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# Local bifurcation of limit cycles and center problem for a class of quintic nilpotent systems

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## Abstract

For a class of fifth degree nilpotent system, the shortened expressions of the first eight quasi-Lyapunov constants are presented. It is shown that the origin is a center if and only if the first eight quasi-Lyapunov constants are zeros. Under a small perturbation, the conclusion that eight limit cycles can be created from the eight-order weakened focus is vigorously proved. It is different from the usual Hopf bifurcation of limit cycles created from an elementary critical point.

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**Keywords:** quintic nilpotent system, quasi-Lyapunov constant, bifurcation of limit cycles, center problem

## 1 Introduction and statement of the main results

Two main open problems in the qualitative theory of planar analytic differential systems are characterizing the local phase portrait at an isolated critical point and the determination and distribution of limit cycles. Recall that a critical point is said to be of *focus-center type* if it is either a focus or a center. In what follows, this problem is called the *focus-center problem* or the *monodromy problem*, which is usually done by the blow-up procedure. Of course, if the linear part of the critical point is non-degenerate (i.e., its determinant does not vanish) the characterization is well known. The problem has also been solved when the linear part is degenerate but not identically null, see [1-3].

On the other hand, once we know that a critical point is of focus-center type, one comes across another classical problem, usually called the *center problem* or the *stability problem*, that is of distinguishing a center from a focus. The Poincaré-Lyapunov theory was developed to solve this problem in the case where the critical point is non-degenerate, see [4,5]. From a theoretical viewpoint, the study of this problem for a concrete family of differential equations goes through the calculation of the so-called *Lyapunov constants*, which gives the necessary conditions for center, see [6,7]. To completely solve the stability problem of polynomial systems of a fixed degree, although the Hilbert basis theorem asserts that the number of needed Lyapunov constants is finite, which is the number is still open.

Probably the most studied degenerated critical points are the *nilpotent* critical points. For these points, zero is a double eigenvalue of the differential matrix, but it is not identically zero. Nevertheless, given an analytic system with a nilpotent monodromic critical point it is not an easy task to know if it is a center or a focus. Analytic systems having a nilpotent critical point at the origin were studied by Andreev [1] in order to obtain their

local phase portraits. However, Andreev's results do not distinguish between a focus and a center. Takens [8] provided a normal form for nilpotent center of foci. Moussu [3] found the  $C^\infty$  normal form for analytic nilpotent centers. Berthier and Moussu in [9] studied the reversibility of the nilpotent centers. Teixeira and Yang [10] analysed the relationship between reversibility and the center-focus problem for systems

$$\begin{aligned}\dot{x} &= -\gamma + X(x, \gamma), \\ \dot{y} &= x + Y(x, \gamma),\end{aligned}\tag{1.1}$$

and

$$\begin{aligned}\dot{x} &= \gamma + X(x, \gamma), \\ \dot{y} &= Y(x, \gamma),\end{aligned}\tag{1.2}$$

where  $X(x, y)$  and  $Y(x, y)$  are real analytic functions without constant and linear terms, defined in a neighborhood of the origin.

It is well known that the dynamical behavior of a dynamical system depends on its parameters. As these parameters are varied, changes may occur in the qualitative structure of the solutions for certain parameter values. These changes are called bifurcations and the parameter values are called a bifurcation set. For a given family of polynomial differential equations usually the number of Lyapunov constants needed to solve the center-focus problem is also related with the so-called *cyclicity* of the point, i.e., the number of limit cycles that appear from it by small perturbations of the coefficients of the given differential equation inside the family considered (see [11] for cases where this relation does not exist). A classical way to produce limit cycles is by perturbing a system which has a center, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the period annulus of the center of the unperturbed system.

For a planar dynamical system, if the origin is an elementary critical point and the linearized system at the origin has a simple pair of pure imaginary eigenvalues  $\pm i\omega$ ,  $\omega > 0$ , then, under a small perturbation of the parameters, a small amplitude limit cycle can be created in a small neighborhood of the origin. This local change of the phase portraits is called Hopf bifurcations.

If the origin is not an elementary critical point, when the parameters are changed, what happens in a small neighborhood of the origin? This bifurcation phenomena is called the bifurcation of multiple critical point. To the best of our knowledge, there are essentially three different ways, the normal form theory [6], the Poincaré return map [12] and Lyapunov functions [13], of studying the center-focus problem of nilpotent critical points, see for instance [3,14,15]. On the other hand, the three tools mentioned above have been also used to generate limit cycles from the critical point, see for instance [15-17], respectively.

In [18,19] it is proved that any analytic nilpotent center is limit of an analytic linear type center, i.e., given any nilpotent center of a system  $X_0$ , there always exists a one-parametric perturbation  $X_\mu$  which has a center for any  $\mu \neq 0$  such that for  $\mu \rightarrow 0$  we have that  $X_\mu \rightarrow X_0$ . Therefore, any nilpotent center can be detected using the same methods that for a nondegenerate center, for instance the Poincaré-Liapunov method.

