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Unification of probability theory on time scales

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To my wife

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Abstract

The theory of time scales was introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis. Probability is a discipline in which appears to be many applications of time scales. Time scales approach to probability theory unifies the standard discrete and continuous random variables. We give some basic random variables on the time scales. We define the distribution functions on time scales and show their properties.

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1 Introduction

A time scale or a measure chain was introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis [1]. We refer the reader to the textbooks [2–4].

Time scale calculus has received a lot of attention [1, 5–7]. In recent years there have been many research activities about applications of time scales. Probability theory is an ideal discipline for applications of time scales since random variables and distributions functions can be described with either discrete or continuous cases.

We give a brief introduction to measure theory on time scales introduced by Guseinov [8] in Section 2. We give the discussion of our original probability results in Section 3. In Section 4 we study the discrete random variables, *i.e.*, binomial, Poisson, geometric, and negative binomial random variables on a discrete time scale $h\mathbb{N}$. In Section 4.5 we define uniform random variables on the time scale, and we give the definition of Gaussian bell in Section 5.

2 Measure on time scales

The Riemann Δ integral has been introduced by Guseinov in [9], the Δ measure and the Lebesgue Δ integral were introduced by Guseinov in [8] and studied by Cabada [10], Ufuktepe and Deniz in [11], and Rzezuchowski in [12]. In this section we set out basic concepts of Δ and ∇ measures.

Let $\mathbb{T}$ be a time scale, $a < b$ be points in $\mathbb{T}$, and $[a, b)$ be a half-closed bounded interval in $\mathbb{T}$, σ and ρ be the forward and backward jump operators respectively on $\mathbb{T}$. Let

$$\Im_1 = \{[a', b') \cap \mathbb{T} : a', b' \in \mathbb{T}, a' \le b'\}$$

be the family of all left closed and right open intervals of $\mathbb{T}$. Then $\Im_1$ is a semiring. Here $[a', a') = \emptyset$. $m_1 : \Im_1 \to [0, \infty]$ is a set function which assigns to each interval its length: $m_1([a', b')) = b' - a'$. So, if $\{I_n\}$ is a sequence of disjoint intervals in $\Im_1$, then $m_1(\bigcup I_n) = \sum m_1(I_n)$.

Let $E \subset \mathbb{T}$. By the Carathéodory extension, the outer measure of E is

$$m_1^*(E) = \inf_{E \subset \bigcup_n I_n} \sum m_1(I_n),$$

where $I_n \in \Im_1$. If there is no such covering of E, then $m_1^*(E) = \infty$.

Definition 2.1 A set $E \subset \mathbb{T}$ is said to be Δ-measurable if for each set A,

$$m_1^*(A) = m_1^*(A \cap E) + m_1^*\big(A \cap E^c\big),$$

where $E^c = \mathbb{T} - E$.

Since we always have $m_1^*(A) \leq m_1^*(A \cap E) + m_1^*(A \cap E^c)$, we see that E is Δ-measurable iff for each A we have $m_1^*(A) \geq m_1^*(A \cap E) + m_1^*(A \cap E^c)$.

If E is Δ-measurable, then E^c is also Δ-measurable. Clearly, $\emptyset$ and $\mathbb{T}$ are Δ-measurable.

Lemma 2.2 *If E_1 and E_2 are Δ-measurable, so is $E_1 \cup E_2$.*

Let $\mathfrak{M}(m_1^*) = \{E \subset \mathbb{T} : E \text{ is } \Delta\text{-measurable}\}$ be a family of Δ-measurable sets.

Corollary 2.3 *$\mathfrak{M}(m_1^*)$ is a σ algebra.*

Definition 2.4 The restriction of m_1^* to $\mathfrak{M}(m_1^*)$ is called the Lebesgue Δ-measure and denoted by μ_Δ.

So, $m_1^*(E) = \mu_\Delta(E)$ if $E \in \mathfrak{M}(m_1^*)$. Similarly, if we take

$$\mathfrak{F}_2 = \big\{\big(a', b'\big] : a', b' \in \mathbb{T}, a' \leq b'\big\},$$

where $(a', a']$ is understood as an empty set, then $m_2 : \mathfrak{F}_2 \to [0, \infty]$ such that $m_2((a', b']) = b' - a'$ is a countably additive measure. Then $\mathfrak{M}(m_2^*)$ is the set of ∇-measurable sets and μ_∇ is the Lebesgue ∇-measure on $\mathbb{T}$.

