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Some identities on the higher-order-twisted q-Euler numbers and polynomials with weight α

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Abstract

In this article, we introduce some properties of higher-order-twisted q-Euler numbers and polynomials with weight α, and we observe some properties of higher-order-twisted q-Euler numbers and polynomials with weight α for several cases. In particular, by using the the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give a new concept of twisted q-Euler numbers and polynomials with weight α.

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1. Introduction

Let p be a fixed odd prime. Throughout this article $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$, will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$ (see [1-14]). When one speaks of q-extension, q can be regarded as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$; it is always clear from context. If $q \in \mathbb{C}$, we assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|1 - q|_p < 1$ (see [1-14]).

In this article, we use the notation of q-number as follows (see [1-14]):

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \to 1} [x]_q = x$ for any x with $|x|_p \leq 1$ in the p-adic case.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows (see [6,7]):

$$
\begin{aligned}
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) &= \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N - 1} f(x)(-q)^x, \\
&= \lim_{N \to \infty} \frac{[2]_q}{1 + q^{p^N}} \sum_{x=0}^{p^N - 1} f(x)(-q)^x.
\end{aligned}
\tag{1}
$$

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From (1), we note that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),$$

where $f_1(x) = f(x + 1)$.

It is well known that the ordinary Euler polynomials are defined by

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$.

In the special case, $x = 0$, $E_n(0) = E_n$ are called the nth Euler numbers (see [1-14]).

By (2), we get the following recurrence relation as follows:

$$E_0 = 1, \text{ and } (E + 1)^n + E = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{2}$$

Recently, (h, q)-Euler numbers are defined by

$$E_{0,q}^{(h)} = \frac{2}{1 + q^h}, \text{ and } q^h(qE_q^{(h)} + 1)^n + E_q^{(h)} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing $\left(E_q^{(h)}\right)^n$ by $E_{n,q}^{(h)}$ (see [1-16]).

Note that $\lim_{q \to 1} E_{n,q}^{(h)} = E_n$.

Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\varphi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto w^x$.

For $\alpha \in \mathbb{N}$ and $w \in T_p$, the twisted q-Euler numbers with weight α are also defined by

$$\tilde{E}_{0,q,w}^{(\alpha)} = \frac{[2]_q}{wq + 1}, \text{ and } wq\left(q^\alpha \tilde{E}_{q,w}^{(\alpha)} + 1\right)^n + \tilde{E}_{n,q,w}^{(\alpha)} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing $\left(\tilde{E}_{q,w}^{(\alpha)}\right)^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}$ (see [2,5]).

The main purpose of this article is to present a systemic study of some families of higher-order-twisted q-Euler numbers and polynomials with weight α. In Section 2, we investigate higher-order-twisted q-Euler numbers and polynomials with weight α and establish interesting properties. In Sections 3, 4, and 5, we observe some properties for special cases.

2. Higher-order-twisted q-Euler numbers and polynomials with weight α

For $h \in \mathbb{Z}$, $\alpha, k \in \mathbb{N}$, $w \in T_p$ and $n \in \mathbb{Z}_+$, let us consider the expansion of higher-order-twisted q-Euler polynomials with weight α as follows:

$$\tilde{E}_{n,q,w}^{(\alpha)}(h, k|x)$$
$$= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} w \sum_{i=1}^{k} x_i \left[\sum_{i=1}^{k} x_i + x \right]_{q^\alpha}^{n} q^{x_1(h-1)+\cdots+x_k(h-k)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \tag{3}$$

From (1) and (3), we note that

$$\tilde{E}_{n,q,w}^{(\alpha)}(h, k|x) = \frac{[2]_q^k}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(1 + wq^{\alpha l + h}) \cdots (1 + wq^{\alpha l + h - k + 1})}. \tag{4}$$

In the special case, $x = 0$ $\tilde{E}_{n,q,w}^{(\alpha)}(h, k|0) = \tilde{E}_{n,q,w}^{(\alpha)}(h, k)$ are called the higher-order-twisted q-Euler numbers with weight α.

By (3), we get

$$\tilde{E}_{n,q,w}^{(\alpha)}(h, k) = (q^\alpha - 1)\tilde{E}_{n+1,q,w}^{(\alpha)}(h - \alpha, k) + \tilde{E}_{n,q,w}^{(\alpha)}(h - \alpha, k). \tag{5}$$

From (5) and mathematical induction, we get the following theorem.

