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The growth of solutions of systems of complex q -shift difference equations

Hong-Yan Xu^{1*}, Ting-Bin Cao² and Bing-Xiang Liu¹

*Correspondence:
xhyhhh@126.com

¹Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi 333403, China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the properties of systems of two types of complex q -shift difference equations with meromorphic solutions from the point of view of Nevanlinna theory. Some results obtained in this paper improve and extend the previous theorems given by Gao, and five examples show the extension of solutions of the system of complex difference equation.

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1 Introduction and main results

In this paper, the problem of the existence and growth of solutions of complex difference equations will be studied. The fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used in this paper (see [1–3]). In addition, for a meromorphic function f , $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set E of finite logarithmic measure $\lim_{r \rightarrow \infty} \int_{[1, r] \cap E} \frac{dt}{t} < \infty$, $S_1(r, f)$ denotes any quantity satisfying $S_1(r, f) = o(T(r, f))$ for all r on a set F of logarithmic density 1 (or for all r outside of a possible exceptional set F' of logarithmic density 0, the logarithmic density of a set F is defined by

$$\limsup_{r \rightarrow \infty} \frac{1}{\log r} \int_{[1, r] \cap F} \frac{1}{t} dt.$$

Throughout this paper, the set E of finite logarithmic measure and F of logarithmic density will be not necessarily the same at each occurrence.

In recent years, there has been an increasing interest in studying difference equations, difference product and q -difference in the complex plane \mathbb{C} , considerable attention was paid to the growth of solutions of difference equations, value distribution and the uniqueness of differences analogues of Nevanlinna's theory [4–13]. Chiang and Feng [6], Halburd and Korhonen [14] established a difference analogue of the logarithmic derivative lemma independently, and Barnett, Halburd, Korhonen and Morgan [4] also established an analogue of the logarithmic derivative lemma on q -difference operators. After their works, many people applied it to prove a number of results on meromorphic solutions of complex difference equations.

In 2000, Ablowitz, Halburd and Herbst [15] studied some class of complex difference equations,

$$f(z + 1) + f(z - 1) = \frac{a_0(z) + a_1(z)f + \dots + a_p(z)f^p}{b_0(z) + b_1(z)f + \dots + b_q(z)f^q}, \tag{1}$$

$$f(z + 1)f(z - 1) = \frac{a_0(z) + a_1(z)f + \dots + a_p(z)f^p}{b_0(z) + b_1(z)f + \dots + b_q(z)f^q}, \tag{2}$$

where the coefficients are meromorphic functions, and obtained the following results.

Theorem 1.1 (see [15]) *If the difference equation (1) (or (2)) with polynomial coefficients $a_i(z)$, $b_i(z)$ admits a transcendental meromorphic solution of finite order, then $d = \max\{p, q\} \leq 2$.*

In 2001, Heittokangas *et al.* [16] further investigated some complex difference equations which are similar to (1) and (2) and obtained the following results which are the improvement of Theorems 1.1 and 1.2.

Theorem 1.2 (see [16, Proposition 8 and Proposition 9]) *Let $c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$. If the equations*

$$\sum_{i=1}^n f(z + c_i) = R(z, f(z)), \quad \prod_{i=1}^n f(z + c_i) = R(z, f(z)),$$

$$R(z, f(z)) := \frac{P(z, f(z))}{Q(z, f(z))} = \frac{a_0(z) + a_1(z)f + \dots + a_s(z)f^s}{b_0(z) + b_1(z)f + \dots + b_t(z)f^t}$$

with rational coefficients $a_i(z)$, $b_i(z)$ admit a transcendental meromorphic solution of finite order, then $d = \max\{s, t\}$.

In 2002, Gundersen *et al.* [17] studied the growth of meromorphic solutions of q -difference equations and obtained the result as follows.

Theorem 1.3 (see [17, Theorem 3.2]) *Suppose that f is a transcendental meromorphic solution of an equation of the form*

$$f(cz) = R(z, f(z)) = \frac{\sum_{i=0}^s a_i(z)f(z)^i}{\sum_{j=0}^t b_j(z)f(z)^j}$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$ and a constant $c \neq 0$ ($|c| > 1$), assuming that $d := \max\{s, t\} \geq 1$, $a_s(z) \neq 0$, $b_t(z) \neq 0$, and that $R(z, f(z))$ is irreducible in f . Then $\rho(f) = \frac{\log d}{\log |c|}$.

In 2010, Zheng and Chen [18] further considered the growth of meromorphic solutions of q -difference equations and obtained some results which extended the theorems given by Heittokangas *et al.* [16].

Theorem 1.4 (see [18]) *Suppose that f is a transcendental meromorphic solution of the equation of the form*

$$\frac{\sum_{\lambda \in I} d_\lambda(z)f(qz)^{i_{\lambda_1}}f(q^2z)^{i_{\lambda_2}} \dots f(q^n z)^{i_{\lambda_n}}}{\sum_{\mu \in J} e_\mu(z)f(qz)^{j_{\mu_1}}f(q^2z)^{j_{\mu_2}} \dots f(q^n z)^{j_{\mu_n}}} = R(z, f(z)), \tag{3}$$

where $I = \{(i_{\lambda_1}, i_{\lambda_2}, \dots, i_{\lambda_n})\}$, $J = \{j_{\mu_1}, j_{\mu_2}, \dots, j_{\mu_n}\}$ are two finite index sets satisfying

$$\max_{\lambda, \mu} \{i_{\lambda_1} + i_{\lambda_2} + \dots + i_{\lambda_n}, j_{\mu_1} + j_{\mu_2} + \dots + j_{\mu_n}\} = \sigma,$$

$q \in \mathbb{C} \setminus \{0\}$, $|q| > 1$, $d \geq 2$, and all coefficients of (3) are of growth $S(r, f)$. If $d = \max\{s, t\} > 2n\sigma$, then for sufficiently large r , $T(r, f) \geq K \left(\frac{d}{2n\sigma}\right)^{\frac{\log r}{n \log |q|}}$, where $K (>0)$ is a constant. Thus, the lower order of f satisfies $\mu(f) \geq \frac{\log d - \log 2n\sigma}{n \log |q|}$.

In 2011, Liu and Qi [11] investigated the properties of meromorphic solutions of a q -shift difference equation and obtained some results as follows.

Theorem 1.5 (see [11, Theorem 4.1]) *Suppose that f is a transcendental meromorphic solution of*

$$f(cz + \eta) = R(z, f(z)) = \frac{\sum_{i=0}^s a_i(z) f(z)^i}{\sum_{j=0}^t b_j(z) f(z)^j},$$

where the coefficients $a_i(z)$, $b_j(z)$ are rational, and $|c| > 1$. Assuming $R(z, f(z))$ is irreducible in f , $a_s(z)b_t(z) \neq 0$. If $s > t + 1$ and $m := s - t$, then $\lambda(1/f) \geq \log |m| / \log |c|$, provided that f has infinitely many poles.

