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A note on Carlitz q-Bernoulli numbers and polynomials

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Abstract

In this article, we first aim to give simple proofs of known formulae for the generalized Carlitz q-Bernoulli polynomials $\beta_{m,\chi}(x, q)$ in the p-adic case by means of a method provided by Kim and then to derive a complex, analytic, two-variable q-L-function that is a q-analog of the two-variable L-function. Using this function, we calculate the values of two-variable q-L-functions at nonpositive integers and study their properties when q tends to 1.

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1. Introduction

Let p be a fixed prime. We denote by $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of a q-extension, q can be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1 - q|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log_p q)$ for $|x|_p \le 1$.

Let d be a fixed positive integer. Let

$$\begin{aligned} X = X_d &= \lim_{\overleftarrow{N}} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1.2}$$

Hence $\lim_{q\to 1} [x]_q = x$ for any $x \in \mathbb{C}$ in the complex case and any x with $|x|_p \le 1$ in the present p-adic case. This is the hallmark of a q-analog: The limit as $q \to 1$ recovers the classical object.

In 1937, Vandiver [1] and, in 1941, Carlitz [2] discussed generalized Bernoulli and Euler numbers. Since that time, many authors have studied these and other related subjects (see, e.g., [3-6]). The final breakthrough came in the 1948 article by Carlitz [7]. He defined inductively new q-Bernoulli numbers $\beta_m = \beta_m(q)$ by

$$\beta_0(q) = 1, \quad q(q\beta(q) + 1)^m - \beta_m(q) = \begin{cases} 1 \text{ if } m = 1 \\ 0 \text{ if } m > 1, \end{cases} \tag{1.3}$$

with the usual convention of β^i by β_i. The q-Bernoulli polynomials are defined by

$$\beta_m(x, q) = (q^x\beta(q) + [x]_q)^m = \sum_{i=0}^{m} \binom{m}{i} \beta_i(q)q^{ix}[x]_q^{m-i}. \tag{1.4}$$

In 1954, Carlitz [8] generalized a result of Frobenius [3] and showed many of the properties of the q-Bernoulli numbers $\beta_m(q)$. In 1964, Carlitz [9] extended the Bernoulli, Eulerian, and Euler numbers and corresponding polynomials as a formal Dirichlet series. In what follows, we shall call them the Carlitz q-Bernoulli numbers and polynomials.

Some properties of Carlitz q-Bernoulli numbers $\beta_m(q)$ were investigated by various authors. In [10], Koblitz constructed a q-analog of p-adic L-functions and suggested two questions. Question (1) was solved by Satoh [11]. He constructed a complex analytic q-L-series that is a q-analog of Dirichlet L-function and interpolates Carlitz q-Bernoulli numbers, which is an answer to Koblitz's question. By using a q-analog of the p-adic Haar distribution (see (1.6) below), Kim [12] answered part of Koblitz's question (2) and constructed q-analogs of the p-adic log gamma functions $G_{p,q}(x)$ on $\mathbb{C}_p \setminus \mathbb{Z}_p$.

In [11], Satoh constructed the generating function of the Carlitz q-Bernoulli numbers $F_q(t)$ in $\mathbb{C}$ which is given by

$$F_q(t) = \sum_{m=0}^{\infty} q^m e^{[m]_q t}(1 - q - q^m t) = \sum_{m=0}^{\infty} \beta_m(q)\frac{t^m}{m!}, \tag{1.5}$$

where q is a complex number with $0 < |q| < 1$. He could not explicitly determine $F_q(t)$ in $\mathbb{C}_p$, see [11, p.347].

In [12], Kim defined the q-analog of the p-adic Haar distribution $\mu_{\text{Haar}}(a + p^N \mathbb{Z}_p) = 1/p^N$ by

$$\mu_q(a + p^N\mathbb{Z}_p) = \frac{q^a}{[p^N]_q}. \tag{1.6}$$

Using this distribution, he proved that the Carlitz q-Bernoulli numbers $\beta_m(q)$ can be represented as the p-adic q-integral on $\mathbb{Z}_p$ by μ_q, that is,

$$\beta_m(q) = \int_{\mathbb{Z}_p} [a]_q^m d\mu_q(a), \tag{1.7}$$

and found the following explicit formula

$$\beta_m(q) = \frac{1}{(q-1)^m} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \frac{i+1}{[i+1]_q}, \tag{1.8}$$

where $m \geq 0$ and $q \in \mathbb{C}_p$ with $0 < |1 - q|_p < p^{-\frac{1}{p-1}}$.