Theorem 1. For $\alpha, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we have

$$\sum_{i=0}^{n-1}(-1)^{i-1}(q^\alpha - 1)^{n-1-i}\tilde{E}_{n-i,q,w}^{(\alpha)}(h, k)$$

$$= (q^\alpha - 1)^{n-1}\tilde{E}_{n,q,w}^{(\alpha)}(h - \alpha, k) + (-1)^n E_{1,q,w}^{(\alpha)}(h - \alpha, k).$$

For complex number $q \in \mathbb{C}_p$, $m \in \mathbb{Z}_+$, we get the following;

$$q^{\alpha(x_1+\cdots+x_{k+1})m} = \left(1 - (1 - q^{\alpha(x_1+\cdots+x_{k+1})})\right)^m$$

$$= \sum_{l=0}^{m}\binom{m}{l}(-1)^l\left(1 - q^{\alpha(x_1+\cdots+x_{k+1})}\right)^l$$

$$= \sum_{l=0}^{m}\binom{m}{l}(-1)^l(1 - q^\alpha)^l\frac{(1 - q^{\alpha(x_1+\cdots+x_{k+1})})^l}{(1 - q^\alpha)^l}$$

$$= \sum_{l=0}^{m}\binom{m}{l}(-1)^l(1 - q^\alpha)^l[x_1 + x_2 + \cdots + x_{k+1}]_{q^\alpha}^l.$$

From (3), (4), and above property, we have

$$\tilde{E}_{0,q,w}^{(\alpha)}(m\alpha, k + 1)$$

$$= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{j=1}^{k+1}x_j}q^{\sum_{j=1}^{k+1}(m\alpha-j)x_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_{k+1})$$

$$= \sum_{l=0}^{m}\binom{m}{l}(q^\alpha - 1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{j=1}^{k+1}x_j}\left[\sum_{j=1}^{k+1}x_j\right]_{q^\alpha}^l q^{-\sum_{j=1}^{k+1}jx_j}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_{k+1}) \tag{6}$$

$$= \sum_{l=0}^{m}\binom{m}{l}(q^\alpha - 1)^l\tilde{E}_{l,q,w}^{(\alpha)}(0, k + 1)$$

$$= \frac{[2]_q^{k+1}}{(1 + wq^{\alpha m})(1 + wq^{\alpha m-1})\ldots(1 + wq^{\alpha m-k})}.$$

From (3), we can derive the following equation.

$$\sum_{j=0}^{i}\binom{i}{j}(q^\alpha - 1)^j\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{s=1}^{k}x_s}\left[\sum_{s=1}^{k}x_s\right]_{q^\alpha}^{n-i+j}q^{\sum_{s=1}^{k}(h-\alpha-s)x_s}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)$$

$$= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} w^{\sum_{s=1}^{k}x_s}\left[\sum_{s=1}^{k}x_s\right]_{q^\alpha}^{n-i}q^{\sum_{s=1}^{k}(h-s)x_s}q^{\alpha\left(\sum_{s=1}^{k}x_s\right)(i-1)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \tag{7}$$

$$= \sum_{j=0}^{i-1}(q^\alpha - 1)^j\binom{i-1}{j}\tilde{E}_{n-i+j,q,w}^{(\alpha)}(h, k).$$

By (3), (4), (5), and (6), we see that

$$\sum_{j=0}^{i} (q^{\alpha} - 1)^j \binom{i}{j} \tilde{E}_{n-1+j,q,w}^{(\alpha)}(h - \alpha, k) = \sum_{j=0}^{i-1} (q^{\alpha} - 1)^j \binom{i-1}{j} \tilde{E}_{n-i+j,q,w}^{(\alpha)}(h, k).$$

Therefore, we obtain the following theorem.