Proposition 2.5 *Let $\{E_n\}$ be an infinite decreasing sequence of Δ-measurable sets, that is, a sequence $E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$, $E_i \in \mathfrak{F}_1$ for each i, $\bigcap E_i \in \mathfrak{F}_1$ and $m_1^*(E_1) < \infty$. Then*

$$m_1^*\left(\bigcap_{n=1}^{\infty} E_i\right) = \lim_{n \to \infty} m_1^*(E_n).$$

Proof [13]. □

Proposition 2.6 (Properties of m_1^*)

(i) $m_1^*(\emptyset) = 0$;

(ii) *If $E \subset F$, then $m_1^*(E) \leq m_1^*(F)$;*

(iii) *If* $\{E_n\}_{n=1}^{\infty}$ *is a sequence of elements of* $\mathfrak{F}_1$, *then*

$$m_1^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m_1^*(E_n).$$

Proof Similar to the proof in [13]. □

Theorem 2.7 *For each* $t_0 \in \mathbb{T} - \{\max \mathbb{T}\}$, *the single point set* $\{t_0\}$ *is* Δ*-measurable and its* Δ*-measure is given by* $\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0$.

Proof Case 1. Let t_0 be right scattered. Then $\{t_0\} = [t_0, \sigma(t_0)) \in \mathfrak{F}_1$. So, $\{t_0\}$ is Δ-measurable and $\mu_\Delta(\{t_0\}) = \sigma(t_0) - t_0$.

Case 2. Let t_0 be right dense. Then there exists a decreasing sequence $\{t_k\}$ of points of $\mathbb{T}$ such that $t_0 \le t_k$ and $t_k \downarrow t_0$. Since $\{t_0\} = \bigcap_{k=1}^{\infty}[t_0, t_k) \in \mathfrak{F}_1$. Therefore, $\{t_0\}$ is Δ-measurable. By Proposition 2.5,

$$\begin{aligned} \mu_\Delta\bigl(\{t_0\}\bigr) &= \mu_\Delta\left(\bigcap_{k=1}^{\infty}[t_0, t_k)\right) \\ &= \lim_{n\to\infty} \mu_\Delta\bigl([t_0, t_n)\bigr) \\ &= \lim_{n\to\infty} t_n - t_0 = 0, \end{aligned}$$

which is the desired result since t_0 is right dense. □

Every kind of interval can be obtained from an interval of the form $[a, b)$ by adding or subtracting the end points a and b. Then each interval of $\mathbb{T}$ is Δ-measurable.

Theorem 2.8 *If* $a, b \in \mathbb{T}$ *and* $a \le b$, *then*
(i) $\mu_\Delta([a, b)) = b - a$;
(ii) $\mu_\Delta((a, b)) = b - \sigma(a)$;
(iii) *If* $a, b \in \mathbb{T} - \max \mathbb{T}$, *then* $\mu_\Delta((a, b]) = \sigma(b) - \sigma(a)$ *and* $\mu_\Delta([a, b]) = \sigma(b) - a$.

Proof [8]. □

Theorem 2.9 *For each* $t_0 \in \mathbb{T} - \{\min \mathbb{T}\}$, *the* ∇*-measure of the single point set* $\{t_0\}$ *is given by* $\mu_\nabla(\{t_0\}) = t_0 - \rho(t_0)$.

Proof Similar to μ_Δ case. □

Theorem 2.10 *If* $a, b \in \mathbb{T}$ *and* $a \le b$, *then*
(i) $\mu_\nabla((a, b]) = b - a$;
(ii) $\mu_\nabla((a, b)) = \rho(b) - a$;
(iii) *If* $a, b \in \mathbb{T} - \min \mathbb{T}$, *then* $\mu_\nabla([a, b)) = \rho(b) - \rho(a)$ *and* $\mu_\nabla([a, b]) = b - \rho(a)$.

Proof The equalities can be obtained by the same technique with μ_Δ case. □

Lemma 2.11 $\lambda^*(E) \le m_1^*(E)$, *where* $\lambda^*(E)$ *is the outer measure of* E.

Example

(i) $\mathbb{T} = \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \dots\}$ and $A = \{0,1,3\}$, then $\mu_\Delta(A) = 1$.

(ii) Let $\mathbb{T} = \{0,1,2,3,4,\dots\}$ and $A = \{1,3,5,6\}$, then $\mu_\Delta(A) = 4$.

3 Probability on time scales

Let $\mathbb{T}$ be any time scale (which may be finite or infinite) and $A \subset \mathbb{T}$, then A can be written as

$$A = \bigcup_{i=1}^{m} [a_i, b_i]_T \cup \{t_1, t_2, \dots, t_n\},$$

where m and n are nonnegative integers which may be finite or infinite, a_i is right dense, b_i is left dense for all $i = 1,2,\dots,m$, all interior points of $[a_i, b_i]_T$ are dense points, and t_i is an isolated point for $j = 1,2,\dots,n$. So, the Lebesgue Δ-measure of $A \neq \emptyset$ is

$$\mu_\Delta(A) = \sum_{i=1}^{m} \bigl(\sigma(b_i) - a_i\bigr) + \sum_{j=1}^{n} \bigl(\sigma(t_j) - t_j\bigr),$$

if $A = \emptyset$, then $\mu_\Delta(A) = 0$.