In 2012, Gao [19] investigated a complex difference equation which is similar to (3) and obtained the following theorem.

Theorem 1.6 (see [19, Theorem 1.1]) *Let $c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$. If the equation*

$$\frac{\sum_{\lambda \in I} d_\lambda(z) f(z)^{i_{\lambda_0}} f(z + c_1)^{i_{\lambda_1}} f(z + c_2)^{i_{\lambda_2}} \dots f(z + c_n)^{i_{\lambda_n}}}{\sum_{\mu \in J} e_\mu(z) f(z)^{j_{\mu_0}} f(z + c_1)^{j_{\mu_1}} f(z + c_2)^{j_{\mu_2}} \dots f(z + c_n)^{j_{\mu_n}}} = R(z, f(z)) \tag{4}$$

has a transcendental meromorphic solution of finite order, then

$$\max\{s, t\} \leq \Gamma_1 + \Gamma_2,$$

where $\Gamma_1 = \max\{i_{\lambda_0} + i_{\lambda_1} + \dots + i_{\lambda_n}\}$, $\Gamma_2 = \max\{j_{\mu_0} + j_{\mu_1} + \dots + j_{\mu_n}\}$, $d_\lambda(z)$, $e_\mu(z)$, $a_i(z)$ and $b_j(z)$ are small functions of f .

Thus, the following questions arise naturally:

Question A Will the assertion of Theorem 1.6 remain valid if the transcendental meromorphic solution has infinite order?

Question B What will happen when $f(z + c_i)$ is replaced by $f(q^i z + c_i)$ for $i = 1, 2, \dots, n$ in Theorem 1.6?

In this paper, we investigate the above questions and obtain the following theorem.

Theorem 1.7 Let $q, c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$, and $d_\lambda(z)$, $e_\mu(z)$, $a_i(z)$ and $b_j(z)$ be of growth $o(T(r, f))$ without an exceptional set as $r \rightarrow \infty$. If f is a transcendental meromorphic solution of the equation of the form

$$\frac{\sum_{\lambda \in I} d_\lambda(z) f(z)^{i_{\lambda_0}} f(qz + c_1)^{i_{\lambda_1}} f(q^2z + c_2)^{i_{\lambda_2}} \cdots f(q^n z + c_n)^{i_{\lambda_n}}}{\sum_{\mu \in J} e_\mu(z) f(z)^{j_{\mu_0}} f(qz + c_1)^{j_{\mu_1}} f(q^2z + c_2)^{j_{\mu_2}} \cdots f(q^n z + c_n)^{j_{\mu_n}}} = R(z, f(z)) \tag{5}$$

and satisfies one of the following conditions:

- (i) $q = 1$ and $\rho_2(f) < 1$;
- (ii) $q \neq 1$ and $\rho(f) = 0$.

Then the conclusion of Theorem 1.6 still holds.

Remark 1.1 Theorem 1.7 is an improvement of Theorem 1.6. And Theorem 1.7 is an answer to Question A and Question B.

In this paper, we also investigated the extension of a meromorphic solution of the following system of complex q -shift difference equations of the form:

$$\begin{cases} P_1(z, f_1(qz + \eta)) = \sum_{J_1} a_{J_1}(z) (\prod_{j \in J_1} f_2(z + c_j)), \\ P_2(z, f_2(qz + \eta)) = \sum_{J_2} a_{J_2}(z) (\prod_{j \in J_2} f_1(z + c_j)), \end{cases} \tag{6}$$

and

$$\begin{cases} \Omega_1(z, f_1, f_2) = R_1(z, f_1), \\ \Omega_2(z, f_1, f_2) = R_2(z, f_2), \end{cases} \tag{7}$$

where $q, \eta, c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$, $\{J_\iota\}$ is a collection of all subsets of $\{1, 2, \dots, n_\iota\}$, $\iota = 1, 2$, two q -shift difference polynomials $\Omega_1(z, f_1, f_2)$, $\Omega_2(z, f_1, f_2)$ are defined by

$$\begin{aligned} \Omega_1(z, f_1, f_2) &= \Omega_1(z, f_1(z), f_2(z), f_1(qz + c_1), f_2(qz + c_1), \dots, f_1(q^n z + c_n), f_2(q^n z + c_n)) \\ &= \sum_{(i)} a_{(i)} \prod_{k=1}^2 f_k^{i_{k0}} (f_k(qz + c_1))^{i_{k1}} \cdots (f_k(q^n z + c_n))^{i_{kn}}, \\ \Omega_2(z, f_1, f_2) &= \Omega_2(z, f_1(z), f_2(z), f_1(qz + c_1), f_2(qz + c_1), \dots, f_1(q^n z + c_n), f_2(q^n z + c_n)) \\ &= \sum_{(j)} b_{(j)} \prod_{k=1}^2 f_k^{j_{k0}} (f_k(qz + c_1))^{j_{k1}} \cdots (f_k(q^n z + c_n))^{j_{kn}}, \end{aligned}$$

and $R_1(z, f_1)$, $R_2(z, f_2)$ are defined by

$$\begin{aligned} R_1(z, f_1) &= \frac{P_1(z, f_1)}{Q_1(z, f_1)} = \frac{\sum_{i=0}^{s_1} a_i^1(z) f_1^i}{\sum_{j=0}^{t_1} b_j^1(z) f_1^j}, \\ R_2(z, f_2) &= \frac{P_2(z, f_2)}{Q_2(z, f_2)} = \frac{\sum_{i=0}^{s_2} a_i^2(z) f_2^i}{\sum_{j=0}^{t_2} b_j^2(z) f_2^j}, \end{aligned}$$

and all coefficients $\{a_i^1(z)\}, \{a_i^2(z)\}, \{b_i^1(z)\}, \{b_i^2(z)\}, \{a_{(i)}(z)\}, \{b_{(i)}(z)\}$ and $a_{j_1}(z), a_{j_2}(z)$ are small functions of f_1, f_2 , and $a_{s_l}^l b_{t_l}^l \neq 0$ for $l = 1, 2$. Let

$$\begin{aligned} \Gamma_{11} &= \max_{(i)} \left\{ \sum_{v=0}^n i_{1v} \right\}, & \Gamma_{12} &= \max_{(i)} \left\{ \sum_{v=0}^n i_{2v} \right\}, \\ \Gamma_{21} &= \max_{(j)} \left\{ \sum_{v=0}^n j_{1v} \right\}, & \Gamma_{22} &= \max_{(j)} \left\{ \sum_{v=0}^n j_{2v} \right\}, \end{aligned}$$

and

$$S(r) := \sum_{l=1}^2 \left(\sum_{i=1}^n T(r, a_i^l) + \sum_{j=1}^n T(r, b_j^l) \right) + \sum_{(i)} T(r, a_{(i)}) + \sum_{(j)} T(r, b_{(j)}).$$

The order and hyper order of a meromorphic solution (f_1, f_2) of the system (6) are defined by

$$\rho = \rho(f_1, f_2) = \max\{\rho(f_1), \rho(f_2)\}, \quad \rho_2 = \rho_2(f_1, f_2) = \max\{\rho_2(f_1), \rho_2(f_2)\},$$

where

$$\rho(f_k) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f_k)}{\log r}, \quad \rho_2(f_k) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f_k)}{\log r}, \quad k = 1, 2.$$

To state our main results, we require the following definition.