Theorem 2. For $\alpha, k \in \mathbb{N}$ and $n, i \in \mathbb{Z}_+$, we have

$$\sum_{j=0}^{i} \binom{i}{j} (q^{\alpha} - 1)^j \tilde{E}_{n-i+j,q,w}^{(\alpha)}(h - \alpha, k) = \sum_{j=0}^{i-1} (q^{\alpha} - 1)^j \binom{i-1}{j} \tilde{E}_{n-i+j,q,w}^{(\alpha)}(h, k).$$

By simple calculation, we easily see that

$$\sum_{j=0}^{m} \binom{m}{j} (q^{\alpha} - 1)^j \tilde{E}_{j,q,w}^{(\alpha)}(0, k) = \frac{[2]_q^k}{(1 + wq^{\alpha m})(1 + wq^{\alpha m - 1}) \cdots (1 + wq^{\alpha m - k + 1})}.$$

3. Polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(0, k|x)$

We now consider the polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(0, k|x)$ (in q^x) by

$$\tilde{E}_{n,q,w}^{(\alpha)}(0, k|x)$$
$$= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} w^{x_1 + \cdots + x_k} \left[x + \sum_{i=1}^{k} x_i \right]_{q^{\alpha}}^{n} q^{-\sum_{j=1}^{k} j x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \tag{8}$$

By (8) and (4), we get

$$(q^{\alpha} - 1)^n \tilde{E}_{n,q,w}^{(\alpha)}(0, k|x) = [2]_q^k \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l x} (-1)^{n-1} \frac{1}{(1 + wq^{\alpha l}) \cdots (1 + wq^{\alpha l - k + 1})}. \tag{9}$$

From (8) and (9), we can derive the following equation.

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_k} q^{\sum_{j=1}^{k} (\alpha n - j) x_j + \alpha n x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$
$$= \sum_{j=0}^{n} \binom{n}{j} [\alpha]_q^j (q - 1)^j \tilde{E}_{j,q,w}^{(\alpha)}(0, k|x),$$

and

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_k} q^{\sum_{j=1}^{k} (\alpha n - j) x_j + \alpha n x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$
$$= \frac{[2]_q^k q^{\alpha n x}}{(1 + wq^{\alpha n}) \cdots (1 + wq^{\alpha n - k + 1})}. \tag{10}$$

Therefore, by (9) and (10), we obtain the following theorem.

Theorem 3. For $\alpha \in \mathbb{N}$ and $n, k \in \mathbb{Z}_+$, we have

$$\tilde{E}_{n,q,w}^{(\alpha)}(0, k|x) = \frac{[2]_q^k}{[\alpha]_q^n (1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{(-wq^{\alpha l - k + 1} : q)_k},$$

and

$$\sum_{l=0}^{n} \binom{n}{l} [\alpha]_q^l (q-1)^l \tilde{E}_{l,q,w}^{(\alpha)}(0, k|x) = \frac{q^{\alpha n x} [2]_q^k}{(-w q^{\alpha n - k + 1} : q)_k},$$

where $(a : q)_0 = 1$ and $(a : q)_k = (1 - a)(1 - aq) \ldots (1 - aq^{k-1})$.

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_k} \left[x + \sum_{j=1}^{k} x_j \right]_{q^\alpha}^n q^{-\sum_{j=1}^{k} j x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1, \cdots, a_k = 0}^{d-1} w^{a_1 + \cdots + a_k} q^{-\sum_{j=2}^{k} (j-1) a_j} (-1)^{\sum_{j=1}^{k} a_j} \times \tag{11}$$

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{d(x_1 + \cdots + x_k)} \left[\frac{x + \sum_{j=1}^{k} a_j}{d} + \sum_{j=1}^{k} x_j \right]_{q^{\alpha d}}^n q^{-d \sum_{j=1}^{k} j x_j} d\mu_{-q^d}(x_1) \cdots d\mu_{-q^d}(x_k)$$

Thus, by (11), we obtain the following theorem.