Definition 3.1 Let $\mathbb{T}$ be a time scale and $\Im_T$ be a field of subsets of $\mathbb{T}$. Suppose that P_Δ is a Δ-measure defined on $\Im_T$. Then P_Δ is a probability measure if $P_\Delta(\mathbb{T}) = 1$. In this case, the triple $\{\mathbb{T}, \Im_T, P_\Delta\}$ is called a Δ-probability space.

Definition 3.2 Let Ω_T be a sample space and $A \subset \Omega_T$, then

$$P_\Delta(A) = \frac{\mu_\Delta(A)}{\mu_\Delta(\Omega_T)}$$

is called Δ-probability of A. Similarly,

$$P_\nabla(A) = \frac{\mu_\nabla(A)}{\mu_\nabla(\Omega_T)}$$

is called ∇-probability of A.

Proposition 3.3 *P_Δ and P_∇ are probability functions.*

Proof Let $A \subset \Omega_T$. By using $0 \le \mu_\Delta(A) \le \mu_\Delta(\Omega_T)$, we get $0 \le P_\Delta(A) \le 1$. $P_\Delta(\Omega_T) = 1$ is clear by the definition. Let $A_1, A_2, \dots$ be countable disjoint subsets of Ω_T, then

$$P_\Delta\left(\bigcup_{i=1}^{\infty} A\right) = \frac{\mu_\Delta(\bigcup_{i=1}^{\infty} A_i)}{\mu_\Delta(\Omega_T)} = \frac{\sum_{i=1}^{\infty} \mu_\Delta(A_i)}{\mu_\Delta(\Omega_T)} = \sum_{i=1}^{\infty} P_\Delta(A_i).$$

The proof of P_∇ is similar. □

Example Let $\Omega_T = \{1,2,3,4,\dots,n\}$ and $A = \{1,2,3,\dots,m\}$, where $m < n$. Then

$$P_\Delta(A) = \frac{\mu_\Delta(A)}{\mu_\Delta(\Omega_T)} = \frac{m}{n},$$

which is equivalent to the counting probability.

Proposition 3.4 *For any $A, B \subset \Omega_T$, we have $P_\Delta(A) \leq P_\Delta(B)$ if $A \subset B$.*

Proof Let $A \subset B \subset \mathbb{T}$. Then

$$P_\Delta(A) = \frac{\mu_\Delta(A)}{\mu_\Delta(\Omega_T)} \leq \frac{\mu_\Delta(B)}{\mu_\Delta(\Omega_T)} = P_\Delta(B). \qquad \square$$

4 Discrete random variables on time scales

Definition 4.1 A random variable $X_{\mathbb{T}}$ is a real-valued function defined on $\Im_T$.

In this section we consider the binomial, Poisson, geometric, and negative binomial random variables on $\mathbb{T} = h\mathbb{N}$, where $h > 0$.

4.1 Binomial random variable on $h\mathbb{N}$

Consider the time scale $\mathbb{T} = h\mathbb{N}$, $h > 0$. Lets there are n Bernoulli trials and but each Bernoulli trial has h independent Bernoulli trials with probability of at least k times success p_h are performed independently, then $X_{\mathbb{T}}$, the number of trials for x successes, is called the binomial random variable on the time scale with parameters n, k, and p_h. The probability function of this random variable is defined as follows:

$$p(x) = P(X_{\mathbb{T}} = x_t) = \binom{\frac{n}{\mu(x)}}{x} p_h^x q_h^{\frac{n}{\mu(x)} - x}, \tag{4.1}$$

where $x_t = hx$, $\mu(x)$ is a grainness function, $p_h = \sum_{i=k}^{h} \binom{h}{i} p^i q^{h-i}$ and $q_h = 1 - p_h$ is called a binomial random variable on the time scale. Since $\mu(x) = x + h - x = h$, we take h instead of $\mu(x)$.

Example Consider a jury trial in which it takes eight out of twelve juror groups to convict; that is, in order for the defendant to be convicted, at least eight of the juror groups must vote him guilty. Also, consider each group consists of three members. If at least two of three members vote that the defendant is guilty, then the decision of the group is guilty. If we assume that each juror group acts independently and each person makes the right decision with probability θ, what is the probability that the jury renders a correct decision?

Let the defendant be innocent. The probability of the juror's right decision is

$$p_1 = \sum_{i=5}^{12} \binom{12}{i} \theta_h^i (1 - \theta_h)^{12-i},$$

where $\theta_h = \binom{3}{2}\theta^2(1-\theta) + \binom{3}{3}\theta^3$.