Here we are glad to highlight the work of Liu and Li [20], where a new definition of the focal value, quasi-Lyapunov constant, are given for the three-order nilpotent critical point. Meanwhile, the equivalence of quasi-Lyapunov constant with focal value is

proved. A linear recursive formula to compute quasi-Lyapunov constants is also presented. Afterward, they proved that if the three-order nilpotent origin is a  $m$ -order weakened focus, then, by a small perturbation for the unperturbed system, there exist  $m$  limit cycles in a neighborhood of the origin. At the same time, the origin becomes an elementary critical point and two complex singular points.

Let  $N(n)$  be the maximum possible number of limit cycles bifurcating from nilpotent critical points for analytic vector fields of degree  $n$ . The authors of [16] got  $N(3) \geq 2$ ,  $N(5) \geq 5$ ,  $N(7) \geq 9$ ; The authors of [15] got  $N(3) \geq 3$ ,  $N(5) \geq 5$ ; For a family of Kukles system with six parameters, the authors of [17] got  $N(3) \geq 3$ . The authors of [21,22] got  $N(3) \geq 7$  and  $N(3) \geq 8$ , respectively.

The aim of this article is to use the integral factor method introduced in [20], in order to compute what will be called *quasi-Lyapunov constants* (see Section 2) for a three-order nilpotent critical point in the following quintic system:

$$\begin{aligned}\frac{dx}{dt} &= \gamma + a_{50}x^5\gamma + a_{41}x^4\gamma + a_{32}x^3\gamma^2 + a_{14}x\gamma^4 + a_{05}\gamma^5, \\ \frac{dy}{dt} &= -2x^3 + b_{21}x^2\gamma + b_{12}x\gamma^2 + b_{03}\gamma^3.\end{aligned}\tag{1.3}$$

In addition, by applying them we give the center condition and lower bound for the cyclicity of the origin, i.e.,  $N(5) \geq 8$ .

Our main results are summarized in the following two theorems:

**Theorem 1.1.** *System (1.3) has a center at the origin if and only if*

$$b_{21} = b_{03} = a_{32} = a_{14} = a_{50} = 0.\tag{1.4}$$

Consider the following perturbed system of (1.3)

$$\begin{aligned}\frac{dx}{dt} &= \delta x + \gamma + a_{50}x^5 + a_{41}x^4\gamma + a_{32}x^3\gamma^2 + a_{14}x\gamma^4 + a_{05}\gamma^5, \\ \frac{dy}{dt} &= 2\delta\gamma - 2x^3 + b_{21}x^2\gamma + b_{12}x\gamma^2 + b_{03}\gamma^3.\end{aligned}\tag{1.5}$$

When  $0 < \delta \ll 1$ , in a small neighborhood of the origin, system (1.5) has exact three critical points:  $O(0, 0)$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$ , where  $O(0, 0)$  is an elementary critical point, another two critical points are complex with

$$x_{1,2} = \pm i\delta + o(\delta), \quad y_{1,2} = \mp i\delta^2 + o(\delta^2),\tag{1.6}$$

when  $\delta \rightarrow 0$ , three critical points coincide to become the three-order nilpotent critical point  $O(0, 0)$  of system (1.3).

**Theorem 1.2.** *Assume that the origin of system (1.3) is an eight-order weakened nilpotent focus. Then, under a small perturbation of system (1.3), for a small parameter  $\delta$ , in a neighborhood of the origin of system (1.5), there exist 8 limit cycles enclosing the elementary node  $O(0, 0)$ .*

Theorems 1.1 and 1.2 will be proved in Sections 3 and 4, respectively.

## 2 Preliminaries

In this section, we summarize some definitions and results about the center-focus problem of three-order nilpotent critical points of the planar dynamical systems that we shall use later on. For more details and proofs about these results see [20].

In canonical coordinates the Lyapunov system with the origin as a nilpotent critical point can be written in the form:

$$\begin{aligned}\frac{dx}{dt} &= \gamma + \sum_{i+j=2}^{\infty} a_{ij}x^i\gamma^j = X(x, \gamma), \\ \frac{d\gamma}{dt} &= \sum_{i+j=2}^{\infty} b_{ij}x^i\gamma^j = Y(x, \gamma).\end{aligned}\quad (2.1)$$

Suppose that the function  $\gamma = \gamma(x)$  satisfies  $X(x, \gamma) = 0$ ,  $\gamma(0) = 0$ . Lyapunov proved (see for instance [23]) that the origin of system (2.1) is a monodromic critical point (i. e., a center or a focus) if and only if

$$\begin{aligned}Y(x, \gamma(x)) &= \alpha x^{2n+1} + o(x^{2n+1}), \quad \alpha < 0, \\ \left[ \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x} \right]_{\gamma=\gamma(x)} &= \beta x^n + o(x^n), \\ \beta^2 + 4(n+1)\alpha &< 0,\end{aligned}\quad (2.2)$$

where  $n$  is a positive integer. The monodromy problem in the case of a nilpotent singular point was also solved in [24].

**Definition 2.1.** Let  $\gamma = f(x) = -a_{20}x^2 + o(x^2)$  be the unique solution of the function equation  $X(x, f(x)) = 0$ ,  $f(0) = 0$  at a neighborhood of the origin. If there are an integer  $m$  and a nonzero real number  $\alpha$ , such that

$$Y(x, f(x)) = \alpha x^m + o(x^m), \quad (2.3)$$

we say that the origin is a high-order singular point of system (2.1) with the multiplicity  $m$ .

