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A pexider difference for a pexider functional equation

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Abstract

We deal with a Pexider difference

$$f(2x+y)+f(2x-y)-g(x+y)-g(x-y)-2g(2x)+2g(x)$$

where f and g map be a given abelian group $(G, +)$ into a sequentially complete Hausdorff topological vector space. We also investigate the Hyers-Ulam stability of the following Pexiderized functional equation

$$f(2x+y)+f(2x-y)=g(x+y)+g(x-y)+2g(2x)-2g(x)$$

in topological vector spaces.

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1. Introduction and preliminaries

In 1940, Ulam [1] proposed the general stability problem: *Let G_1 be a group, G_2 be a metric group with the metric d. Given ε >0, does there exists δ >0 such that if a function h: $G_1 \to G_2$ satisfies the inequality*

$$d\left(h(xy)-h(x)h(y)\right)<\delta, \quad (x,y\in G_1),$$

then there is a homomorphism H: $G_1 \to G_2$ with

$$d(h(x),H(x))<\varepsilon, \quad (x\in G_1)?$$

Hyers [2] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1950, Aoki [3] extended the theorem of Hyers by considering the unbounded cauchy difference inequality

$$\left\|f(x+y)-f(x)-f(y)\right\|\leq\varepsilon(\|x\|^p+\|y\|^p) \quad (\varepsilon>0,\ p\,\varepsilon\,[0,1)).$$

In 1978, Rassias [4] also generalized the Hyers' theorem for linear mappings under the assumption $t \mapsto f(tx)$ is continuous in t for each fixed x.

Recently, Adam and Czerwik [5] investigated the problem of the Hyers-Ulam stability of a generalized quadratic functional equation in linear topological spaces. Najati and Moghimi [6] investigated the Hyers-Ulam stability of the functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x)$$

in quasi-Banach spaces. In this article, we prove that the Pexiderized functional equation

$$f(2x+y)+f(2x-y)=g(x+y)+g(x-y)+2g(2x)-2g(x)$$

is stable for functions *f*, *g* defined on an abelian group and taking values in a topological vector space.

Throughout this article, let G be an abelian group and X be a sequentially complete Hausdorff topological vector space over the field $\mathbb{Q}$ of rational numbers.

A mapping f: $G \to X$ is said to be *quadratic* if and only if it satisfies the following functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x\ y \in G$. A mapping $f\ G \to X$ is said to be *additive* if and only if it satisfies $f(x+y)=f(x)+f(y)$ for all $x\ y\ \varepsilon\ G$. For a given f: $G \to X$, we will use the following notation

$$Df(x,y):=f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-2f(2x)+2f(x).$$

For given sets $A\ B \subseteq X$ and a number $k \in \mathbb{R}$, we define the well known operations

$$A+B:=\{a+b: a\in A, b\in B\},\quad kA:=\{ka: a\in A\}.$$

We denote the convex hull of a set $U \subseteq X$ by conv(U) and by $\overline{U}$ the sequential closure of U. Moreover it is well know that:

(1) If $A \subseteq X$ are bounded sets, then conv(A) and $\overline{A}$ are bounded subsets of X.

(2) If A, $B \subseteq X$ and $\alpha\ \beta \in \mathbb{R}$, then $\alpha\ conv(A) + \beta\ \mathrm{conv}(B) = \mathrm{conv}(\alpha A + \beta B)$.

(3) Let X_1 and X_2 be linear spaces over $\mathbb{R}$. If f: $X_1 \to X_2$ is a additive (quadratic) function, then $f(rx) = rf(x)$ ($f(rx) = r^2 f(x)$), for all $x \in X_1$ and all $r \in \mathbb{Q}$.

2. Main results

We start with the following lemma.

Lemma 2.1. *Let G be a 2-divisible abelian group and $B \subseteq X$ be a nonempty set. If the functions f, g: $G \to X$ satisfy*

$$f(2x+y)+f(2x-y)-g(x+y)-g(x-y)-2g(2x)+2g(x)\in B \tag{2.1}$$

for all x, y $\in$ G, then

$$Df(x,y)\ \in\ 2\ \mathrm{conv}(B-B), \tag{2.2}$$

$$Dg(x,y)\in \mathrm{conv}(B-B) \tag{2.3}$$

for all x, y $\in$ G.