Let the defendant be guilty. Then the probability of the juror's correct decision is

$$p_2 = \sum_{i=8}^{12} \binom{12}{i} \theta_h^i (1 - \theta_h)^{12-i}.$$

If α represents the probability that the defendant is guilty, then $\alpha p_1 + (1-\alpha)p_2$ is the desired result.

While evaluating the expected value and the variance of the discrete random variables, we will make use of the following proposition.

Proposition 4.2 *If $X_{\mathbb{T}}$ is a discrete random variable that takes on one of the values x_i, $i \geq 1$, with the respective probabilities $p(x_i)$, then for any real-valued function g, $E[g(X)] = \sum_i g(x_i)p(x_i)$.*

The expected value of a binomial random variable is given as follows:

$$\begin{aligned}
E\big[X_{\mathbb{T}}^s\big] &= \sum_{i=0}^{\frac{n}{h}} i^s P(X_{\mathbb{T}} = i) = \sum_{i=0}^{\frac{n}{h}} i^s \binom{\frac{n}{h}}{i} p_h^i q_h^{\frac{n}{h}-i} \\
&= \sum_{i=1}^{\frac{n}{h}} i^s \binom{\frac{n}{h}}{i} p_h^i q_h^{\frac{n}{h}-i} \\
&= p_h \sum_{i=1}^{\frac{n}{h}} i \binom{\frac{n}{h}}{i} i^{s-1} p_h^{i-1} q_h^{\frac{n}{h}-i} = p_h \sum_{i=1}^{\frac{n}{h}} \frac{n}{h} \binom{\frac{n}{h}-1}{i-1} i^{s-1} p_h^{i-1} q_h^{\frac{n}{h}-i} \\
&= \frac{np_h}{h} \sum_{i=0}^{\frac{n}{h}-1} \binom{\frac{n}{h}-1}{i} (i+1)^{s-1} p_h^i q_h^{\frac{n}{h}-1-i} = \frac{np_h}{h} E\big[(Y+1)^{s-1}\big],
\end{aligned}$$

where Y is a random variable with parameters $\frac{n}{h} - 1$ and p_h. If we set $s = 1$, we get

$$E[X_{\mathbb{T}}] = \frac{np_h}{h}. \tag{4.2}$$

Remark 4.3 When we take $h = 1$, then the time scale is a set of natural numbers and the expected value is as in the classical probability theory.

If $s = 2$,

$$\begin{aligned}
E\big[X_{\mathbb{T}}^2\big] &= \frac{np_h}{h} E[Y+1] \\
&= \frac{np_h}{h} \big(E[Y] + 1\big) \\
&= \frac{np_h}{h} \left(\left(\frac{n}{h} - 1 \right) p_h + 1 \right).
\end{aligned}$$

So, the variance of a binomial random variable is

$$\begin{aligned}
\operatorname{Var}(X_{\mathbb{T}}) &= E\big[X_{\mathbb{T}}^2\big] - E[X_{\mathbb{T}}]^2 \\
&= \frac{np_h q_h}{h}.
\end{aligned}$$

Remark 4.4 When we take $h = 1$, then the time scale is a set of natural numbers and the variance is as in the classical probability theory.

4.2 Poisson random variable on $h\mathbb{N}$

Definition 4.5 Let $\mathbb{T} = h\mathbb{N}$ be the time scale. A random variable $X_{\mathbb{T}}$ with possible values $0, h, 2h, \ldots$ is called a Poisson random variable on the time scale with the parameter $\lambda > 0$,

$$p(t) = P(X_{\mathbb{T}} = t) = e^{-\lambda} \frac{\lambda^{\frac{t}{\mu(t)}}}{\mu(t)(\frac{t}{\mu(t)})!}. \tag{4.3}$$

Since

$$\begin{aligned}\int_{t\in\mathbb{T}} p(t)\Delta t &= \int_0^h p(t)\Delta t + \int_h^{2h} p(t)\Delta t + \cdots \\ &= p(0)h + p(h)h + p(2h)h + \cdots \\ &= h\bigl(p(0) + p(h) + p(2h) + \cdots\bigr) \\ &= h\sum_{i=0}^{\infty} p(ih) \\ &= h\sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^{\frac{ih}{\mu(t)}}}{h(\frac{ih}{\mu(t)})!} \\ &= h\sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^{\frac{ih}{h}}}{h(\frac{ih}{h})!} \\ &= e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = 1,\end{aligned}$$

then $p(t)$ is a probability mass function.