By using the results in [23], we attain the following conclusion.

**Lemma 2.1.** The origin of system (2.1) is a three-order singular point which is a saddle point or a center, if and only if  $b_{20} = 0$ ,  $(2a_{20} - b_{11})^2 + 8b_{30} < 0$ .

When the condition in Lemma 2.1 holds, we can assume that

$$a_{20} = \mu, \quad b_{20} = 0, \quad b_{11} = 2\mu, \quad b_{30} = -2. \quad (2.4)$$

Otherwise, by letting  $(2a_{20} - b_{11})^2 + 8b_{30} = -16\lambda^2$ ,  $2a_{20} + b_{11} = 4\lambda\mu$  and making the transformation  $\xi = \lambda x$ ,  $\eta = \lambda\gamma + \frac{1}{4}(2a_{20} - b_{11})\lambda x^2$ , we obtain the mentioned result.

From (2.4), system (2.1) becomes the following real autonomous planar system

$$\begin{aligned}\frac{dx}{dt} &= \gamma + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^i\gamma^j = X(x, \gamma), \\ \frac{d\gamma}{dt} &= -2x^3 + 2\mu x\gamma + \sum_{i+2j=4}^{\infty} b_{ij}x^i\gamma^j = Y(x, \gamma).\end{aligned}\quad (2.5)$$

Write that

$$X(x, \gamma) = \gamma + \sum_{k=2}^{\infty} X_k(x, \gamma), \quad Y(x, \gamma) = \sum_{k=2}^{\infty} Y_k(x, \gamma), \quad (2.6)$$

where for  $k = 1, 2, \dots$ ,

$$X_k(x, y) = \sum_{i+j=k} a_{ij} x^i y^j, \quad Y_k(x, y) = \sum_{i+j=k} b_{ij} x^i y^j. \quad (2.7)$$

By using the transformation of generalized polar coordinates

$$x = r \cos \theta, \quad y = r^2 \sin \theta, \quad (2.8)$$

system (2.5) becomes

$$\begin{aligned} \frac{dr}{dt} &= \frac{\cos \theta [\sin \theta (1 - 2\cos^2 \theta) + \mu(\cos^2 \theta + 2\sin^2 \theta)]}{1 + \sin^2 \theta} r^2 + o(r^2), \\ \frac{d\theta}{dt} &= \frac{-r}{2(1 + \sin^2 \theta)(\cos^4 \theta + \sin^2 \theta)} + o(r). \end{aligned} \quad (2.9)$$

Thus, we have

$$\frac{dr}{d\theta} = \frac{-\cos \theta [\sin \theta (1 - 2\cos^2 \theta) + \mu(\cos^2 \theta + 2\sin^2 \theta)]}{2(\cos^4 \theta + \sin^2 \theta)} r + o(r). \quad (2.10)$$

Let

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k \quad (2.11)$$

be a solution of (2.10) satisfying the initial condition  $r|_{\theta=0} = h$ , where  $h$  is small and

$$\begin{aligned} v_1(\theta) &= (\cos^4 \theta + \sin^2 \theta)^{-\frac{1}{4}} \exp\left(\frac{-\mu}{2} \arctan \frac{\sin \theta}{\cos^2 \theta}\right), \\ v_1(k\pi) &= 1, \quad k = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (2.12)$$

Because for all sufficiently small  $r$ , we have  $d\theta/dt < 0$ . In a small neighborhood, we can define the successor function of system (2.5) as follows:

$$\Delta(h) = \tilde{r}(-2\pi, h) - h = \sum_{k=2}^{\infty} v_k(-2\pi) h^k. \quad (2.13)$$

We have the following result:

**Lemma 2.2.** *For any positive integer  $m$ ,  $v_{2m+1}(-2\pi)$  has the form*

$$v_{2m+1}(-2\pi) = \sum_{k=1}^m \varsigma_k^{(m)} v_{2k}(-2\pi), \quad (2.14)$$

where  $\varsigma_k^{(m)}$  is a polynomial of  $v_j(\pi)$ ,  $v_j(2\pi)$ ,  $v_j(-2\pi)$ , ( $j = 2, 3, \dots, 2m$ ) with rational coefficients.

It is different from the center-focus problem for the elementary critical points, we know from Lemma 2.2 that when  $k > 1$  for the first non-zero  $v_k(-2\pi)$ ,  $k$  is an even integer.

**Definition 2.2.** (1) *For any positive integer  $m$ ,  $v_{2m}(-2\pi)$  is called the  $m$ -th focal value of system (2.5) in the origin.*

(2) If  $v_2(-2\pi) \neq 0$ , then, the origin of system (2.5) is called 1-order weakened focus. In addition, if there is an integer  $m > 1$ , such that  $v_2(-2\pi) = v_4(-2\pi) = \dots = v_{2m-2}(-2\pi) = 0$ , but  $v_{2m}(-2\pi) \neq 0$ , then, the origin is called a  $m$ -order weakened focus of system (2.5).