Recently, Kim and Rim [13] constructed the generating function of the Carlitz q-Bernoulli numbers $F_q(t)$ in $\mathbb{C}_p$:

$$F_q(t) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{j+1}{[j+1]_q} (-1)^j \left(\frac{1}{1-q}\right)^j \frac{t^j}{j!}, \tag{1.9}$$

where $q \in \mathbb{C}_p$ with $0 < |1-q|_p < p^{-\frac{1}{p-1}}$.

This article is organized as follows.

In Section 2, we consider the generalized Carlitz q-Bernoulli polynomials in the p-adic case by means of a method provided by Kim. We obtain the generating functions of the generalized Carlitz q-Bernoulli polynomials. We shall provide some basic formulas for the generalized Carlitz q-Bernoulli polynomials which will be used to prove the main results of this article.

In Section 3, we construct the complex, analytic, two-variable q-L-function that is a q-analog of the two-variable L-function. Using this function, we calculate the values of two-variable q-L-functions at nonpositive integers and study their properties when q tends to 1.

2. Generalized Carlitz q-Bernoulli polynomials in the p-adic (and complex) case

For any uniformly differentiable function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic q-integral on $\mathbb{Z}_p$ is defined to be the limit $\frac{1}{[p^N]_q} \sum_{a=0}^{p^N-1} f(a) q^a$ as $N \to \infty$. The uniform differentiability guarantees the limit exists. Kim [12,14-16] introduced this construction, denoted $I_q(f)$, where $|1 - q|_p < p^{-1/(p-1)}$.

The construction of $I_q(f)$ makes sense for many q in $\mathbb{C}_p$ with the weaker condition $|1 - q|_p < 1$. Indeed, when $|1 - q|_p < 1$ the function q^x is uniformly differentiable and the space of uniformly differentiable functions $\mathbb{Z}_p \to \mathbb{C}_p$ is closed under multiplication, so we can make sense of its p-adic q-integral $I_q(f)$ for $|1 - q|_p < 1$.

Lemma 2.1. *For $q \in \mathbb{C}_p$ with $0 < |1 - q|_p < 1$ and $x \in \mathbb{Z}_p$, we have*

$$\lim_{N\to\infty} \frac{1}{1-q^{p^N}} \sum_{a=0}^{p^N-1} q^{ax} = \frac{x}{1-q^x}.$$

Proof. We assume that $q \in \mathbb{C}_p$ satisfies the condition $0 < |1 - q|_p < 1$. Then it is known that

$$q^x = \sum_{m=0}^{\infty} \binom{x}{m} (q-1)^m$$

for any $x \in \mathbb{Z}_p$ (see [[17], Lemma 3.1 (iii)]). Therefore, we obtain

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{1-q^{p^N}} \sum_{a=0}^{p^N-1} q^{ax} &= \frac{1}{1-q^x} \lim_{N\to\infty} \frac{\left(q^{p^N}\right)^x - 1}{q^{p^N}-1} \\
&= \frac{1}{1-q^x} \lim_{N\to\infty} \frac{\sum_{m=1}^{\infty} \binom{x}{m} \left(q^{p^N}-1\right)^m}{q^{p^N}-1} \\
&= \frac{1}{1-q^x} \lim_{N\to\infty} \sum_{m=0}^{\infty} \binom{x}{m+1} \left(q^{p^N}-1\right)^m \\
&= \frac{x}{1-q^x}.
\end{aligned}$$

This completes the proof.

