1].

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Authors' contributions
H-XF carried out the main part of this manuscript. JM participated discussion and corrected the main theorem. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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