(3) If for all positive integer  $m$ , we have  $v_{2m}(-2\pi) = 0$ , then, the origin of system (2.5) is called a center.

**Definition 2.3.** Let  $f_k, g_k$  be two bounded functions with respect to  $\mu$  and all  $a_{ij}, b_{ij}, k = 1, 2, \dots$ . If for some integer  $m$ , there exist  $\xi_1^{(m)}, \xi_2^{(m)}, \dots, \xi_{m-1}^{(m)}$ , which are continuous bounded functions with respect to  $\mu$  and all  $a_{ij}, b_{ij}, i = 1, 2, \dots$ , such that

$$f_m = g_m + \left( \xi_1^{(m)} f_1 + \xi_2^{(m)} f_2 + \dots + \xi_{m-1}^{(m)} f_{m-1} \right). \quad (2.15)$$

We say that  $f_m$  is equivalent to  $g_m$ , denoted by  $f_m \sim g_m$ .

If  $f_1 = g_1$  and for all positive integers  $m$ ,  $f_m \sim g_m$ , we say that the function sequences  $\{f_m\}$  and  $\{g_m\}$  are equivalent, denoted by  $\{f_m\} \sim \{g_m\}$ .

We know from Lemma 2.2 and Definition 2.2 that for the sequence  $\{v_k(-2\pi)\}$ ,  $k \geq 2$ , we have  $v_{2k+1}(-2\pi) \sim 0$ ,  $k = 1, 2, \dots$

We next state the results concerning with bifurcation of limit cycles of system (2.5). Consider the perturbed system of (2.5)

$$\frac{dx}{dt} = \delta x + X(x, y), \quad \frac{dy}{dt} = 2\delta y + Y(x, y), \quad (2.16)$$

where  $X(x, y), Y(x, y)$  are given by (2.6). Clearly, when  $0 < |\delta| \ll 1$ , in a neighborhood of the origin, there exist one elementary node at the origin and two complex critical points of system (2.16) at  $(x_1, y_1)$  and  $(x_2, y_2)$ , where

$$x_{1,2} = \frac{-\delta}{\mu \pm i} + o(\delta), \quad y_{1,2} = \frac{\pm i\delta^2}{(\mu \pm i)^2} + o(\delta^2). \quad (2.17)$$

When  $\delta \rightarrow 0$ , one elementary node and two complex critical points coincide to become a three-order critical point. Let

$$r = \tilde{r}(\theta, h, \delta) = v_0(\theta, \delta) + \sum_{k=1}^{\infty} v_k(\theta, \delta) h^k, \quad (2.18)$$

be a solution of system (2.16) satisfying the initial condition  $r|_{\theta=0} = h$ , where  $h$  is sufficiently small and

$$v_0(0, \delta) = 0, \quad v_1(0, \delta) = 1, \quad \dots, \quad v_k(0, \delta) = 0, \quad k = 2, 3, \dots \quad (2.19)$$

We have that

$$v_0(\theta, \delta) = A(\theta)\delta + o(\delta), \quad (2.20)$$

where

$$A(\theta) = \frac{-v_1(\theta, 0)}{2} \int_0^\theta \frac{(1 + \sin^2 \theta) d\theta}{v_1(\theta, 0)(\cos^4 \theta + \sin^2 \theta)}. \quad (2.21)$$

Hence, when  $0 < h \ll 1$ ,  $|\theta| < 4\pi$ ,  $\delta = o(h)$ ,  $\tilde{r}(\theta, h, \delta) = v_1(\theta, 0)h + o(h)$  and

$$v_0(-2\pi, \delta) = A(-2\pi)\delta + o(\delta), \quad (2.22)$$

where

$$A(-2\pi) = \frac{1}{2} \int_0^{2\pi} \frac{1 + \sin^2 \theta}{(\cos^4 \theta + \sin^2 \theta)^{\frac{3}{4}}} \exp \left( \frac{\mu}{2} \arctan \frac{\sin \theta}{\cos^2 \theta} \right) d\theta > 0. \quad (2.23)$$

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= \delta x + \gamma + \sum_{k+j=2}^{\infty} a_{kj}(\gamma) x^k \gamma^j, \\ \frac{dy}{dt} &= 2\delta y + \sum_{k+j=2}^{\infty} b_{kj}(\gamma) x^k \gamma^j, \end{aligned} \quad (2.24)$$

where  $\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_{m-1}\}$  is  $(m-1)$ -dimensional parameter vector. Let  $\gamma_0 = \{\gamma_1^{(0)}, \gamma_2^{(0)}, \dots, \gamma_{m-1}^{(0)}\}$  be a point at the parameter space. Suppose that for  $\|\gamma - \gamma_0\| \ll 1$ , the functions of the right hand of system (2.24) are power series of  $x, y$  with a non-zero convergence radius and have continuous partial derivatives with respect to  $\gamma$ . In addition,

$$a_{20}(\gamma) \equiv \mu, \quad b_{20}(\gamma) \equiv 0, \quad b_{11}(\gamma) \equiv 2\mu, \quad b_{30}(\gamma) \equiv -2. \quad (2.25)$$

For an integer  $k$ , letting  $v_{2k}(-2\pi, \gamma)$  be the  $k$ -order focal value of the origin of system  $(2.24)_{\delta=0}$ .