Definition 1.8 (see [19]) If (f_1, f_2) is a meromorphic solution of the equation system (6), and its component f_k satisfies

$$\limsup_{r \rightarrow \infty, r \notin E} \frac{S(r)}{T(r, f_k)} = 0, \quad k = 1, 2,$$

then f_k is called admissible, where E is an exceptional set with finite logarithmic measure.

Theorem 1.9 For the system (6), $\{J_i\}$ are two collections of all non-empty subsets of $\{1, 2, \dots, n_i\}$ for $i = 1, 2$, c_j ($j = 1, 2, \dots, n$) are distinct complex constants, and $P_i(z, u)$ is a polynomial in u of $\deg_u = s_i$ (>0), its coefficients of $P_i(z, u)$ are all small functions of f_1, f_2 . Let (f_1, f_2) be a meromorphic solution of the system (6) such that f_1, f_2 are non-rational meromorphic, and all the coefficients of (6) are small functions relative to f_1, f_2 . Thus,

(i) if $0 < |q| < 1$ and $s_1 s_2 \geq n_1 n_2$, we have

$$\mu(f_1) + \mu(f_2) \geq \frac{\log s_1 s_2 - \log n_1 n_2}{-\log |q|}; \tag{8}$$

(ii) if $|q| > 1$ and $s_1 s_2 \leq n_1 n_2$, then

$$\rho(f_1) + \rho(f_2) \leq \frac{\log n_1 n_2 - \log s_1 s_2}{\log |q|}; \tag{9}$$

(iii) if $q = 1$ and $s_1 s_2 \geq n_1 n_2$, then $\mu(f_1) + \mu(f_2) \geq \infty$, where $\mu(f)$ is the lower order of f .

Some examples will show that the conclusions (8) and (9) in Theorem 1.9 are sharp.

Example 1.1 The function $(f_1(z), f_2(z)) = (e^z, e^{-z})$ satisfies the system

$$\begin{cases} [f_1(z+c)]^2[f_1(z-c)]^2 + f_1(z+2c) + f_1(z-2c) \\ \quad = a_1(z)(f_2(qz+\eta))^8 + b_1(z)(f_2(qz+\eta))^2, \\ [f_2(z+c)]^2[f_2(z-c)]^2 + f_2(z+2c) + f_2(z-2c) \\ \quad = a_2(z)(f_1(qz+\eta))^8 + b_2(z)(f_1(qz+\eta))^2, \end{cases}$$

where c, η are any nonzero complex constants, $q = -\frac{1}{2}$, and

$$a_1(z) = e^{8\eta}, \quad b_1(z) = e^{2\eta}(e^{2c} + e^{-2c}), \quad a_2(z) = e^{-8\eta}, \quad b_2(z) = e^{-2\eta}(e^{2c} + e^{-2c}).$$

Thus, we have

$$\mu(f_1) + \mu(f_2) = 1 + 1 = \frac{\log s_1 s_2 - \log n_1 n_2}{-\log |q|},$$

where $s_1 = s_2 = 8$, $n_1 = n_2 = 4$ and $|q| = \frac{1}{2} < 1$. This example shows that the equality in (8) can be arrived at.

Example 1.2 The function $(f_1(z), f_2(z)) = (e^{z^2}, e^{(z+1)^2})$ satisfies the system

$$\begin{cases} f_1(z+c) + f_1(z-c) = a_1(z)(f_2(qz+\eta))^4, \\ f_2(z+c) + f_2(z-c) = a_2(z)(f_1(qz+\eta))^4, \end{cases}$$

where c is any nonzero complex constant, $q = \frac{1}{2}$, $\eta = -1$, and

$$a_1(z) = e^{c^2}(e^{2zc} + e^{-2zc}), \quad a_2(z) = e^{(c+1)^2}(e^{2z(c+1)} + e^{-2z(c+1)-4c})e^{4z-4}.$$

We note that $a_1(z), a_2(z)$ are small functions relative to $e^{z^2}, e^{(z+1)^2}$. Thus, we have

$$\mu(f_1) + \mu(f_2) = 4 > 2 = \frac{\log s_1 s_2 - \log n_1 n_2}{-\log |q|},$$

where $s_1 = s_2 = 4$, $n_1 = n_2 = 2$ and $|q| = \frac{1}{2} < 1$. This example shows that the inequality (8) is true.

Example 1.3 The function $(f_1(z), f_2(z)) = (e^z, e^{-z})$ satisfies the system

$$\begin{cases} f_1(z+c)f_1(z-c)f_1(z+2c)f_1(z-2c) + f_1(z+c)f_1(z-c) \\ \quad = a_1(z)(f_2(qz+\eta))^2 + b_1(z)f_2(qz+\eta), \\ f_2(z+c)f_2(z-c)f_2(z+2c)f_2(z-2c) + f_2(z+c)f_2(z-c) \\ \quad = a_2(z)(f_1(qz+\eta))^2 + b_2(z)f_1(qz+\eta), \end{cases}$$

where c, η are any nonzero complex constants, $q = -2$, and

$$a_1(z) = e^{2\eta}, \quad b_1(z) = e^\eta, \quad a_2(z) = e^{-2\eta}, \quad b_2(z) = e^{-\eta}.$$

Thus, we have

$$\rho(f_1) + \rho(f_2) = 2 = \frac{\log n_1 n_2 - \log s_1 s_2}{\log |q|},$$

where $n_1 = n_2 = 4$, $s_1 = s_2 = 2$ and $|q| = 2 > 1$. This example shows that the equality in (9) can be arrived at.