The Poisson probability function is the limit of a binomial probability function, the expected value of a binomial random variable with parameters $(\frac{n}{h}, p_h)$ is $\frac{n}{h}p_h = \lambda$. It is reasonable to expect that the mean of the Poisson random variable with the parameter λ is λ as follows:

$$\begin{aligned}E[X_{\mathbb{T}}] &= \int_{t\in\mathbb{T}} tp(t)\Delta t = \int_{t\in\mathbb{T}} te^{-\lambda}\frac{\lambda^{\frac{t}{h}}}{h(\frac{t}{h})!}\Delta t \\ &= \frac{e^{-\lambda}}{h}\sum_{n=0}^{\infty}\int_{nh}^{\sigma(nh)} t\frac{\lambda^{\frac{t}{h}}}{(\frac{t}{h})!}\Delta t = \frac{e^{-\lambda}}{h}\sum_{n=0}^{\infty} nh\frac{\lambda^n}{n!} \\ &= \lambda e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!} = \lambda.\end{aligned}$$

For the variance of the Poisson random variable on $h\mathbb{N}$, we first compute

$$\begin{aligned}E\bigl[X_{\mathbb{T}}^2\bigr] &= \int_{t\in\mathbb{T}} t^2 p(t)\Delta t \\ &= \int_{t\in\mathbb{T}} t^2 e^{-\lambda}\frac{\lambda^{\frac{t}{\mu(t)}}}{\mu(t)(\frac{t}{\mu(t)})!}\Delta t \\ &= \int_{t\in\mathbb{T}} t^2 e^{-\lambda}\frac{\lambda^{\frac{t}{h}}}{h(\frac{t}{h})!}\Delta t \\ &= e^{-\lambda}\lambda\int_{t\in\mathbb{T}} t\frac{\lambda^{\frac{t}{h}-1}}{(\frac{t}{h}-1)!}\Delta t \\ &= e^{-\lambda}\lambda\sum_{n=0}^{\infty}\int_{nh}^{\sigma(nh)} t\frac{\lambda^{\frac{t}{h}-1}}{(\frac{t}{h}-1)!}\Delta t\end{aligned}$$

$$\begin{aligned}
&= e^{-\lambda}\lambda\left(\int_0^h t\frac{\lambda^{\frac{t}{h}-1}}{(\frac{t}{h}-1)!}\Delta t + \int_h^{2h} t\frac{\lambda^{\frac{t}{h}-1}}{(\frac{t}{h}-1)!}\Delta t + \cdots\right) \\
&= e^{-\lambda}\lambda\left(0\frac{\lambda^0}{0!} + 1\frac{\lambda^1}{1!} + 2\frac{\lambda^2}{2!} + \cdots\right) \\
&= e^{-\lambda}\lambda\sum_{k=0}^{\infty}(k+1)\frac{\lambda^k}{k!} \\
&= e^{-\lambda}\lambda\left(\sum_{k=0}^{\infty}k\frac{\lambda^k}{k!} + \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\right) \\
&= e^{-\lambda}\lambda\left(\lambda e^{\lambda} + e^{\lambda}\right) = \lambda(\lambda+1).
\end{aligned}$$

Therefore,

$$\operatorname{Var}(X_{\mathbb{T}}) = E\left[X_{\mathbb{T}}^2\right] - E[X_{\mathbb{T}}]^2 = \lambda. \tag{4.4}$$

Example An energy company produces batteries and sells five in a box. The probability that a battery is defective is 0.1. We assume if a box contains at least two defective batteries, then this box is also defective. Find the probability that a sample of ten boxes contains at most one defective.

By a binomial random variable, the desired probability is

$$\binom{10}{0}p_h^0 q_h^{10} + \binom{10}{1}p_h q_h^9,$$

where

$$p_h = \sum_{i=2}^{5}\binom{5}{i}(0.1)^i(0.9)^{5-i}$$

and $q_h = 1 - p_h$. So, the desired probability is 0.806708, whereas the Poisson approximation yields the value

$$\sum_{i=0}^{1} e^{-10p_h}\frac{(10p_h)^i}{i!} \simeq 0.803535.$$

4.3 Geometric random variable on $h\mathbb{N}$

Suppose that independent trial groups, each having the probability p_h, $0 < p_h < 1$, of being a success, are performed until a success occurs. If we let $X_{\mathbb{T}}$ equal the number of trials required, then we define

$$P(X_{\mathbb{T}} = n) = (1-p_h)^{\frac{n}{h}-1}p_h, \quad n = h, 2h, \ldots. \tag{4.5}$$

Equation (4.5) follows because in order for $X_{\mathbb{T}}$ to equal $\frac{n}{h}$, it is necessary and sufficient that the first $\frac{n}{h}-1$ trial groups are failures and the $\frac{n}{h}$th trial group is a success. Equation (4.5) then follows, since the outcomes of the successive trial groups are assumed to be independent.