Definition 2.2 ([12, §2, p. 323]). Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$ and let $x \in \mathbb{Z}_p$. For $q \in \mathbb{C}_p$ with $0 < |1 - q|_p < 1$ and an integer $m \geq 0$, the generalized Carlitz q-Bernoulli polynomials $\beta_{m,\chi}(x, q)$ are defined by

$$\begin{aligned}\beta_{m,\chi}(x,q) &= \int_X \chi(a)[x+a]_q^m d\mu_q(a) \\ &= \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{dp^N-1} \chi(a)[x+a]_q^m q^a.\end{aligned} \tag{2.1}$$

Remark 2.3. If $\chi = \chi^0$, the trivial character and $x = 0$, then (2.1) reduces to (1.7) since $d = 1$. In particular, Kim [12] defined a class of p-adic interpolation functions $G_{p,q}(x)$ of the Carlitz q-Bernoulli numbers $\beta_m(q)$ and gave several interesting applications of these functions.

By Lemma 2.1, we can prove the following explicit formula of $\beta_{m,\chi}(x, q)$ in $\mathbb{C}_p$.

Proposition 2.4. *For $q \in \mathbb{C}_p$ with $0 < |1 - q|_p < 1$ and an integer $m \geq 0$, we have*

$$\beta_{m,\chi}(x,q) = \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{[d(i+1)]_q}.$$

Proof. For $m \geq 0$, (2.1) implies

$$\begin{aligned}\beta_{m,\chi}(x,q) &= \lim_{N\to\infty} \frac{1}{[d]_q} \frac{1}{[p^N]_{q^d}} \sum_{k=0}^{d-1} \sum_{a=0}^{p^N-1} \chi(k+da)[x+k+da]_q^m q^{k+da} \\ &= \lim_{N\to\infty} \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) \frac{q^k}{1-q^{dp^N}} \sum_{q=0}^{p^N-1} \left(1-q^{x+k+da}\right)^m q^{da} \\ &= \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \\ &\quad \times \lim_{N\to\infty} \frac{1}{1-(q^d)^{p^N}} \sum_{a=0}^{p^N-1} \left(q^d\right)^{a(i+1)} \\ &= \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{1-q^{d(i+1)}}\end{aligned}$$

(where we use Lemma 2.1).

This completes the proof.

Remark 2.5. We note here that similar expressions to those of Proposition 2.4 with $\chi = \chi^0$ are given by Kamano [[18], Proposition 2.6] and Kim [12, §2]. Also, Ryoo et al. [19, Theorem 4] gave the explicit formula of $\beta_{m,\chi}(0, q)$ in $\mathbb{C}$ for $m \geq 0$.

Lemma 2.6. *Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$. Then for $q \in \mathbb{C}$ with $|q| < 1$,*

$$\sum_{m=0}^{\infty} \chi(m) q^{mx} = \frac{1}{1-q^{dx}} \sum_{k=0}^{d-1} \chi(k) q^{kx}.$$

Proof. If we write $m = ad + k$, where $0 \leq k \leq d - 1$ and $a = 0,1, 2,\ldots$, we have the desired result.

We now consider the case:

$$q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p, \quad 0 < |q| < 1, \quad 0 < |1-q|_p < 1. \tag{2.2}$$

For instance, if we set

$$q = \frac{1}{1-pz} \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$$

for each $z \neq 0 \in \mathbb{Z}$ and $p > 3$, we find $0 < |q| < 1$, $0 < |1 - q|_p < 1$.

Let $F_{q,\chi}(t, x)$ be the generating function of $\beta_{m,\chi}(x, q)$ defined in Definition 2.2. From Proposition 2.4, we have

$$\begin{aligned} F_{q,\chi}(t,x) &= \sum_{m=0}^{\infty} \beta_{m,\chi}(x,q)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\frac{1}{(1-q)^m}\sum_{k=0}^{d-1}\chi(k)q^k\sum_{i=0}^{m}\binom{m}{i}(-1)^i q^{i(x+k)}\frac{i+1}{[d(i+1)]_q}\right)\frac{t^m}{m!} \\ &= P_{q,\chi}(t,x) + Q_{q,\chi}(t,x), \end{aligned} \tag{2.3}$$

where

$$P_{q,\chi}(t,x) = \sum_{m=0}^{\infty}\frac{1}{(1-q)^m}\sum_{k=0}^{d-1}\chi(k)q^k\sum_{i=0}^{m}\binom{m}{i}(-1)^i q^{i(x+k)}\frac{i}{[d(i+1)]_q}\frac{t^m}{m!}$$

and

$$Q_{q,\chi}(t,x) = \sum_{m=0}^{\infty}\frac{1}{(1-q)^m}\sum_{k=0}^{d-1}\chi(k)q^k\sum_{i=0}^{m}\binom{m}{i}(-1)^i q^{i(x+k)}\frac{1}{[d(i+1)]_q}\frac{t^m}{m!}.$$