Example 1.4 The function $(f_1(z), f_2(z)) = (e^{z^2}, e^{(z+1)^2})$ satisfies the system

$$\begin{cases} d_1(z)f_1(z+c)f_1(z-c)f_1(z+2c)f_1(z-2c)f_1(z+3c)f_1(z-3c) \\ \quad \times f_1(z+4c)f_1(z-4c) + d_2(z)f_1(z+c)f_1(z-c)f_1(z+\frac{c}{2})f_1(z-\frac{c}{2}) \\ \quad = a_1(z)(f_2(qz+\eta))^2 + b_1(z)f_2(qz+\eta), \\ e_1(z)f_2(z+c)f_2(z-c)f_2(z+2c)f_2(z-2c)f_2(z+3c)f_2(z-3c) \\ \quad \times f_2(z+4c)f_2(z-4c) + e_2(z)f_2(z+c)f_2(z-c)f_2(z+\frac{c}{2})f_2(z-\frac{c}{2}) \\ \quad = a_2(z)(f_1(qz+\eta))^2 + b_2(z)f_1(qz+\eta), \end{cases}$$

where c is a nonzero constant, $q = 2$, $\eta = -1$,

$$d_1(z) = e^{-60c^2}, \quad d_2(z) = e^{-\frac{5}{2}c^2}, \quad e_1(z) = e^{-60c^2-8}e^{-16z}, \quad e_2(z) = e^{-\frac{5}{2}c^2-4}e^{-8z},$$

and

$$a_1(z) = 1, \quad b_1(z) = 1, \quad a_2(z) = e^{8z-2}, \quad b_2(z) = e^{4z-1}.$$

We note that $a_i(z)$, $b_i(z)$, $d_i(z)$, $e_i(z)$, $i = 1, 2$ are small functions relative to e^{z^2} , $e^{(z+1)^2}$. Thus, we have

$$\rho(f_1) + \rho(f_2) = 4 \leq \log_2 25 = \frac{\log n_1 n_2 - \log s_1 s_2}{\log |q|},$$

where $n_1 = n_2 = 10$, $s_1 s_2 = 2$ and $|q| = 2 > 1$. This example shows that the inequality in (9) is true.

Example 1.5 The function $(f_1(z), f_2(z)) = (e^{e^z}, e^{e^{-z}})$ satisfies the following system:

$$\begin{cases} f_1(z+2\log 2) + f_1(z+3\log 2) + f_1(z+2\log 2)f_1(z+3\log 2) \\ \quad = f_2(-z-2\log 2) + (f_2(-z-2\log 2))^2 + (f_2(-z-2\log 2))^3, \\ f_2(z+2\log 2) + f_2(z+3\log 2) + f_2(z+2\log 2)f_2(z+3\log 2) \\ \quad = f_2(-z-2\log 2) + (f_2(-z-2\log 2))^2 + (f_2(-z-2\log 2))^3. \end{cases}$$

We have $\mu(f_1) + \mu(f_2) = \infty$. Thus, it shows that (iii) in Theorem 1.9 is true when $s_1 = s_2 = n_1 = n_2 = 3$, $c_1 = 2 \log 2$, $c_2 = 3 \log 2$, $q = -1$ and $\eta = -2 \log 2$.

Theorem 1.10 Let (f_1, f_2) be an admissible meromorphic solution of (7), $\max\{s_1, t_1\} > \Gamma_{11}$, $\max\{s_2, t_2\} > \Gamma_{22}$ and satisfy one of the following conditions:

- (i) $q = 1$ and $\rho_2(f_1, f_2) < 1$;
- (ii) $q \neq 1$ and $\rho(f_1, f_2) = 0$.

Then

$$(\max\{s_1, t_1\} - \Gamma_{11})(\max\{s_2, t_2\} - \Gamma_{22}) \leq \Gamma_{12}\Gamma_{21}.$$

Theorem 1.11 *Let (f_1, f_2) be an admissible meromorphic solution of (7), the coefficients $\{a_i^1(z)\}$, $\{a_i^2(z)\}$, $\{b_i^1(z)\}$, $\{b_i^2(z)\}$, $\{a_{(i)}(z)\}$ and $\{b_{(i)}(z)\}$ of (7) be meromorphic functions and not necessarily small functions of f_1, f_2 and*

$$\max\{s_1, t_1\} > \Gamma_{11}, \quad \max\{s_2, t_2\} > \Gamma_{22}.$$

If f_1, f_2 and q satisfy one of the following conditions:

- (i) $q = 1$ and $\rho_2(f_1, f_2) < 1$;
- (ii) $q \neq 1$ and $\rho(f_1, f_2) = 0$.

Then the solutions in the pair (f_1, f_2) are either both admissible or both non-admissible.

Remark 1.2 It is easy to see that Theorem 1.7, Theorem 1.10 and Theorem 1.11 are the improvement of some results in [19].

2 Some lemmas

Lemma 2.1 (Valiron-Mohon'ko [20]) *Let $f(z)$ be a meromorphic function. Then, for all irreducible rational functions in f ,*

$$R(z, f(z)) = \frac{\sum_{i=0}^m a_i(z)f(z)^i}{\sum_{j=0}^n b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Lemma 2.2 ([21, Theorem 5.1]) *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$, $\varsigma < 1$, ε be a number small enough. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{1-\varsigma-\varepsilon}}\right) = S(r, f)$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.3 ([21, Lemma 8.3]) *Let $T : [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing continuous function and let $s \in (0, +\infty)$. If the hyper order of T is strictly less than one, that is,*

$$\limsup_{r \rightarrow \infty} \frac{\log \log T(r)}{\log r} < 1,$$

and $\delta \in (0, 1 - \zeta)$, then

$$T(r + s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right)$$

for all r runs to infinity outside of a set of finite logarithmic measure.

From Lemma 2.3, we can get the following lemma easily.

Lemma 2.4 *Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and c be a non-zero complex constant. Then we have*

$$T(r, f(z + c)) = T(r, f(z)) + S(r, f), \quad N(r, f(z + c)) = N(r, f(z)) + S(r, f).$$

Lemma 2.5 ([22, p.36] or [13, Theorem 1.1 and Theorem 1.3]) *Let $f(z)$ be a transcendental meromorphic function of zero order and q be a nonzero complex constant. Then*

$$T(r, f(qz)) = T(r, f(z)) + S_1(r, f), \quad N(r, f(qz)) = N(r, f(z)) + S_1(r, f),$$

on a set of logarithmic density 1 or for all r outside of a possible exceptional set of logarithmic density 0.

Lemma 2.6 *Let $f(z)$ be a transcendental meromorphic function of zero order and q, c be two nonzero complex constants. Then*

$$T(r, f(qz + c)) = T(r, f(z)) + S_1(r, f) \quad \text{and} \quad N(r, f(qz + c)) \leq N(r, f) + S_1(r, f).$$

Proof By using the idea of [22, p.47], we can get the conclusions of Lemma 2.6 easily. Now, we will give another method to prove this lemma as follows.