Since

$$\sum_{n/h=1}^{\infty}(1-p_h)^{\frac{n}{h}-1}p_h = p_h\sum_{n/h=1}^{\infty}(1-p_h)^{\frac{n}{h}-1}$$
$$= p_h\frac{1}{1-(1-p_h)} = 1,$$

it follows that with probability one, a success group will eventually occur.

Definition 4.6 Any random variable $X_{\mathbb{T}}$ whose probability mass function is given by Equation (4.5) is said to be a geometric random variable with the parameter p_h.

By letting $\frac{n}{h} = k$, the expected value geometric random variable is given as follows:

$$\begin{aligned} E[X_{\mathbb{T}}] &= \sum_{k=1}^{\infty} k(1-p_h)^{k-1}p_h \\ &= p_h\frac{d}{dq_h}\left(\sum_{k=0}^{\infty} q_h^k\right) \\ &= p_h\frac{d}{dq_h}\left(\frac{1}{1-q_h}\right) \\ &= p_h\frac{1}{(1-q_h)^2} = \frac{1}{p_h}. \end{aligned} \tag{4.6}$$

To determine the variance of a geometric random variable, we first compute $E[X^2]$.

$$\begin{aligned} E\left[X^2\right] &= \sum_{k=1}^{\infty} k^2(1-p_h)^{k-1}p_h = p_h\sum_{k=1}^{\infty} k^2q_h^{k-1} = p_h\sum_{k=1}^{\infty}\frac{d}{dq_h}\left(kq_h^k\right) \\ &= p_h\frac{d}{dq_h}\left(\sum_{k=1}^{\infty} kq_h^k\right) = p_h\frac{d}{dq_h}\left(\frac{q_h}{1-q_h}E[X_{\mathbb{T}}]\right) = p_h\frac{d}{dq_h}\left(q_h(1-q_h)^{-2}\right) \\ &= p_h\left(\frac{1}{p_h^2}+\frac{2(1-p_h)}{p_h^3}\right) = \frac{2}{p_h^2}-\frac{1}{p_h}. \end{aligned}$$

Hence,

$$\operatorname{Var}(X_{\mathbb{T}}) = E\left[X_{\mathbb{T}}^2\right] - \left(E[X_{\mathbb{T}}]\right)^2 = \frac{1-p_h}{p_h^2}. \tag{4.7}$$

4.4 Negative binomial random variable on $h\mathbb{N}$

Negative binomial random variables on $h\mathbb{N}$ are generalizations of geometric random variables on $h\mathbb{N}$. Suppose that a sequence of Bernoulli independent trials, each trial is repeated h Bernoulli trials with probability of at least k-times success p_h, are performed. Let $X_{\mathbb{T}}$ be the number of experiments until ith success occurs, then it is called a negative binomial random variable if

$$P(X_{\mathbb{T}} = n) = \binom{\frac{n}{h}-1}{i-1}p_h^i q_h^{\frac{n}{h}-i}, \quad n = ih, (i+1)h, (i+2)h, \ldots. \tag{4.8}$$

The expected value and the variance of a negative binomial random variable

$$\begin{aligned}E\big[X_{\mathbb{T}}^s\big] &= \sum_{m=i}^{\infty} m^s \binom{m-1}{i-1} p_h^i (1-p_h)^{m-i} \\ &= \frac{i}{p_h}\sum_{m=i}^{\infty} m^{s-1}\binom{m}{i} p_h^{i+1}(1-p_h)^{m-i} \\ &= \frac{i}{p_h}\sum_{m=i+1}^{\infty}(m-1)^{s-1}\binom{m-1}{i} p_h^{i+1}(1-p_h)^{m-(i+1)} \\ &= \frac{i}{p_h}E\big[(Y-1)^{s-1}\big],\end{aligned}$$

where $m=\frac{n}{h}$, and Y is a negative binomial random variable on $h\mathbb{N}$ with parameters $i+1$, p_h. We use the identity

$$m\binom{m-1}{i-1} = i\binom{m}{i}$$

in the third line of the preceding equation. Setting $s=1$ in $E[X_{\mathbb{T}}^s]$, we get the expected value of a negative binomial random variable on $h\mathbb{N}$,

$$E[X_{\mathbb{T}}] = \frac{i}{p_h}. \tag{4.9}$$

Setting $s=2$ in $E[X_{\mathbb{T}}^s]$ and using Equation (4.9) gives that

$$E\big[X_{\mathbb{T}}^2\big] = \frac{i}{p_h}E[Y-1] = \frac{i}{p_h}\left(\frac{i+1}{p_h}-1\right).$$

Therefore,

$$\begin{aligned}\operatorname{Var}(X_{\mathbb{T}}) &= \frac{i}{p_h}\left(\frac{i+1}{p_h}-1\right) - \left(\frac{i}{p_h}\right)^2 \\ &= \frac{i(1-p_h)}{p_h^2}.\end{aligned}$$

Example A student takes multiple choice exams which have five questions with three choices. The student is successful if he/she gives at least three correct answers in an exam. What is the probability of the third success of the student in the tenth exam by guessing?