Then, noting that

$$e^{\frac{t}{1-q}} = \sum_{i=0}^{\infty}(-1)^i(q-1)^{-i}\frac{t^i}{i!},$$

we see that

$$\begin{aligned} P_{q,\chi}(t,x) &= \sum_{m=0}^{\infty}\frac{1}{(1-q)^m}\sum_{k=0}^{d-1}\chi(k)q^k\sum_{i=0}^{m}\binom{m}{i}(-1)^i q^{i(x+k)}\frac{i}{[d(i+1)]_q}\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\frac{1}{(1-q)^n}\frac{t^n}{n!}\sum_{j=0}^{\infty}\frac{1}{(q-1)^j}\sum_{k=0}^{d-1}\chi(k)q^{j(x+k)+k}\frac{j}{[d(j+1)]_q}\frac{t^j}{j!} \\ &= e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\left(\frac{1}{q-1}\right)^j\sum_{k=0}^{d-1}\chi(k)q^{j(x+k)+k}\frac{j}{[d(j+1)]_q}\frac{t^j}{j!}. \end{aligned} \tag{2.4}$$

Moreover, (2.4) now becomes

$$\begin{aligned} P_{q,\chi}(t,x) &= e^{\frac{t}{1-q}} \sum_{j=1}^{\infty} \left(\frac{1}{q-1}\right)^j \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{1}{[d(j+1)]_q} \frac{t^j}{(j-1)!} \\ &= e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1}\right)^j q^{(j+1)x} \sum_{k=0}^{d-1} \chi(k) \frac{q^{k(j+2)}}{q^{d(j+2)}-1} \frac{t^{j+1}}{j!} \\ &= -te^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1}\right)^j q^{(j+1)x} \sum_{n=0}^{\infty} \chi(n) q^{n(j+2)} \frac{t^j}{j!} \\ &\quad \text{(where we use Lemma 2.6)} \\ &= -te^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \chi(n) q^{x+2n} \sum_{j=0}^{\infty} \left(\frac{-q^{n+x}}{1-q}\right)^j \frac{t^j}{j!} \\ &= -te^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \chi(x) q^{x+2n} e^{\frac{(-q^{n+x})t}{1-q}} \\ &= -t \sum_{n=0}^{\infty} \chi(x) q^{x+2n} e^{[n+x]_q t} \end{aligned} \tag{2.5}$$

(cf. [13,16,20]). Similar arguments apply to the case $Q_{q,\chi}(t, x)$. We can rewrite

$$Q_{q,\chi}(t,x) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1}\right)^j \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{1}{[d(j+1)]_q} \frac{t^j}{j!} \tag{2.6}$$

and

$$Q_{q,\chi}(t,x) = (1-q) \sum_{n=0}^{\infty} \chi(n) q^n e^{[n+x]_q t}. \tag{2.7}$$

Then, by (2.4), (2.5), (2.6), and (2.7), we have the following theorem.

Theorem 2.7. *Let* $q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$, $0 < |q| < 1$, $0 < |1-q|_p < 1$. *Then the generalized Carlitz q-Bernoulli polynomials* $\beta_{m,\chi}(x, q)$ *for* $m \le 0$ *is given by equating the coefficients of powers of t in the following generating function:*

$$\begin{aligned} F_{q,\chi}(t,x) &= e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1}\right)^{j-1} \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{j+1}{q^{d(j+1)}-1} \frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty} \chi(n) q^n e^{[n+x]_q t} (1-q-q^{n+x}t). \end{aligned} \tag{2.8}$$

Remark 2.8. If $\chi = \chi 0$, the trivial character, and $x = 0$, (2.8) reduces to (1.5).