Proof. Putting $y = 0$ in (2.1), we get

$$2f(2x)-2g(2x)\in B \tag{2.4}$$

for all $x \in G$. If we replace x by $\frac{1}{2}x$ in (2.4), then we have

$$f(x) - g(x) \in \frac{1}{2}B \tag{2.5}$$

for all $x \in G$. It follows from (2.5) and (2.1) that

$$\begin{aligned} Df(x, y) &= f(2x+y) + f(2x-y) - g(x+y) - g(x-y) - 2g(2x) + 2g(x) \\ &\quad - [f(x+y) - g(x+y)] - [f(x-y) - g(x-y)] \\ &\quad - [2f(2x) - 2g(2x)] + [2f(x) - 2g(x)] \\ &\in 2\,\mathrm{conv}(B - B). \end{aligned}$$

Moreover, we have

$$\begin{aligned} Dg(x, y) &= f(2x+y) + f(2x-y) - g(x+y) - g(x-y) - 2g(2x) + 2g(x) \\ &\quad - [f(2x+y) - g(2x+y)] - [f(2x-y) - g(2x-y)] \\ &\in \mathrm{conv}(B - B). \end{aligned}$$

Theorem 2.2. *Let G be a 2-divisible abelian group and $B \subseteq X$ be a bounded set. Suppose that the odd functions f, g: $G \to X$ satisfy (2 1) for all $x, y \in G$. Then there exists exactly one additive function $\mathcal{A} : G \to X$ such that*

$$\mathcal{A}(x) - f(x) \in 4\overline{\mathrm{conv}(B - B)}, \quad \mathcal{A}(x) - g(x) \in 2\overline{\mathrm{conv}(B - B)} \tag{2.6}$$

for all $x \in G$. Moreover the function $\mathcal{A}$ is given by

$$\mathcal{A}(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} g(2^n x)$$

for all $x \in G$. Moreover, the convergence of the sequences are uniform on G.

Proof. By Lemma 2.1, we get (2.2). Setting $y = x$, $y = 3x$ and $y = 4x$ in (2.2), we get

$$f(3x) - 3f(2x) + 3f(x) \in 2\,\mathrm{conv}(B - B), \tag{2.7}$$

$$f(5x) - f(4x) - f(2x) + f(x) \in 2\mathrm{conv}(B - B), \tag{2.8}$$

$$f(6x) - f(5x) + f(3x) - 3f(2x) + 2f(x) \in 2\mathrm{conv}(B - B) \tag{2.9}$$

for all $x \in G$. It follows from (2.7), (2.8), and (2.9) that

$$f(6x) - f(4x) - f(2x) \in 6\,\mathrm{conv}(B - B)$$

for all $x \in G$. So

$$f(3x) - f(2x) - f(x) \in 6\,\mathrm{conv}(B - B) \tag{2.10}$$

for all $x \in G$. Using (2.7) and (2.10), we obtain

$$\frac{1}{2}f(2x) - f(x) \in 2\mathrm{conv}(B - B)$$

for all $x \in G$. Therefore

$$\begin{aligned}\frac{1}{2^n}f(2^nx)-\frac{1}{2^m}f(2^mx)&=\sum_{k=m}^{n-1}\left[\frac{1}{2^{k+1}}f(2^{k+1}x)-\frac{1}{2^k}f(2^kx)\right]\\&\in\sum_{k=m}^{n-1}\frac{2}{2^k}\operatorname{conv}(B-B)\\&\subseteq\frac{4}{2^m}\operatorname{conv}(B-B)\end{aligned}\tag{2.11}$$

for all $x \in G$ and all integers $n > m \geq 0$. Since B is bounded, we conclude that conv $(B - B)$ is bounded. It follows from (2.11) and boundedness of the set conv$(B - B)$ that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is (uniformly) Cauchy in X for all $x \in G$. Since X is a sequential complete topological vector space, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is convergent for all $x \in G$, and the convergence is uniform on G. Define

$$\mathcal{A}_1 : G \to X, \quad \mathcal{A}_1(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^nx).$$

Since conv$(B - B)$ is bounded, it follows from (2.2) that

$$D\mathcal{A}_1(x,y)=\lim_{n\to\infty}\frac{1}{2^n}Df(2^nx,2^ny)=0$$

for all $x\ y \in G$. So $\mathcal{A}_1$ is additive (see [6]). Letting $m = 0$ and $n\to\infty$ in (2.11), we get

$$\mathcal{A}_1(x)-f(x)\in 4\overline{\operatorname{conv}(B-B)}\tag{2.12}$$

for all $x \in G$. Similarly as before applying (2.3) we have an additive mapping $\mathcal{A}_2 : G \to X$ defined by $\mathcal{A}_2(x) := \lim_{n\to\infty}\frac{1}{2^n}g(2^nx)$ which is satisfying

$$\mathcal{A}_2(x)-g(x)\in 2\overline{\operatorname{conv}(B-B)}\tag{2.13}$$

for all $x \in G$. Since B is bounded, it follows from (2.5) that $\mathcal{A}_1 = \mathcal{A}_2$. Letting $\mathcal{A} := \mathcal{A}_1$, we obtain (2.6) from (2.12) and (2.13).