Theorem 4. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have

$$\tilde{E}_{n,q,w}^{(\alpha)}(0, k|x)$$

$$= \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1, \cdots, a_k = 0}^{d-1} (-w)^{a_1 + \cdots + a_k} q^{-\sum_{j=2}^{k} (j-1) a_j} \tilde{E}_{n,q^d,w^d}^{(\alpha)} \left(0, k|\frac{x + a_1 + \cdots + a_k}{d} \right).$$

Moreover,

$$\tilde{E}_{n,q,w}^{(\alpha)}(0, k|dx)$$

$$= \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1, \cdots, a_k = 0}^{d-1} (-w)^{a_1 + \cdots + a_k} q^{-\sum_{j=2}^{k} (j-1) a_j} \tilde{E}_{n,q^d,w^d}^{(\alpha)} \left(0, k|x + \frac{a_1 + \cdots + a_k}{d} \right).$$

By (8), we get

$$\tilde{E}_{n,q,w}^{(\alpha)}(0, k|x = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{E}_{l,q,w}^{(\alpha)}(0, k)$$

$$= \left([x]_{q^\alpha} + q^{\alpha x} \tilde{E}_{q,w}^{(\alpha)}(0, k) \right)^n,$$

where $\tilde{E}_{n,q,w}^{(\alpha)}(0, k|0) = \tilde{E}_{n,q,w}^{(\alpha)}(0, k)$.

Thus, we note that

$$\tilde{E}_{n,q,w}^{(\alpha)}(0, k|x + \gamma = \sum_{l=0}^{n} \binom{n}{l} [\gamma]_{q^\alpha}^{n-l} q^{\alpha l y} \tilde{E}_{l,q,w}^{(\alpha)}(0, k|x)$$

$$= \left([\gamma]_{q^\alpha} + q^{\alpha \gamma} \tilde{E}_{q,w}^{(\alpha)}(0, k|x) \right)^n.$$

4. Polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(h, 1|x)$

Let us define polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(h, 1|x)$ as follows:

$$\tilde{E}_{n,q,w}^{(\alpha)}(h, 1|x) = \int_{\mathbb{Z}_p} w^{x_1} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1). \tag{12}$$

From (12), we have

$$\tilde{E}_{n,q,w}^{(\alpha)}(h, 1 | x) = \frac{[2]_q}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{(1 + w q^{\alpha l + h})}.$$

By the calculation of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we see that

$$
\begin{aligned}
q^{\alpha x} & \int_{\mathbb{Z}_p} w^{x_1} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1) \\
&= (q^\alpha - 1) \int_{\mathbb{Z}_p} w^{x_1} [x + x_1]_{q^\alpha}^{n+1} q^{x_1(h-\alpha-1)} d\mu_{-q}(x_1) + \int_{\mathbb{Z}_p} w^{x_1} [x + x_1]_{q^\alpha}^n q^{x_1(h-\alpha-1)} d\mu_{-q}(x_1).
\end{aligned}
\tag{13}
$$

Thus, by (13), we obtain the following theorem.

Theorem 5. For $\alpha \in \mathbb{N}$ and $h \in \mathbb{Z}$, we have

$$q^{\alpha x} \tilde{E}_{n,q,w}^{(\alpha)}(h, 1 | x) = (q^\alpha - 1) \tilde{E}_{n+1,q,w}^{(\alpha)}(h - \alpha, 1 | x) + \tilde{E}_{n,q,w}^{(\alpha)}(h - \alpha, 1 | x).$$

It is easy to show that

$$
\begin{aligned}
\tilde{E}_{n,q,w}^{(\alpha)}(h, 1 | x) &= \int_{\mathbb{Z}_p} w^{x_1} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1) \\
&= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-1} q^{\alpha l x} \int_{\mathbb{Z}_p} w^{x_1} [x_1]_{q^\alpha}^l q^{x_1(h-1)} d\mu_{-q}(x_1) \\
&= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-1} q^{\alpha l x} \tilde{E}_{l,q,w}^{(\alpha)}(h, 1) \\
&= \left(q^{\alpha x} \tilde{E}_{q,w}^{(\alpha)}(h, 1) + [x]_{q^\alpha} \right)^n, \quad \text{for } n \geq 1,
\end{aligned}
\tag{14}
$$

with the usual convention about replacing $(\tilde{E}_{q,w}^{(\alpha)}(h, 1))^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}(h, 1)$.