3. *q*-analog of the two-variable *L*-function (in $\mathbb{C}$)

From Theorem 2.7, for $k \ge 0$, we obtain the following

$$\begin{aligned} \beta_{k,\chi}(x,q) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k F_{q,\chi}(t,x)\Big|_{t=0} \\ &= (1-q) \sum_{m=0}^{\infty} \chi(m) q^m [m+x]_q^k - k \sum_{m=0}^{\infty} \chi(m) q^{x+2m} [m+x]_q^{k-1}. \end{aligned} \tag{3.1}$$

Hence we can define a q-analog of the L-function as follows:

Definition 3.1. Suppose that χ is a primitive Dirichlet character with conductor $d \in \mathbb{N}$. Let q be a complex number with $0 < |q| < 1$, and let $L_q(s, x, \chi)$ be a function of two-variable $(s, x) \in \mathbb{C} \times \mathbb{R}$ defined by

$$L_q(s, x, \chi) = \frac{1-q}{s-1}\sum_{m=0}^{\infty}\frac{\chi(m)q^m}{[m+x]_q^{s-1}} + \sum_{m=0}^{\infty}\frac{\chi(m)q^{m+2x}}{[m+x]_q^s} \tag{3.2}$$

for $0 < x \leq 1$ (cf. [11,13,14,21-25]).

In particular, the two-variable function $L_q(s, x, \chi)$ is a generalization of the one-variable $L_q(s, \chi)$ of Satoh [11], yielding the one-variable function when the second variable vanishes.

Proposition 3.2. *For $k \in \mathbb{Z}$, $k \geq 1$, the limiting value* $\lim_{s \to k} L_q(1 - s, x, \chi) = L_q(1 - k, x, \chi)$ *exists and is given explicitly by*

$$L_q(1-k, x, \chi) = -\frac{1}{k}\beta_{k,\chi}(x, q).$$

Proof. The proof is clear by Proposition 2.4, Theorem 2.7 and (3.1).

The formula of Proposition 3.2 is slight extension of the result in [19] and [11, Theorem 2].

Theorem 3.3. *For any positive integer k, we have*

$$\begin{aligned}\lim_{q\to 1}\beta_{k,\chi}(x,q) &= \lim_{q\to 1}\frac{1}{(1-q)^m}\sum_{k=0}^{d-1}\chi(k)q^k\sum_{i=0}^{m}\binom{m}{i}(-1)^i q^{i(x+k)}\frac{i+1}{[d(i+1)]_q}\\ &= B_{k,\chi}(x),\end{aligned}$$

where the $B_{k,\chi}(x)$ are the kth generalized Bernoulli polynomials.

Proof. We follow the proof in [[26], Theorem 1] motivated by the study of a simple q-analog of the Riemann zeta function. Recall that the ordinary Bernoulli polynomials $B_k(x)$ are defined by

$$\frac{i}{q^i-1}q^{ix} = \frac{1}{\log q}\frac{i\log q}{e^{i\log q}-1}e^{x(i\log q)} = \frac{1}{\log q}\sum_{k=0}^{\infty}B_k(x)i^k\frac{(\log q)^k}{k!}, \tag{3.3}$$

where it is noted that in this instance, the notation $B_k(x)$ is used to replace $B^k(x)$ symbolically. For each $m \geq 1$, let

$$(e^t-1)^m = \sum_{k=0}^{\infty}d_k^{(m)}\frac{t^k}{k!}. \tag{3.4}$$

Note that

$$(e^t-1)^m = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}e^{it} = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}i^k\right)\frac{t^k}{k!}. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$d_k^{(m)} = \begin{cases}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}i^k, & m \leq k\\ 0, & 0 \leq k < m.\end{cases} \tag{3.6}$$