**Theorem 2.1.** *If for  $\gamma = \gamma_0$ , the origin of system  $(2.24)_{\delta=0}$  is a  $m$ -order weak focus, and the Jacobian*

$$\left. \frac{\partial(v_2, v_4, \dots, v_{2m-2})}{\partial(\gamma_1, \gamma_2, \dots, \gamma_{m-1})} \right|_{\gamma=\gamma_0} \neq 0, \quad (2.26)$$

*then there exist two positive number  $\delta^*$  and  $\gamma^*$ , such that for  $0 < |\delta| < \delta^*$ ,  $0 < \|\gamma - \gamma_0\| < \gamma^*$ , in a neighborhood of the origin, system (2.24) has at most  $m$  limit cycles which enclose the origin (an elementary node)  $O(0, 0)$ . In addition, under the above conditions, there exist  $\tilde{\gamma}, \tilde{\delta}$ , such that when  $\gamma = \tilde{\gamma}, \delta = \tilde{\delta}$ , there exist exactly  $m$  limit cycles of (2.24) in a small neighborhood of the origin.*

We give the following key results, which define the quasi-Lyapunov constants and provide a way of computing them.

**Theorem 2.2.** *For system (2.5), one can construct successively a formal series*

$$M(x, y) = y^2 + \sum_{k+j=3}^{\infty} c_{kj} x^k \gamma^j, \quad (2.27)$$

*such that*

$$\frac{\partial}{\partial x} \left( \frac{X}{M^{s+1}} \right) + \frac{\partial}{\partial y} \left( \frac{Y}{M^{s+1}} \right) = \frac{1}{M^{s+2}} \sum_{m=1}^{\infty} (2m - 4s - 1) \lambda_m x^{2m+4}, \quad (2.28)$$

*i.e.,*

$$\left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) M - (s+1) \left( \frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y \right) = \sum_{m=1}^{\infty} \lambda_m (2m - 4s - 1) x^{2m+4}. \quad (2.29)$$

where  $s$  is a given positive integer,

$$c_{30} = 0, \quad c_{40} = 1, \quad (2.30)$$

and

$$\{v_{2m}(-2\pi)\} \sim \{\sigma_m \lambda_m\}, \quad (2.31)$$

with

$$\sigma_m = \frac{1}{2} \int_0^{2\pi} \frac{(1 + \sin^2 \theta) \cos^{2m+4} \theta}{(\cos^4 \theta + \sin^2 \theta)^2} v_1^{2m-1}(\theta) d\theta > 0. \quad (2.32)$$

We see from (2.27) and (2.30) that when (2.8) holds,  $M = y^2 + x^4 + o(r^4)$ .

**Definition 2.4.** For system (2.5),  $\lambda_m$  is called the  $m$ -th quasi-Lyapunov constant of the origin.

**Theorem 2.3.** For any positive integer  $s$  and a given number sequence

$$\{c_{0\beta}\}, \quad \beta \geq 3, \quad (2.33)$$

one can construct successively the terms with the coefficients  $c_{\alpha\beta}$  satisfying  $\alpha \neq 0$  of the formal series

$$M(x, y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta} x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x, y), \quad (2.34)$$

such that

$$\frac{\partial}{\partial x} \left( \frac{X}{M^{s+1}} \right) + \frac{\partial}{\partial y} \left( \frac{Y}{M^{s+1}} \right) = \frac{1}{M^{s+2}} \sum_{m=3}^{\infty} \omega_m(s, \mu) x^m, \quad (2.35)$$

where for all  $k$ ,  $M_k(x, y)$  is a  $k$ -homogeneous polynomial of  $x, y$  and  $s\mu = 0$ .

Now, (2.35) can be written by

$$\left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) M - (s+1) \left( \frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y \right) = \sum_{m=3}^{\infty} \omega_m(s, \mu) x^m. \quad (2.36)$$

It is easy to see that (2.36) is linear with respect to the function  $M$ , so that we can easily find the following recursive formulae for the calculation of  $c_{\alpha\beta}$  and  $\omega_m(s, \mu)$ .

**Theorem 2.4.** For  $\alpha \geq 1$ ,  $\alpha + \beta \geq 3$  in (2.34) and (2.35),  $c_{\alpha\beta}$  can be uniquely determined by the recursive formula

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha} (A_{\alpha-1, \beta+1} + B_{\alpha-1, \beta+1}). \quad (2.37)$$

For  $m \geq 1$ ,  $\omega_m(s, \mu)$  can be uniquely determined by the recursive formula

$$\omega_m(s, \mu) = A_{m,0} + B_{m,0}, \quad (2.38)$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [k - (s+1)(\alpha - k + 1)] a_{kj} c_{\alpha-k+1, \beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [j - (s+1)(\beta - j + 1)] b_{kj} c_{\alpha-k, \beta-j+1}. \end{aligned} \quad (2.39)$$



Notice that in (2.39), we set

$$\begin{aligned} c_{00} &= c_{10} = c_{01} = 0, \\ c_{20} &= c_{11} = 0, \quad c_{02} = 1, \\ c_{\alpha\beta} &= 0, \quad \text{if } \alpha < 0 \text{ or } \beta < 0. \end{aligned} \quad (2.40)$$