Here $\mathbb{T}=5\mathbb{N}$ and n, the number of questions on exams, is 50. So, by formula (4.8), we have

$$P(X_{\mathbb{T}}=10) = \binom{9}{2} p_h^3 q_h^7 \simeq 0.0639,$$

where $p_h = \binom{5}{3}(\frac{1}{3})^3(\frac{2}{3})^2 + \binom{5}{4}(\frac{1}{3})^4(\frac{2}{3})^1 + \binom{5}{5}(\frac{1}{3})^5(\frac{2}{3})^0 = \frac{17}{81}$ and $q_h = 1-p_h = \frac{64}{81}$.

4.5 Uniform random variable on the time scale

Let $\mathrm{T} = [t_0, t_1] \cup [t_2, t_3] \cup \cdots \cup [t_{n-1}, t_n]$, where $t_0 = a$ and $t_n = b$ and $S_R = \{t_1, t_3, \ldots, t_{2k+1}\}$ be the set of right scattered points. A uniform Δ-probability function on T can be defined as follows:

$$f_\Delta(t) = \begin{cases} \frac{1}{\mu_\Delta([a,b))}, & \text{if } t \in \mathrm{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Since this function satisfies the following condition:

$$\begin{aligned} \int_{-\infty}^{\infty} f_\Delta(t)\Delta(t) &= \int_{-\infty}^{a} f_\Delta(t)\Delta(x) + \int_{a}^{b} f_\Delta(t)\Delta(x) + \int_{b}^{\infty} f_\Delta(t)\Delta(x) \\ &= \int_{a}^{t_1} \frac{1}{\mu_\Delta([a,b))}\Delta(x) + \cdots + \int_{t_{n-1}}^{t_n} \frac{1}{\mu_\Delta([a,b))}\Delta(x) \\ &= \frac{1}{\mu_\Delta([a,b))} \sum_{i=1}^{n} \mu_\Delta\bigl([t_{i-1}, t_i)\bigr) \\ &= \frac{b-a}{b-a} = 1, \end{aligned}$$

this function is a probability function. Also, a uniform Δ-probability distribution function on the time scale is defined as follows:

$$F_\Delta(t) = \begin{cases} 0, & t < a \text{ or } t \in (t_{2i+1}, \sigma(t_{2i+1})), \text{ where } t_{2i+1} \text{ is right scattered}, \\ \frac{\mu_\Delta([a,t)\cap\Omega)}{\mu_\Delta([a,b))}, & \text{if } t \in \mathrm{T}, \\ 1, & t \geq b. \end{cases}$$

This function satisfies all the properties of the distribution function

$$\begin{aligned} \lim_{t\to+\infty} F_\Delta(t) &= \lim_{t\to+\infty} \frac{\mu_\Delta([a,t)\cap\Omega)}{\mu_\Delta([a,b))_\mathrm{T}} \\ &= \frac{\mu_\Delta([a,b))_\mathrm{T}}{\mu_\Delta([a,b))_\mathrm{T}} \\ &= 1, \\ \lim_{t\to-\infty} F_\Delta(t) &= \lim_{t\to-\infty} \frac{\mu_\Delta([a,t)\cap\Omega)}{b-a} \\ &= \lim_{t\to-\infty} \frac{\mu_\Delta(\{a\})_\mathrm{T}}{b-a} \\ &= \frac{\sigma(a)-a}{b-a} = 0. \end{aligned}$$

Remark 4.7 If we take the left closed and right open interval on our time scale T such that $A = [t_i, t_{i+1})$, then the integral over this set $\int_A f_\Delta(t)\Delta(t) = \int_{t_i}^{t_{i+1}} f_\Delta(t)\Delta(t) = \frac{1}{\mu_\Delta([b-a))}\mu_\Delta([t_i, t_{i+1})) = \frac{b-a}{b-a} = 1$, and also, if we take right and left open intervals and since a is right dense, then our result is the same $\int_A f_\Delta(t)\Delta(t) = \int_{t_i}^{t_{i+1}} f_\Delta(t)\Delta(t) + \int_{t_i}^{\sigma(t_{i+1})} f_\Delta(t)\Delta(t) = \frac{1}{\mu_\Delta((b-a))}\mu_\Delta((t_i, t_{i+1})) = \frac{b-\sigma(a)}{b-a} = 1$.