From Lemma 2.4 and Lemma 2.5, we get the first equality of this lemma.

Next, the idea of the proof of the other inequalities of this lemma is from [8, 23]. Since f is of zero order, by Lemma 2.5, we have

$$\begin{aligned} N(r, f(qz + c)) &= N\left(r, f\left(q\left(z + \frac{c}{q}\right)\right)\right) \\ &= N\left(r, f\left(z + \frac{c}{q}\right)\right) + S_1\left(r, f\left(z + \frac{c}{q}\right)\right) \\ &\leq N\left(r + \left|\frac{c}{q}\right|, f\right) + S_1\left(r, f\left(z + \frac{c}{q}\right)\right). \end{aligned} \tag{10}$$

By Lemma 2.5 and since f is a meromorphic function of zero order, we have $S_1(r, f(z + \frac{c}{q})) = S_1(r, f)$. And by Lemma 2.3, we have

$$N\left(r + \left|\frac{c}{q}\right|, f\right) \leq N(r, f) + S(r, f), \tag{11}$$

outside of a possible exceptional set with finite logarithmic measure.

From (10), (11) and $N(r, f) \leq N(r + |\frac{c}{q}|, f)$, we have $N(r + |\frac{c}{q}|, f) = N(r, f) + S(r, f)$ and $N(r, f(qz + c)) \leq N(r, f) + S_1(r, f)$. \square

Lemma 2.7 ([4, 14] or [11, Theorem 2.1]) *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz + \eta)}{f(z)}\right) = S(r, f),$$

on a set of logarithmic density 1 for all r outside of a possible exceptional set of logarithmic density 0.

Lemma 2.8 ([22, p.44] or [19, Lemma 2.5]) *If*

$$\limsup_{r \rightarrow \infty, r \notin E_1} \frac{S(r)}{T(r, f_1)} = 0, \quad T(r, f_2) = O(S(r)) \quad (r \notin E_2),$$

then

$$\limsup_{r \rightarrow \infty, r \notin E_1 \cup E_2} \frac{T(r, f_2)}{T(r, f_1)} = 0,$$

where E_1, E_2 are sets with finite logarithmic measure.

Lemma 2.9 *Let $q, c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$, f_1, f_2 be two transcendental meromorphic functions and $T(r, a_{(i)}) = o(T(r, f_k))$, $k = 1, 2$,*

$$\Omega_1(z, f_1, f_2) = \sum_{(i)} a_{(i)} \prod_{k=1}^2 f_k^{i_{k0}} (f_k(qz + c_1))^{i_{k1}} \cdots (f_k(q^n z + c_n))^{i_{kn}},$$

where (i) is an index set consisting of finite elements.

If $q = 1$ and $\rho_2(f_k) < 1$ for $k = 1, 2$, then

$$T(r, \Omega_1(z, f_1, f_2)) \leq \sum_{k=1}^2 \Gamma_{1k} T(r, f_k) + M + o(T(r, f_1) + T(r, f_2))$$

for all r outside of a possible exceptional set E of finite logarithmic measure.

If $q \neq 1$ and $\rho(f_k) = 0$ for $k = 1, 2$, then

$$T(r, \Omega_1(z, f_1, f_2)) \leq \sum_{k=1}^2 \Gamma_{1k} T(r, f_k) + M + o(T(r, f_1) + T(r, f_2))$$

for all r on a set F of logarithmic density 1 or for all r outside of a possible exceptional set F' of logarithmic density 0, where Γ_{1k} , $k = 1, 2$ are as stated in Section 1, M a real constant and not necessarily the same at each occurrence.

Proof Set $G_{(i)} = a_{(i)}(z) \prod_{k=1}^2 (f_k(z))^{i_{k0}} (f_k(qz + c_1))^{i_{k1}} \cdots (f_k(q^n z + c_n))^{i_{kn}}$.

Let z_1 be a pole of $\Omega_1(z, f_1, f_2)$ of order τ that is not the zero and pole of $a_{(i)}(z)$, then there exists at least one index $i \in \{(i)\}$ such that $G_{(i)}$ has a pole at z_1 of order $\tau_1 \geq \tau$. Thus, there exists a subset $\{\sigma_0, \sigma_1, \dots, \sigma_\mu\}$ of $\{0, 1, 2, \dots, n\}$ such that each one of $f_k(z), f_k(qz +$

$c_1, \dots, f_k(q^n z + c_n)$ ($k = 1, 2$) has a pole at z_1 . If $m_{\sigma_j} (\geq 1)$ ($j = 0, 1, \dots, \mu$) are the order of $f(q^{\sigma_j} z + c_{\sigma_j})$ at z_1 respectively, then we have

$$\begin{aligned} \tau &\leq \tau_1 = m_{\sigma_0} i_{\sigma_0} + m_{\sigma_1} i_{\sigma_1} + \dots + m_{\sigma_\mu} i_{\sigma_\mu} \\ &\leq m_{k0} i_{k0} + m_{k1} i_{k1} + \dots + m_{kn} i_{kn}. \end{aligned}$$

Hence, we have $n(r, \Omega_1(z, f_1, f_2)) \leq \sum_{k=1}^2 i_{k0}(n(r, f_k(z))) + i_{k1}n(r, f_k(qz + c_1)) + \dots + i_{kn}n(r, f_k(q^n z + c_n)) + \sum_{(i)} n(r, a_{(i)})$, that is,

$$\begin{aligned} N(r, \Omega_1(z, f_1, f_2)) &\leq \sum_{k=1}^2 [i_{k0}N(r, f_k(z)) + i_{k1}N(r, f_k(qz + c_1)) + \dots \\ &\quad + i_{kn}N(r, f_k(q^n z + c_n))] + \sum_{(i)} N(r, a_{(i)}) + M. \end{aligned} \tag{12}$$

We can rewrite $G_{(i)}$ in the following form:

$$G_{(i)} = a_{(i)}(z) \prod_{k=1}^2 (f_k(z))^{i_{k0} + i_{k1} + \dots + i_{kn}} \left(\frac{f_k(qz + c_1)}{f_k} \right)^{i_{k1}} \dots \left(\frac{f_k(q^n z + c_n)}{f_k} \right)^{i_{kn}}.$$

It follows

$$\begin{aligned} m(r, G_{(i)}) &\leq m(r, a_{(i)}) + \sum_{k=1}^2 (i_{k0} + i_{k1} + \dots + i_{kn})m(r, f_k) + M \\ &\quad + O \left\{ \sum_{p=1}^n \left(m \left(r, \frac{f_1(q^p z + c_p)}{f_1(z)} \right) + m \left(r, \frac{f_2(q^p z + c_p)}{f_2(z)} \right) \right) \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} m(r, \Omega_1(r, f_1, f_2)) &\leq \sum_{(i)} m(r, a_{(i)}) + \sum_{k=1}^2 \max\{i_{k0} + i_{k1} + \dots + i_{kn}\}m(r, f_k) \\ &\quad + M + O \left\{ \sum_{p=1}^n \left(m \left(r, \frac{f_1(q^p z + c_p)}{f_1(z)} \right) + m \left(r, \frac{f_2(q^p z + c_p)}{f_2(z)} \right) \right) \right\}. \end{aligned} \tag{13}$$

We will consider two cases as follows.