From $q I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0)$, we have

$$
\begin{aligned}
w q^h & \int_{\mathbb{Z}_p} w^{x_1} [x + x_1 + 1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1) + \int_{\mathbb{Z}_p} w^{x_1} [x + x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q}(x_1) \\
&= [2]_q [x]_{q^\alpha}^n.
\end{aligned}
\tag{15}
$$

By (13) and (15), we get

$$w q^h \tilde{E}_{n,q,w}^{(\alpha)}(h, 1 | x + 1) + \tilde{E}_{n,q,w}^{(\alpha)}(h, 1 | x) = [2]_q [x]_{q^\alpha}^n. \tag{16}$$

For $x = 0$ in (16), we have

$$w q^h \tilde{E}_{n,q,w}^{(\alpha)}(h, 1 | 1) + \tilde{E}_{n,q,w}^{(\alpha)}(h, 1) = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{17}$$

Therefore, by (14) and (17), we obtain the following theorem.

Theorem 6. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$w q^h (q^\alpha \tilde{E}_{q,w}^{(\alpha)}(h, 1) + 1)^n \tilde{E}_{n,q,w}^{(\alpha)}(h, 1) = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing $(\tilde{E}_{q,w}^{(\alpha)}(h, 1))^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}(h, 1)$.

From the fermionic p-adic q-integral on $\mathbb{Z}_p$, we easily get

$$\tilde{E}_{0,q,w}^{(\alpha)}(h,1) = \int_{\mathbb{Z}_p} w^{x_1} q^{x_1(h-1)} d\mu_{-q}(x_1) = \frac{[2]_q}{[2]_{wq^h}}.$$

By (12), we see that

$$\begin{aligned}
\tilde{E}_{n,q^{-1},w^{-1}}^{(\alpha)}(h,1|1-x) &= \int_{\mathbb{Z}_p} w^{-1}[1-x+x_1]_{q^{-\alpha}}^n q^{-x_1(h-1)} d\mu_{-q^{-1}}(x_1) \\
&= (-1)^n wq^{\alpha n+h-1} \frac{[2]_q}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l}(-1)^l q^{\alpha lx} \frac{1}{1+wq^{\alpha l+h}} \\
&= (-1)^n wq^{\alpha n+h-1} \tilde{E}_{n,q,w}^{(\alpha)}(h,1|x)
\end{aligned} \tag{18}$$

Therefore, by (18), we obtain the following theorem.

Theorem 7. For $\alpha \in \mathbb{N}$, $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$\tilde{E}_{n,q^{-1},w^{-1}}^{(\alpha)}(h,1|1-x) = (-1)^n wq^{\alpha n+h-1} \tilde{E}_{n,q,w}^{(\alpha)}(h,1|x).$$

In particular, for $x = 1$, we get

$$\begin{aligned}
&\tilde{E}_{n,q,w}^{(\alpha)}(h,1) \\
&= (-1)^n wq^{\alpha n+h-1} \tilde{E}_{n,q,w}^{(\alpha)}(h,1|1) \\
&= (-1)^{n+1} q^{\alpha n-1} \tilde{E}_{n,q,w}^{(\alpha)}(h,1) \text{ if } n \geq 1.
\end{aligned}$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} w^{x_1} q^{x_1(h-1)} [x+x_1]_{q^\alpha}^n d\mu_{-q}(x_1) \\
&= \frac{[d]_{q^\alpha}^n}{[d]_{-q}} \sum_{a=0}^{d-1} w^a q^{ha}(-1)^a \int_{\mathbb{Z}_p} w^{dx_1} \left[\frac{x+a}{d}+x_1\right]_{q^{\alpha d}}^n q^{x_1(h-1)d} d\mu_{-q^d}(x_1).
\end{aligned} \tag{19}$$

Therefore, by (19), we obtain the following theorem.

Theorem 8 (Multiplication formula). For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$\tilde{E}_{n,q,w}^{(\alpha)}(h,1|x) = \frac{[d]_{q^\alpha}^n}{[d]_{-q}} \sum_{a=0}^{d-1} w^a q^{ha}(-1)^q \tilde{E}_{n,q^d,w^d}^{(\alpha)}\left(h,1|\frac{x+a}{d}\right).$$

5. Polynomials $\tilde{E}_{n,q,w}^{(\alpha)}(h,k|x)$ and $k = h$

In (3), we know that

$$\begin{aligned}
&\tilde{E}_{n,q,w}^{(\alpha)}(h,k|x) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x_1+\cdots+x_k+x]_{q^\alpha}^n q^{(h-1)x_1+\cdots+(h-k)x_k} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k).
\end{aligned}$$