It is also clear from the definition that $d_0^{(0)} = 1, d_k^{(0)} = 0$ and $d_k^{(k)} = k!$ for $k \in \mathbb{N}$. From (2.3), (3.3), and (3.6), we obtain

$$\begin{aligned}\beta_{m,\chi}(x,q) &= \frac{q^{-x}}{(q-1)^m}\sum_{k=0}^{d-1}\chi(k)q^{k+x}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}q^{i(k+x)}\frac{i+1}{[d(i+1)]_q}\\ &= \frac{q^{-x}}{(q-1)^{m-1}}\sum_{k=0}^{d-1}\chi(k)\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\\ &\quad\times e^{d(i+1)\log q\frac{(k+x)}{d}}\frac{d(i+1)\log q}{e^{d(i+1)\log q-1}}\frac{1}{d\log q}\\ &= \frac{q^{-x}}{(q-1)^{m-1}}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(i+1)^n\right)\\ &\quad\times d^{n-1}\sum_{k=0}^{d-1}\chi(k)B_n\left(\frac{k+x}{d}\right)\frac{(\log q)^{n-1}}{n!}\\ &= q^{-x}\frac{(\log q)^{m-1}}{(q-1)^{m-1}}d^{m-1}\sum_{k=0}^{d-1}\chi(k)B_m\left(\frac{k+x}{d}\right)\\ &\quad+q^{-x}\sum_{\sigma=1}^{\infty}\sum_{i=0}^{\sigma}\binom{m+\sigma}{i}d_{m+\sigma-i}^{(m)}\frac{1}{(m+\sigma)!}\frac{(\log q)^{m+\sigma-1}}{(q-1)^{m-1}}\\ &\quad\times d^{m+\sigma-1}\sum_{k=0}^{d-1}\chi(k)B_{m+\sigma}\left(\frac{k+x}{d}\right).\end{aligned}$$

Then, because

$$\log q = \log(1+(q-1)) = (q-1) - \frac{(q-1)^2}{2} + \cdots = (q-1) + O((q-1)^2)$$

as $q \to 1$, we find

$$\lim_{q\to 1}\frac{(\log q)^{m+\sigma-1}}{(q-1)^{m-1}} = \begin{cases}1, & \sigma = 0\\ 0, & \sigma \geq 1,\end{cases}$$

so

$$\lim_{q\to 1}\beta_{m,\chi}(x,q) = d^{m-1}\sum_{k=0}^{d-1}\chi(k)B_m\left(\frac{k+x}{d}\right) = B_{m,\chi}(x),$$

where the $B_{m,\chi}(x)$ are the mth generalized Bernoulli polynomials (e.g., [14,19]). This completes the proof.

Corollary 3.4. *For any positive integer k, we have*

$$\lim_{q\to 1}L_q(1-k,x,\chi) = -\frac{1}{k}B_{k,x}(x).$$

Remark 3.5. The formula of Theorem 3.3 is slight extension of the result in [[26], Theorem 1].

Remark 3.6. From Theorem 2.7, the generalized Bernoulli polynomials $B_{m,\chi}(x)$ are defined by means of the following generating function [[27], p. 8]

$$\begin{aligned}F_\chi(t,x) &:= \lim_{q\to 1} F_{q,\chi}(t,x)\\ &= -t\sum_{a=1}^{d}\sum_{l=0}^{\infty}\chi(a+dl)e^{(a+dl)t}e^{xt}\\ &= \sum_{a=1}^{d}\frac{\chi(a)te^{(a+x)t}}{e^{dt}-1}\\ &= \sum_{m=0}^{\infty}B_{m,\chi}(x)\frac{t^m}{m!}.\end{aligned}$$

Remark 3.7. If we substitute $\chi = \chi^0$, the trivial character, in Definition 3.1 and Corollary 3.4, we can also define a q-analog of the Hurwitz zeta function

$$\zeta(s,x) = \sum_{m=0}^{\infty}\frac{1}{(m+x)^s}$$

by

$$\zeta_q(s,x) = L_q(s,x,\chi^0) = \frac{1-q}{s-1}\sum_{m=0}^{\infty}\frac{q^{m+x}}{[m+x]_q^{s-1}} + \sum_{m=0}^{\infty}\frac{q^{2(m+x)}}{[m+x]_q^{s}}$$

and obtain the identity

$$\lim_{q\to 1}\zeta_q(s,x) = \zeta(s,x)$$

for all $s \neq 1$, as well as the formula

$$\lim_{q\to 1}\zeta_q(1-k,x) = -\frac{1}{k}B_k(x)$$

for integers $k \geq 1$ (cf. [11,13,19,22,24,25]).

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Authors' contributions
The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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