Case 1. If $q = 1$ and $\rho_2(f_k) < 1$. By Lemma 2.2, since the coefficients $a_{(i)}(z)$ are small functions of f_1, f_2 , and from (13) and the definitions of $S(r, f), S_1(r, f)$, we have

$$m(r, \Omega_1(z, f_1, f_2)) \leq \Gamma_{11}m(r, f_1) + \Gamma_{12}m(r, f_2) + M + o(T(r, f_1) + T(r, f_2)), \quad r \notin E, \tag{14}$$

where E is a set of finite logarithmic measure and Γ_{11}, Γ_{12} are as stated in Section 1.

Since $q = 1$ and $\rho_2(f_k) < 1$, by Lemma 2.3 and Lemma 2.4, from (12), we have

$$N(r, \Omega_1(z, f_1, f_2)) \leq \sum_{(i)} N(r, a_{(i)}) + \sum_{k=1}^2 \Gamma_{1k}N(r, f_k) + o(T(r, f_1) + T(r, f_2)) + M \tag{15}$$

for all r outside of a possible exceptional set E of finite logarithmic measure. From (14) and (15), we get

$$T(r, \Omega_1(z, f_1, f_2)) \leq \sum_{k=1}^2 \Gamma_{1k} T(r, f_k) + M + o(T(r, f_1) + T(r, f_2))$$

for all r outside of a possible exceptional set E of finite logarithmic measure.

Case 2. If $q \neq 1$ and $\rho(f_k) = 0$ for $k = 1, 2$. By using the same argument as in Case 1, from Lemma 2.6, Lemma 2.7 and Lemma 2.8, we can get

$$T(r, \Omega_1(z, f_1, f_2)) \leq \sum_{k=1}^2 \Gamma_{1k} T(r, f_k) + M + o(T(r, f_1) + T(r, f_2))$$

for all r on a set of logarithmic density 1. □

Lemma 2.10 ([24, Lemma 4]) *Suppose that a meromorphic function f has finite lower order λ . Then, for every constant $c > 1$ and a given ε , there exists a sequence $r_n = r_n(c, \varepsilon) \rightarrow \infty$ such that*

$$T(cr_n, f) \leq c^{\lambda+\varepsilon} T(r_n, f).$$

Lemma 2.11 ([25]) *Let $f(z)$ be a transcendental meromorphic function and $p(z) = p_k z^k + p_{k-1} z^{k-1} + \dots + p_1 z + p_0$ be a complex polynomial of degree $k > 0$. For given $0 < \delta < |p_k|$, let $\lambda = |p_k| + \delta$, $\mu = |p_k| - \delta$, then for given $\varepsilon > 0$ and for r large enough,*

$$(1 - \varepsilon)T(\mu r^k, f) \leq T(r, f \circ p) \leq (1 + \varepsilon)T(\lambda r^k, f).$$

Lemma 2.12 ([26, 27] or [5]) *Let $g : (0, +\infty) \rightarrow R$, $h : (0, +\infty) \rightarrow R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E with finite linear measure, or $g(r) \leq h(r)$, $r \notin H \cup (0, 1]$, where $H \subset (1, \infty)$ is a set of finite logarithmic measure. Then, for any $\alpha > 1$, there exists r_0 such that $g(r) \leq h(\alpha r)$ for all $r \geq r_0$.*

3 The proof of Theorem 1.9

From the assumptions of Theorem 1.9, we get that f_1, f_2 are transcendental meromorphic functions.

Denote $\Psi_i(r) = \max\{T(r, a_j^i(z)) | j = 1, 2, \dots, s_i\}$, $i = 1, 2$, and $C = \max\{|c_1|, |c_2|, \dots, |c_n|\}$. Since $T(r, f(z+c)) \leq (1 + o(1))T(r + |c|, f) + M$ (Ref. [15]), by applying Lemma 2.1 to (6) and using the same argument as in Lemma 2.9, we have

$$\begin{aligned} s_1 T(r, f_1(qz + \eta)) + \Psi_1(r) &= T(r, P_1(z, f_1(qz + \eta))) \\ &= T\left(r, \sum_{j=1}^{m_1} \alpha_{j_1}(z) \left(\prod_{j \in I_1} f_2(z + c_j)\right)\right) \leq \sum_{j=1}^{m_1} T(r, f_2(z + c_j)) + S(r, f_2) \\ &\leq \left(1 + \frac{\varepsilon_1}{2}\right) \sum_{j=1}^{m_1} T(r + C, f_2) + S(r, f_2) \\ &\leq n_1 \left(1 + \frac{\varepsilon_1}{2}\right) T(\beta_1 r, f_2) + S(r, f_2), \end{aligned} \tag{16}$$

$$\begin{aligned}
 s_2 T(r, f_2(qz + \eta)) + \Psi_2(r) &= T(r, P_2(z, f_2(qz + \eta))) \\
 &= T\left(r, \sum_{j_2} \alpha_{j_2}(z) \left(\prod_{j \in j_2} f_1(z + c_j)\right)\right) \leq \sum_{j=1}^{n_2} f_1(z + c_j) + S(r, f_1) \\
 &\leq \left(1 + \frac{\varepsilon_2}{2}\right) \sum_{j=1}^{n_2} T(r + C, f_1) + S(r, f_1) \\
 &\leq n_1 \left(1 + \frac{\varepsilon_2}{2}\right) T(\beta_2 r, f_1) + S(r, f_1) \tag{17}
 \end{aligned}$$

for sufficiently large r and any given $\beta_i > 1$, $\varepsilon_i > 0$, $i = 1, 2$. By Lemma 2.11 and (16), (17), for $\theta_i = |q| - \delta_i$ ($0 < \delta_i < |q|$, $0 < \theta_i < 1$), $i = 1, 2$ and sufficiently large r , we get

$$\begin{aligned}
 s_1(1 - \varepsilon_1) T(\theta_1 r, f_1) &\leq n_1(1 + \varepsilon_1) T(\beta_1 r, f_2), \\
 s_2(1 - \varepsilon_2) T(\theta_2 r, f_2) &\leq n_2(1 + \varepsilon_2) T(\beta_2 r, f_1),
 \end{aligned}$$