To prove the uniqueness of $\mathcal{A}$, suppose that there exists another additive function $\mathcal{A}'$: $G\to X$ satisfying (2.6). So

$$\mathcal{A}'(x)-\mathcal{A}(x)=[\mathcal{A}'(x)-f(x)]+[f(x)-\mathcal{A}(x)]\in 8\overline{\operatorname{conv}(B-B)}$$

for all $x \in G$. Since $\mathcal{A}'$ and $\mathcal{A}$ are additive, replacing x by 2^nx implies that

$$\mathcal{A}'(x)-\mathcal{A}(x)\in\frac{8}{2^n}\overline{\operatorname{conv}(B-B)}$$

for all $x \in G$ and all integers n. Since $\overline{\operatorname{conv}(B-B)}$ is bounded, we infer $\mathcal{A}' = \mathcal{A}$. This completes the proof of theorem.

Theorem 2.3 *Let G be a 2, 3-divisible abelian group and $B \subseteq X$ be a bounded set. Suppose that the even functions f, g: $G\to X$ satisfy (2 1) for all x, y $\in G$. Then there exists exactly one quadratic function $\mathcal{Q} : G \to X$such that*

$$\mathcal{Q}(x)-f(x)+f(0)\in 4\overline{\operatorname{conv}(B-B)},\ \ \mathcal{Q}(x)-g(x)+g(0)\in 2\overline{\operatorname{conv}(B-B)}$$

for all $x \in G$. Moreover, the function $\mathcal{Q}$ is given by

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x) = \lim_{n\to\infty} \frac{1}{4^n} g(2^n x)$$

for all $x \in G$. Moreover, the convergence of the sequences are uniform on G.

Proof. By replacing y by $x + y$ in (2.2), we get

$$\begin{aligned} f(3x+y)+f(x-y) &- f(2x+y) - f(y) \\ &- 2f(2x) + 2f(x) \in 2conv(B-B) \end{aligned} \tag{2.14}$$

for all $x, y \in G$. Replacing y by $-y$ in (2.14), we get

$$\begin{aligned} f(3x-y)+f(x+y) &- f(2x-y) - f(y) \\ &- 2f(2x) + 2f(x) \in 2\mathrm{conv}(B-B) \end{aligned} \tag{2.15}$$

for all $x, y \in G$. It follows from (2.2), (2.14), and (2.15) that

$$\begin{aligned} f(3x+y)+f(3x-y) &- 2f(y) \\ &- 6f(2x) + 6f(x) \in 6\mathrm{conv}(B-B) \end{aligned} \tag{2.16}$$

for all $x, y \in G$. By letting $y = 0$ and $y = 3x$ in (2.16), we get

$$2f(3x) - 6f(2x) + 6f(x) - 2f(0) \in 6\mathrm{conv}(B-B), \tag{2.17}$$

$$f(6x) - 2f(3x) - 6f(2x) + 6f(x) + f(0) \in 6\mathrm{conv}(B-B) \tag{2.18}$$

for all $x \in G$. Using (2.17) and (2.18), we obtain

$$f(6x) - 4f(3x) + 3f(0) \in 12\mathrm{conv}(B-B) \tag{2.19}$$

for all $x \in G$. If we replace x by $\frac{1}{3}x$ in (2.19), then

$$f(2x) - 4f(x) + 3f(0) \in 12\mathrm{conv}(B-B)$$

for all $x \in G$. Therefore

$$\frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^n} f(2^n x) + \frac{3}{4^{n+1}} f(0) \in \frac{3}{4^n}\mathrm{conv}(B-B) \tag{2.20}$$

for all $x \in G$ and all integers n. So

$$\begin{aligned} &\frac{1}{4^n} f(2^n x) - \frac{1}{4^m} f(2^m x) \\ &= \sum_{k=m}^{n-1} \frac{1}{4^{k+1}} f(2^{k+1}x) - \frac{1}{4^k} f(2^k x) \\ &\in - \sum_{k=m}^{n-1} \frac{3}{4^{k+1}} f(0) + \sum_{k=m}^{n-1} \frac{3}{4^k}\mathrm{conv}(B-B) \\ &\subseteq - \sum_{k=m}^{n-1} \frac{3}{4^{k+1}} f(0) + \frac{1}{4^{m-1}}\mathrm{conv}(B-B) \end{aligned} \tag{2.21}$$

for all $x \in G$ and all integers $n > m \geq 0$. It follows from (2.21) and boundedness of the set conv(B - B) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is (uniformly) Cauchy in X for all $x \in G$. The rest of the proof is similar to proof of of Theorem 2.2.

Remark 2.4. If the functions f, g: $G \rightarrow X$ satisfy (2.1), where f is even (odd) and g is odd (even), then it is easy to show that f and g are bounded.

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Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
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