We see from Theorem 2.4 that, by choosing  $\{c_{\alpha\beta}\}$ , such that

$$\omega_{2k+1}(s, \mu) = 0, \quad k = 1, 2, \dots, \quad (2.41)$$

we can obtain a solution group of  $\{c_{\alpha\beta}\}$  of (2.41), thus, we have

$$\lambda_m = \frac{\omega_{2m+4}(s, \mu)}{2m - 4s - 1}. \quad (2.42)$$

Clearly, the recursive formulae presented by Theorem 2.4 is linear with respect to all  $c_{\alpha\beta}$ . Accordingly, it is convenient to realize the computations of quasi-Lyapunov constants by using computer algebraic system like *Mathematica*.

### 3 Proof of Theorem 1.1

Now we start the preparation of the proof of Theorem 1.1. Obviously, the origin of system (1.3) is a three-order nilpotent critical point which is a center or a focus. Straightforward computation by using the recursive formulae shown in Theorem 2.4 and computer algebraic system *Mathematica* gives the following result. For detailed recursive formulae, please see Appendix.

**Theorem 3.1.** *The first eight quasi-Lyapunov constants of the origin of system (1.3) are as follows:*

$$\begin{aligned} \lambda_1 &= \frac{1}{3}b_{21}, \\ \lambda_2 &\sim \frac{1}{5}(5a_{50} + 6b_{03}), \\ \lambda_3 &\sim \frac{1}{21}(6a_{32} - a_{50}b_{12}), \\ \lambda_4 &\sim \frac{1}{135}(36a_{14} - 45a_{41}a_{50} - 8a_{50}b_{12}^2), \\ \lambda_5 &\sim \frac{1}{693}a_{50}(80a_{50}^2 - 201a_{41}b_{12} - 40b_{12}^3), \\ \lambda_6 &\sim \frac{1}{10530}a_{50}(5400a_{05} + 90a_{41}^2 + 1551a_{41}b_{12}^2 + 280b_{12}^4), \\ \lambda_7 &\sim -\frac{1}{3742200}a_{50}b_{12}(3541005a_{41}^2 + 1107732a_{41}b_{12}^2 + 83680b_{12}^4), \\ \lambda_8 &\sim -\frac{1}{2246933520}a_{41}a_{50}(1506240a_{41}^2 - 231634641a_{41}b_{12}^2 - 4194300b_{12}^4), \end{aligned} \quad (3.1)$$

where in the above expression of  $\lambda_k$ , we have already let  $\lambda_1 = \lambda_2 = \dots = \lambda_{k-1} = 0$ ,  $k = 2, 3, 4, 5, 6, 7, 8$ .

**Lemma 3.1.** *To guarantee the origin of system (1.3) is a center, the necessary condition is that  $a_{50} = 0$ .*

*Proof.* From (3.1), we have  $\lambda_5 = 0$  if and only if  $a_{50} = 0$  or

$$80a_{50}^2 - 201a_{41}b_{12} - 40b_{12}^3 = 0. \quad (3.2)$$

Notice the expressions of  $\lambda_7$  and  $\lambda_8$ , we calculate the following resultant:

$$\text{Resultant}[\lambda_7/a_{50}, \lambda_8/a_{50}, a_{41}] = \frac{3633949684618013}{71987660827238130975000} b_{12}^{15}. \quad (3.3)$$

Consequently, using (3.2)  $\lambda_7 = \lambda_8 = 0$  yields  $a_{50} = 0$ , as we wanted to prove.

It follows easily from Theorem 3.1 and Lemma 3.1 that

**Theorem 3.2.** *The first eight quasi-Lyapunov constants of the origin of system (1.3) are zeros if and only if condition (1.4) is satisfied.*

On the other hand, when condition (1.4) holds, system (1.3) goes over to

$$\begin{aligned} \frac{dx}{dt} &= y(1 + a_{41}x^4 + a_{05}y^4), \\ \frac{dy}{dt} &= -x(2x^2 - b_{12}y^2), \end{aligned} \quad (3.4)$$

the vector field defined by system (3.4) is symmetrical with respect to the origin. Therefore, we have

**Theorem 3.3.** *The origin of system (1.3) is a center if and only if the first eight quasi-Lyapunov constants vanish, i.e., this situation happens if and only if condition (1.4) holds.*

All the above discussion allows to finish the proof of Theorem 1.1.

#### 4 Proof of Theorem 1.2

We proceed to show that 8 limit cycles can be bifurcated in this instance. We found that the highest possible order for a weakened focus at the origin is eight. First of all, we need to find the conditions under which the nilpotent origin of system (1.3) is a eight-order weakened focus.

From the fact  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$ ,  $\lambda_8 \neq 0$ , the following statement holds.