outside of a possible exceptional set E_1, E_2 of finite linear measure, respectively. From Lemma 2.12, for any given $\gamma_i > 1$ ($i = 1, 2$) and sufficiently large r , we can obtain

$$\begin{aligned}
 s_1(1 - \varepsilon_1) T(\theta_1 r, f_1) &\leq n_1(1 + \varepsilon_1) T(\beta_1 \gamma_1 r, f_2), \\
 s_2(1 - \varepsilon_2) T(\theta_2 r, f_2) &\leq n_2(1 + \varepsilon_2) T(\beta_2 \gamma_2 r, f_1),
 \end{aligned}$$

that is,

$$\frac{s_1(1 - \varepsilon_1)}{n_1(1 + \varepsilon_1)} T(r, f_1) \leq T\left(\frac{\beta_1 \gamma_1}{\theta_1} r, f_2\right), \quad \frac{s_2(1 - \varepsilon_2)}{n_2(1 + \varepsilon_2)} T(r, f_2) \leq T\left(\frac{\beta_2 \gamma_2}{\theta_2} r, f_1\right). \tag{18}$$

Case 3.1 Since $\beta_i > 1$, $\gamma_i > 1$, $0 < \theta_i < 1$, then we have $\frac{\beta_i \gamma_i}{\theta_i} > 1$, $i = 1, 2$. From (18), and by Lemma 2.10, for any given $\varepsilon > 0$, there exists a sequence $r_n \rightarrow \infty$ such that

$$\begin{aligned}
 \frac{s_1(1 - \varepsilon_1)}{n_1(1 + \varepsilon_1)} T(r_n, f_1) &\leq \left(\frac{\beta_1 \gamma_1}{\theta_1}\right)^{\mu(f_2) + \varepsilon} T(r_n, f_2), \\
 \frac{s_2(1 - \varepsilon_2)}{n_2(1 + \varepsilon_2)} T(r_n, f_2) &\leq \left(\frac{\beta_2 \gamma_2}{\theta_2}\right)^{\mu(f_1) + \varepsilon} T(r_n, f_1)
 \end{aligned}$$

for $r_n > r_0$. From the above inequalities, we have

$$\frac{s_1(1 - \varepsilon_1)}{n_1(1 + \varepsilon_1)} \frac{s_2(1 - \varepsilon_2)}{n_2(1 + \varepsilon_2)} \leq \left(\frac{\beta_1 \gamma_1}{\theta_1}\right)^{\mu(f_2) + \varepsilon} \left(\frac{\beta_2 \gamma_2}{\theta_2}\right)^{\mu(f_1) + \varepsilon}. \tag{19}$$

Thus, let $\varepsilon' \rightarrow 0$, $\delta_i \rightarrow 0$, $\beta_i \rightarrow 1$ and $\gamma_i \rightarrow 1$ for $i = 1, 2$ and $\varepsilon' = \max\{\varepsilon, \varepsilon_1, \varepsilon_2\}$, since $0 < |q| < 1$ and $s_1 s_2 \geq n_1 n_2$, from (19), we can get

$$\mu(f_1) + \mu(f_2) \geq \frac{\log s_1 s_2 - \log n_1 n_2}{-\log |q|}.$$

Thus, (8) holds.

Case 3.2 Suppose that $|q| > 1$. By using the same argument as above, we can get

$$s_1(1 - \varepsilon_1)T(\theta'_1 r, f_1) \leq n_1 \left(1 + \frac{\varepsilon_1}{2}\right) T(r + C, f_2) + S(r, f_2),$$

$$s_2(1 - \varepsilon_2)T(\theta'_2 r, f_2) \leq n_2 \left(1 + \frac{\varepsilon_2}{2}\right) T(r + C, f_1) + S(r, f_1),$$

where $\theta'_i = |q| - \delta_i$ ($\delta_i > 0$ is chosen to be such that $\theta'_i > 1$), and r is sufficiently large. We can choose sufficiently small $\varepsilon_i > 0$ such that $\frac{1}{\theta'_i} + \varepsilon_i < 1$. Thus, it follows

$$s_1(1 - \varepsilon_1)T(r, f_1) \leq n_1(1 + \varepsilon_1)T\left(\frac{r + C}{\theta'_1}, f_2\right) + S(r, f_2)$$

$$\leq n_1(1 + \varepsilon_1)T\left(\left(\frac{1}{\theta'_1} + \varepsilon_1\right)r, f_2\right) + S(r, f_2),$$

$$s_2(1 - \varepsilon_2)T(r, f_2) \leq n_2(1 + \varepsilon_2)T\left(\frac{r + C}{\theta'_2}, f_1\right) + S(r, f_1)$$

$$\leq n_2(1 + \varepsilon_2)T\left(\left(\frac{1}{\theta'_2} + \varepsilon_2\right)r, f_1\right) + S(r, f_1),$$

outside of a possible exceptional set E_i ($i = 1, 2$) of finite logarithmic measure, respectively.

Since $n_1 n_2 \geq s_1 s_2$, $\frac{1}{\theta'_i} < 1$, $i = 1, 2$, and f_1, f_2 are transcendental, by applying Lemma 3.1 in [17] and Lemma 2.12, for $\varepsilon_i \rightarrow 0$ and $\delta_i \rightarrow 0$, we have

$$\rho(f_1) \leq \frac{\log n_1 n_2 - \log s_1 s_2}{2 \log |q|}, \quad \rho(f_2) \leq \frac{\log n_1 n_2 - \log s_1 s_2}{2 \log |q|},$$

which implies that (9) is true.

Case 3.3 $q = 1$ and $s_1 s_2 > n_1 n_2$. By using the same argument as in Case 3.1, we can get that $\mu(f_1) + \mu(f_2) \geq \infty$.

From Case 3.1, Case 3.2 and Case 3.3, the proof of Theorem 1.9 is completed.

4 Proofs of Theorems 1.7, 1.10 and 1.11

4.1 The proof of Theorem 1.7

Two cases will be considered as follows.