Thus, we get

$$(q^\alpha-1)^n \tilde{E}_{n,q,w}^{(\alpha)}(h,k|x) = [2]_q^k \sum_{l=0}^n \binom{n}{l}(-1)^{n-l} \frac{q^{\alpha lx}}{(1+wq^{\alpha l+h})\cdots(1+wq^{\alpha l+h-k+1})},$$

and

$$
wq^h \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x + 1 + \sum_{i=1}^{k} x_i]_{q^\alpha}^n q^{\sum_{i=1}^{k}(h-i)x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
$$

$$
= - \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x + \sum_{i=1}^{k} x_i]_{q^\alpha}^n q^{\sum_{i=1}^{k}(h-i)x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \tag{20}
$$

$$
+ [2]_q \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_2+\cdots+x_k} [x + \sum_{i=2}^{k} x_i]_{q^\alpha}^n q^{\sum_{i=2}^{k}(h-i)x_i} d\mu_{-q}(x_2) \cdots d\mu_{-q}(x_k).
$$

Therefore, by (3) and (20), we obtain the following theorem.

Theorem 9. For $h \in \mathbb{Z}$, $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we have

$$
wq^h \tilde{E}_{n,q,w}^{(\alpha)}(h, k|x + 1) + \tilde{E}_{n,q,w}^{(\alpha)}(h, k|x) = [2]_q \tilde{E}_{n,q,w}^{(\alpha)}(h - 1, k - 1|x).
$$

Note that

$$
q^{\alpha x} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+w_k} [x + \sum_{i=1}^{k} x_i]_{q^\alpha}^n q^{\sum_{i=1}^{k}(h-i)x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
$$

$$
= (q^\alpha - 1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x + \sum_{i=1}^{k} x_i]_{q^\alpha}^{n+1} q^{\sum_{i=1}^{k}(h-\alpha-i)x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \tag{21}
$$

$$
+ \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x + \sum_{i=1}^{k} x_i]_{q^\alpha}^n q^{\sum_{i=1}^{k}(h-\alpha-i)x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
$$

$$
= (q^\alpha - 1) \tilde{E}_{n+1,q,w}^{(\alpha)}(h - \alpha, k|x) + \tilde{E}_{n,q,w}^{(\alpha)}(h - \alpha, k|x).
$$

Therefore, by (21), we obtain the following theorem.

Theorem 10. For $n \in \mathbb{Z}_+$, we have

$$
q^{\alpha x} \tilde{E}_{n,q,w}^{(\alpha)}(h, k|x) = (q^\alpha - 1) \tilde{E}_{n+1,q,w}^{(\alpha)}(h - \alpha, k|x) + \tilde{E}_{n,q,w}^{(\alpha)}(h - \alpha, k|x).
$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then we get

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{x_1+\cdots+x_k} [x + \sum_{j=1}^{k} x_j]_{q^\alpha}^n q^{\sum_{j=1}^{k}(h-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
$$

$$
= \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1,\cdots,a_k=0}^{d-1} w^{a_1+\cdots+a_k} q^{h\sum_{j=1}^{k} aj - \sum_{j=2}^{k}(j-1)a_j} (-1)^{\sum_{j=1}^{k} a_j} \times \tag{22}
$$

$$
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{d(x_1+\cdots+x_k)} \left[\frac{x + \sum_{j=1}^{k} a_j}{d} + \sum_{j=1}^{k} x_j \right]_{q^{\alpha d}}^n q^{d\sum_{j=1}^{k}(h-j)x_j} d\mu_{-q^d}(x_1) \cdots d\mu_{-q^d}(x_k).
$$

Therefore, by (22), we obtain the following theorem.