**Theorem 4.1.** *The origin is a weakened focus of maximum order eight for system (1.3). It is of order eight if and only if one of the following four sets of conditions holds:*

$$\begin{aligned} a_{41} &= \frac{2(-92311 + 3\sqrt{32272369})}{1180335} b_{12}^2, & a_{50} &= \frac{1}{10} \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}, \\ b_{21} &= 0, & b_{03} &= -\frac{1}{12} \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}, \\ a_{32} &= \frac{1}{60} \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}} b_{12}^{\frac{5}{2}}, & a_{14} &= \frac{(37823 + 9\sqrt{32272369}) \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}}}{14164020} b_{12}^{\frac{7}{2}}, \\ a_{05} &= -\frac{8189365715 + 1597749\sqrt{32272369}}{1114552569780} b_{12}^4, & & b_{12} > 0; \end{aligned} \quad (4.1)$$

$$\begin{aligned} a_{41} &= \frac{2(-92311\sqrt{32272369})}{1180335} b_{12}^2, & a_{50} &= -\frac{1}{10} \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}, \\ b_{21} &= 0, & b_{03} &= \frac{1}{12} \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}, \\ a_{32} &= -\frac{1}{60} \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}} b_{12}^{\frac{5}{2}}, & a_{14} &= -\frac{(37823 + 9\sqrt{32272369}) \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}}}{14164020} b_{12}^{\frac{7}{2}}, \\ a_{05} &= -\frac{8189365715 + 1597749\sqrt{32272369}}{1114552569780} b_{12}^4, & & b_{12} > 0; \end{aligned} \quad (4.2)$$

$$\begin{aligned}
 a_{41} &= \frac{2(-92311 - 3\sqrt{32272369})}{1180335} b_{12}^2, & a_{50} &= \frac{1}{10} \sqrt{\frac{1684063 - 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}, \\
 b_{21} &= 0, & b_{03} &= -\frac{1}{12} \sqrt{\frac{1684063 - 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}, \\
 a_{32} &= \frac{1}{60} \sqrt{\frac{1684063 - 201\sqrt{32272369}}{157378}} b_{12}^{\frac{5}{2}}, & a_{14} &= \frac{(37823 - 9\sqrt{32272369}) \sqrt{\frac{1684063 - 201\sqrt{32272369}}{157378}}}{14164020} b_{12}^{\frac{7}{2}}, \\
 a_{05} &= \frac{-8189365715 + 1597749\sqrt{32272369}}{1114552569780} b_{12}^4, & b_{12} &> 0;
 \end{aligned} \tag{4.3}$$

$$\begin{aligned}
 a_{41} &= \frac{2(-92311 - 3\sqrt{32272369})}{1180335} b_{12}^2, & a_{50} &= -\frac{1}{10} \sqrt{\frac{1684063 - 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}, \\
 b_{21} &= 0, & b_{03} &= \frac{1}{12} \sqrt{\frac{1684063 - 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}, \\
 a_{32} &= -\frac{1}{60} \sqrt{\frac{1684063 - 201\sqrt{32272369}}{157378}} b_{12}^{\frac{5}{2}}, & a_{14} &= \frac{(-37823 + 9\sqrt{32272369}) \sqrt{\frac{1684063 - 201\sqrt{32272369}}{157378}}}{14164020} b_{12}^{\frac{7}{2}}, \\
 a_{05} &= \frac{-8189365715 + 1597749\sqrt{32272369}}{1114552569780} b_{12}^4, & b_{12} &> 0.
 \end{aligned} \tag{4.4}$$

*Proof.* Observe that  $\lambda_5 = \lambda_7 = 0$ ,  $\lambda_8 \neq 0$ , we get

$$80a_{50}^2 - 201a_{41}b_{12} - 40b_{12}^3 = 0, \tag{4.5}$$

and

$$3541005a_{41}^2 + 1107732a_{41}b_{12}^2 + 83680b_{12}^4 = 0, \tag{4.6}$$

with  $a_{50}b_{12}a_{41} \neq 0$ .

More explicitly, (4.5) can be put in the form

$$a_{50}^2 = \frac{1}{80} b_{12} (201a_{41} + 40b_{12}^2). \tag{4.7}$$

The equivalent expression of (4.6) is

$$3541005 \left( \frac{a_{41}}{b_{12}^2} \right)^2 + 1107732 \frac{a_{41}}{b_{12}^2} + 83680 = 0. \tag{4.8}$$

By solving the Equation (4.8), we obtain that

$$a_{41} = \frac{2(-92311 \pm 3\sqrt{32272369})}{1180335} b_{12}^2. \tag{4.9}$$

Hence (4.5) is simplified as

$$a_{50}^2 = \frac{1684063 \pm 201\sqrt{32272369}}{15737800} b_{12}^3, \tag{4.10}$$

where  $\frac{1684063 + 201\sqrt{32272369}}{15737800} \approx 0.179563 > 0$ ,  $\frac{1684063 - 201\sqrt{32272369}}{15737800} \approx 0.0344525 > 0$

then  $b_{12} > 0$ .