Case 4.1 Suppose that $q = 1$ and $\rho_2(f) < 1$. By applying Lemma 2.1 and Lemma 2.9 for equation (5), we can get

$$\max\{s, t\}T(r, f) \leq (\Gamma_1 + \Gamma_2)T(r, f) + O(1) + o(T(r, f))$$

for all r outside of a possible exceptional set E of finite logarithmic measure. From the above inequality, we have

$$\max\{s, t\} \leq \Gamma_1 + \Gamma_2 + \frac{O(1) + o(T(r, f))}{T(r, f)}$$

for all r outside of a possible exceptional set E of finite logarithmic measure. Since f is transcendental, we can get

$$\max\{s, t\} \leq \Gamma_1 + \Gamma_2.$$

Case 4.2 Suppose that $q \neq 1$ and $\rho(f) = 0$. By using the same argument as in Case 4.1, we can get

$$\max\{s, t\}T(r, f) \leq (\Gamma_1 + \Gamma_2)T(r, f) + O(1) + o(T(r, f))$$

for all r on a set of logarithmic density 1. Since f is transcendental, from the definition of logarithmic density, we can get

$$\max\{s, t\} \leq \Gamma_1 + \Gamma_2.$$

Thus, this completes the proof of Theorem 1.7.

4.2 The proof of Theorem 1.10

Two cases will be considered to prove Theorem 1.10.

Case 4.3 If $q = 1$ and $\rho_2(f_1, f_2) < 1$. By applying Lemma 2.1 and Lemma 2.9 for the system of complex equations (7), and from the assumptions of Theorem 1.10, we have

$$\max\{s_1, t_1\}T(r, f_1) \leq \sum_{k=1}^2 \Gamma_{1k}T(r, f_k) + O(1) + o(T(r, f_1) + T(r, f_2)), \quad r \notin H_1, \quad (20)$$

$$\max\{s_2, t_2\}T(r, f_2) \leq \sum_{k=1}^2 \Gamma_{2k}T(r, f_k) + O(1) + o(T(r, f_1) + T(r, f_2)), \quad r \notin H_2, \quad (21)$$

where H_1, H_2 are the sets of finite logarithmic measure. Since (f_1, f_2) is a transcendental meromorphic solution of (5), from (20) and (21), we have

$$\begin{aligned} & [\max\{s_1, t_1\} - \Gamma_{11} + o(1)][\max\{s_2, t_2\} - \Gamma_{22} + o(1)]T(r, f_1)T(r, f_2) \\ & \leq (\Gamma_{12} + o(1))(\Gamma_{21} + o(1))T(r, f_1)T(r, f_2) \end{aligned} \quad (22)$$

for all r outside of a possible exceptional set $H_3 = H_1 \cup H_2$ of finite logarithmic measure. Thus, from (22), we can get

$$[\max\{s_1, t_1\} - \Gamma_{11}][\max\{s_2, t_2\} - \Gamma_{22}] \leq \Gamma_{12}\Gamma_{21}.$$

Case 4.4 Suppose that $q \neq 1$ and $\rho(f_1, f_2) = 0$. By using a similar method as that in Case 4.3, we have

$$\max\{s_1, t_1\}T(r, f_1) \leq \sum_{k=1}^2 \Gamma_{1k}T(r, f_k) + O(1) + o(T(r, f_1) + T(r, f_2)), \quad r \notin F_1,$$

$$\max\{s_2, t_2\}T(r, f_2) \leq \sum_{k=1}^2 \Gamma_{2k}T(r, f_k) + O(1) + o(T(r, f_1) + T(r, f_2)), \quad r \notin F_2,$$

where F_1, F_2 are the sets of logarithmic density 0. Thus, we get

$$\begin{aligned} & [\max\{s_1, t_1\} - \Gamma_{11} + o(1)][\max\{s_2, t_2\} - \Gamma_{22} + o(1)]T(r, f_1)T(r, f_2) \\ & \leq (\Gamma_{12} + o(1))(\Gamma_{21} + o(1))T(r, f_1)T(r, f_2) \end{aligned} \tag{23}$$

for all r outside of a possible exceptional set $F_3 = F_1 \cup F_2$ of logarithmic density 0. Thus, from the definition of logarithmic density, we can get

$$[\max\{s_1, t_1\} - \Gamma_{11}][\max\{s_2, t_2\} - \Gamma_{22}] \leq \Gamma_{12}\Gamma_{21}.$$

Thus, the proof of Theorem 1.10 is completed.

4.3 The proof of Theorem 1.11

Two cases will be considered as follows.

Case 4.5 If $q = 1$ and $\rho_2(f_1, f_2) < 1$. Since all coefficients $\{a_i^1(z)\}, \{a_i^2(z)\}, \{b_i^1(z)\}, \{b_i^2(z)\}, \{a_{(i)}(z)\}$ and $\{b_{(i)}(z)\}$ are not necessarily small functions, by using the same argument as in Lemma 2.9, from Lemma 2.1, we have

$$\max\{s_1, t_1\} \leq \sum_{k=1}^2 \Gamma_{1k} T(r, f_k) + S(r) + O(1) + o(T(r, f_1) + T(r, f_2)), \quad r \notin H_3, \tag{24}$$

$$\max\{s_2, t_2\} \leq \sum_{k=1}^2 \Gamma_{2k} T(r, f_k) + S(r) + O(1) + o(T(r, f_1) + T(r, f_2)), \quad r \notin H_4, \tag{25}$$

where H_3, H_4 are two sets of finite logarithmic measure, $S(r)$ is as stated in Section 1.

Suppose that the component f_1 is admissible and f_2 is non-admissible, from (24) we have

$$\max\{s_1, t_1\} \leq \Gamma_{11} + o(1) + \Gamma_{12} \frac{T(r, f_2)}{T(r, f_1)} + \frac{S(r)}{T(r, f_1)}$$

for all r outside of a possible exceptional set of finite logarithmic measure. Thus, from Lemma 2.8, we have

$$\max\{s_1, t_1\} \leq \Gamma_{11}$$

outside of a possible exceptional set $E_3 = H_3 \cup E_2 \cup E_1$ with finite logarithmic measure, which contradicts the assumptions of Theorem 1.11.

Suppose that the component f_2 is admissible and f_1 is non-admissible, from (25) we have

$$\max\{s_2, t_2\} \leq \Gamma_{22} + o(1) + \Gamma_{21} \frac{T(r, f_1)}{T(r, f_2)} + \frac{S(r)}{T(r, f_2)}$$

for all r outside of a possible exceptional set of finite logarithmic measure. Thus, from Lemma 2.8, we have

$$\max\{s_2, t_2\} \leq \Gamma_{22}$$

outside of a possible exceptional set $E_4 = H_4 \cup E_2 \cup E_1$ with finite logarithmic measure, which contradicts the assumptions of Theorem 1.11.

Case 4.6 If $q = 1$ and $\rho(f_1, f_2) = 0$. By using the same argument as in Case 4.5, noting the definitions of logarithmic measure and logarithmic density, we can prove that the conclusion of Theorem 1.11 is true.

Thus, the proof of Theorem 1.11 is completed.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HXY completed the main part of this article, HXY, TBC and BXL corrected the main theorems. All authors read and approved the final manuscript.

Author details

¹Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi 333403, China.

²Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, China.

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