5 Gaussian bell on time scales

The continuous Gaussian bell $f(t) = e^{-\frac{t^2}{2}}$ satisfies the initial value problem

$$f'(t) = -tf(t), \qquad f(0) = 1.$$

The Gaussian bell is an even function, then the infinite time scale should be symmetric with respect to zero and contain zero. We denote the positive part of $\mathbb{T}$ by $\mathbb{T}_+$. So, the Gaussian bell $f_{\mathbb{T}}$ on the time scale should satisfy the following relation $\forall t \in \mathbb{T}_+$:

$$f_{\mathbb{T}}^{\Delta}(t) = p(t)f(t), \qquad f_{\mathbb{T}}(0) = 1,$$

where $p(t)$ must be $-t$ in the case $\mathbb{T} = \mathbb{R}$. Erbe and Peterson [7] defined the Gaussian bell on the time scale as follows.

Definition 5.1 On the time scale $\mathbb{T}$, we define the Gaussian bell $f_{\mathbb{T}}$ to be the unique solution of the initial value problem for all $t \in \mathbb{T}_+$

$$f_{\mathbb{T}}^{\Delta}(t) = \ominus(t \odot 1)f(t), \qquad f_{\mathbb{T}}(0) = 1 \tag{5.1}$$

and $f_{\mathbb{T}}(-t) = f_{\mathbb{T}}(t)$.

By using the definition of $\odot$ (circle dot) and $\ominus$ (circle minus) operations [3], if $\mu(t) > 0$,

$$\ominus(t \odot 1) = \frac{(1 + \mu(t))^{-t} - 1}{\mu(t)}. \tag{5.2}$$

By the definition of Δ derivative, Equations (5.1) and (5.2), we get

$$f_{\mathbb{T}}\big(\sigma(t)\big) = \big(1 + \mu(t)\big)^{-t} f_{\mathbb{T}}(t), \quad \forall t \in \mathbb{T}_+, f_{\mathbb{T}}(0) = 1. \tag{5.3}$$

Since $f_{\mathbb{T}}(t)$ satisfies the differential equation of the continuum Gaussian bell $f(t)$ at right-dense points $t \in \mathbb{T}_+$, thus it has a non-positive derivative at those points. We can conclude that $f_{\mathbb{T}}(t)$ is non-increasing on $\mathbb{T}_+$. On the discrete time scale $\mathbb{T}$, *i.e.*, a time scale $\mathbb{T}$ containing no continuum intervals, we can write $f_{\mathbb{T}}$ as

$$f_{\mathbb{T}}(t) = \prod_{x \in [0,t)} \big(1 + \mu(x)\big)^{-x}, \quad \forall t \in \mathbb{T}_+. \tag{5.4}$$

Example Consider $\mathbb{T} = h\mathbb{Z}$, $h > 0$. So, $\mu(t) = h$, substituting $t = hn$, we get

$$f_{\mathbb{T}}(hn) = \prod_{x \in [0,t)} (1 + h)^{-x} = (1 + h)^{-h \sum_{k=0}^{n-1} k} = (1 + h)^{\frac{hn(1-n)}{2}}, \quad \forall n \in \mathbb{N}_0, \tag{5.5}$$

which implies $f_{\mathbb{T}}(t) = [(1 + h)^{\frac{1}{h}}]^{\frac{-t(t-h)}{2}}$, $\forall t \in \mathbb{T}_+$.

For large t, $e^{\frac{t^2}{2}} \le f_{\mathbb{T}}(t)$, $f_{\mathbb{T}}(t)$ converges to the continuum Gaussian bell as $h \to 0$; $\forall t \in [hn, h(n+1)]$, $n \in \mathbb{N}_0$,

$$\lim_{h \to 0} f_{\mathbb{T}}(ht) = \lim_{h \to 0} \big[(1 + h)^{\frac{1}{h}}\big]^{-\frac{th^2(t-1)}{2}} = e^{-\frac{t^2}{2}}, \quad \forall t \in \mathbb{R}.$$

Example Let $\mathbb{T} = \bigcup_{n=0}^{\infty}[2n, 2n+1]$. Then the Gaussian bell on $\mathbb{T}$ is

$$f_{\mathbb{T}}(t) = \left(\frac{e}{2}\right)^{n^2} e^{\frac{n-t^2}{2}}, \quad \forall t \in [2n, 2n+1]. \tag{5.6}$$

Mathematical induction is used for showing Equation (5.6).

In general, a probability distribution function and expected value of a random variable on a time scale can be defined as follows:

$$F_{\Delta}(x) = \int_{-\infty}^{x} p(t)\Delta t,$$
$$E(x) = \int_{-\infty}^{\infty} tp(t)\Delta t.$$

By using an exponential function on time scales, we can define an exponential probability density function in a general case and we can define a moment generating function by using Laplace transformations on time scales. Then future works can be stochastic processes on time scales and stochastic dynamic equations.

Competing interests
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