The fact  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_6 = 0$  follows that

$$\begin{aligned}
 b_{21} &= 0, \quad b_{03} = -\frac{5}{6} a_{50}, \quad a_{32} = \frac{1}{6} a_{50} b_{12}, \quad a_{14} = \frac{1}{36} a_{50} (45a_{41} + 8b_{12}^2), \\
 a_{05} &= -\frac{1}{5400} (90a_{41}^2 + 1551a_{41}b_{12}^2 + 280b_{12}^4).
 \end{aligned} \tag{4.11}$$

Without loss of generality, we assume that

$$a_{41} = \frac{2(-92311 + 3\sqrt{32272369})}{1180335} b_{12}^2, \quad a_{50} = \frac{1}{10} \sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}} b_{12}^{\frac{3}{2}}. \quad (4.12)$$

After substituting (4.12) into (4.11) we can obtain condition (4.1). At the moment, some easy computations lead us to

$$\lambda_8 = -\frac{(-9759524327217471 - 2029352919017\sqrt{32272369})(-92311 + 3\sqrt{32272369})\sqrt{\frac{1684063 + 201\sqrt{32272369}}{157378}}}{22808202724416130287435000} b_{12}^{\frac{15}{2}} \quad (4.13)$$

$$\approx -0.00029769 b_{12}^{\frac{15}{2}} \neq 0.$$

Following similar steps we can obtain the other three families of statement (4.2)-(4.4). Hence the claim is proved.

In arriving at our conclusions, we only need to show that, when one of the four sets of conditions in Theorem 4.1 holds, the Jacobian of the first eight quasi-Lyapunov constants of system (1.3) with respect to  $b_{21}$ ,  $b_{03}$ ,  $a_{32}$ ,  $a_{14}$ ,  $a_{50}$ ,  $a_{05}$ ,  $a_{41}$  are not equal to zero. An easy computation shows that

$$\begin{aligned} \frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)}{\partial(b_{21}, b_{03}, a_{32}, a_{14}, a_{50}, a_{05}, a_{41})} \Big|_{(4.1)} &= \frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)}{\partial(b_{21}, b_{03}, a_{32}, a_{14}, a_{50}, a_{05}, a_{41})} \Big|_{(4.2)} \\ &= -\frac{32(10924089213604647 + 2069952083969\sqrt{32272369})}{114161509909380574265625} b_{12}^9 \approx -6.35822 \times 10^{-6} b_{12}^9 \neq 0, \end{aligned} \quad (4.14)$$

$$\begin{aligned} \frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)}{\partial(b_{21}, b_{03}, a_{32}, a_{14}, a_{50}, a_{05}, a_{41})} \Big|_{(4.3)} &= \frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)}{\partial(b_{21}, b_{03}, a_{32}, a_{14}, a_{50}, a_{05}, a_{41})} \Big|_{(4.4)} \\ &= \frac{32(-10924089213604647 + 2069952083969\sqrt{32272369})}{114161509909380574265625} b_{12}^9 \approx 2.3407 \times 10^{-7} b_{12}^9 \neq 0. \end{aligned} \quad (4.15)$$

The above considerations imply the conclusion of Theorem 1.2.

## Appendix

We present here the *Mathematica* code for computing the quasi-Lyapunov constants at the nilpotent origin for system (1.3) based on the algorithm of Theorem 2.4:

$$c_{0,0} = 0, \quad c_{1,0} = 0, \quad c_{0,1} = 0, \quad c_{2,0} = 0, \quad c_{1,1} = 0, \quad c_{0,2} = 1;$$

when  $\alpha < 0$ , or  $\beta < 0$ ,  $c_{\alpha,\beta} = 0$ ;

else

$$\begin{aligned} c_{\alpha,\beta} &= (a_{50}(5 - (1+s)(-5+\alpha))c_{-5+\alpha,1+\beta} + a_{41}(4 - (1+s)(-4+\alpha))c_{-4+\alpha,\beta} \\ &\quad + 2(1+s)(2+\beta)c_{-4+\alpha,2+\beta} + a_{32}(3 - (1+s)(-3+\alpha))c_{-3+\alpha,-1+\beta} \\ &\quad + b_{21}(1 - (1+s)(1+\beta))c_{-3+\alpha,1+\beta} + b_{12}(2 - (1+s)\beta)c_{-2+\alpha,\beta} \\ &\quad + a_{14}(1 - (1+s)(-1+\alpha))c_{-1+\alpha,-3+\beta} + b_{03}(3 - (1+s)(-1+\beta)) \\ &\quad \times c_{-1+\alpha,-1+\beta} - a_{05}(1+s)\alpha c_{\alpha,-4+\beta})/(s+1)/\alpha, \\ \omega_m &= a_{50}(5 - (-4+m)(1+s))c_{-4+m,0} + a_{41}(4 - (-3+m)(1+s))c_{-3+m,-1} \\ &\quad + 2(1+s)c_{-3+m,1} + a_{32}(3 - (-2+m)(1+s))c_{-2+m,-2} + b_{21}c_{-2+m,0} \\ &\quad + b_{12}(3+s)c_{-1+m,-1} + a_{14}(1 - m(1+s))c_{m,-4} + b_{03}(3 + 2(1+s))c_{m,-2} \\ &\quad - a_{05}(1+m)(1+s)c_{1+m,-5}, \\ \lambda_m &= \frac{\omega_{2m+4}}{2m - 4s - 1}. \end{aligned}$$

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# Authors' contributions

YW completed the main part of this article, LL corrected the main theorems, FL enhanced the revised version. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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