Theorem 11. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have

$$
\tilde{E}_{n,q,w}^{(\alpha)}(h, k|dx)
$$

$$
= \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k} \sum_{a_1,\cdots,a_k=0}^{d-1} w^{a_1+\cdots+a_k} q^{h\sum_{j=1}^{k} aj - \sum_{j=2}^{k}(j-1)a_j} (-1)^{\sum_{j=1}^{k} a_j} \tilde{E}_{n,q^d,w^d}^{(\alpha)} \left(h, k|x + \frac{a_1 + \cdots + a_k}{d} \right).
$$

Let $\tilde{E}_{n,q,w}^{(\alpha)}(k, k|x) = \tilde{E}_{n,q,w}^{(\alpha)}(k|x)$. Then we get

$$(q^{\alpha} - 1)^n \tilde{E}_{n,q,w}^{(\alpha)}(k|x) = \sum_{l=0}^{n} \binom{n}{l}(-1)^{n-l} q^{\alpha l x} \frac{[2]_q^k}{(1 + wq^{\alpha l+k}) \cdots (1 + wq^{\alpha l+1})},$$

and

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{-(x_1 + \cdots + x_k)} [k - x + \sum_{i=1}^{k} x_i]_{q^{-\alpha}}^n q^{-\sum_{i=1}^{k}(k-i)x_i} d\mu_{-q^{-1}}(x_1) \cdots d\mu_{-q^{-1}}(x_k)$$

$$= \frac{q^{\binom{k}{2}}}{(1 - q^{-\alpha})^n} [2]_q^k \sum_{l=0}^{n} \binom{n}{l}(-1)^l q^{\alpha l x} \frac{1}{(1 + wq^{\alpha l+1}) \cdots (1 + wq^{\alpha l+k})}$$

$$= (-1)^n q^{n\alpha} q^{\binom{k}{2}} \frac{[2]_q^k}{(1 - q^{\alpha})^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{\alpha l x}}{(1 + wq^{\alpha l+1}) \cdots (1 + wq^{\alpha l+k})}$$

$$= (-1)^n q^{\alpha n + \binom{k}{2}} \tilde{E}_{n,q,w}^{(\alpha)}(k|x).$$

(23)

Therefore, by (23), we obtain the following theorem.

Theorem 12. For $n \in \mathbb{Z}_+$, we have

$$\tilde{E}_{n,q^{-1},w^{-1}}^{(\alpha)}(k|k - x) = (-1)^n w^k q^{\alpha n + \binom{k}{2}} \tilde{E}_{n,q,w}^{(\alpha)}(k|x).$$

Let $x = k$ in Theorem 12. Then we see that

$$\tilde{E}_{n,q^{-1},w^{-1}}^{(\alpha)}(k|0) = (-1)^n w^k q^{\alpha n + \binom{k}{2}} \tilde{E}_{n,q,w}^{(\alpha)}(k|k).$$

(24)

From (15), we note that

$$wq^k \tilde{E}_{n,q,w}^{(\alpha)}(k|x + 1) + \tilde{E}_{n,q,w}^{(\alpha)}(k|x) = [2]_q \tilde{E}_{n,q,w}^{(\alpha)}(k - 1|x).$$

(25)

It is easy to show that

$$(q^{\alpha} - 1)^n \tilde{E}_{n,q,w}^{(\alpha)}(k|0) = \sum_{l=0}^{n} \binom{n}{l}(-1)^{l+n} \frac{[2]_q^k}{(1 + wq^{\alpha l+1}) \cdots (1 + wq^{\alpha l+k})}.$$

By simple calculation, we get

$$\sum_{l=0}^{n} \binom{n}{l}(q^{\alpha} - 1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{\sum_{i=1}^{k} x_k} [\sum_{i=1}^{k} x_k]_{q^{\alpha}}^l q^{\sum_{l=i}^{k}(k-i)x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= \frac{[2]_q^k}{(1 + wq^{\alpha n+k})(1 + wq^{\alpha n+k-1}) \cdots (1 + wq^{\alpha n+1})}.$$

(26)

From (26), we note that

$$\sum_{l=0}^{n} \binom{n}{l}(q^{\alpha} - 1)^l \tilde{E}_{l,q,w}^{(\alpha)}(k|0) = \frac{[2]_q^k}{(1 + wq^{\alpha n+k})(1 + wq^{\alpha n+k-1}) \cdots (1 + wq^{\alpha n+1})},$$

and

$$\tilde{E}_{n,q,w}^{(\alpha)}(k|x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{\sum_{i=1}^{k} x_k}[x + \sum_{i=1}^{k} x_k]_{q^\alpha}^n q^{\sum_{i=1}^{k}(k-i)x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= \sum_{l=0}^{n} \binom{n}{l}[x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{E}_{l,q,w}^{(\alpha)}(k|0)$$

$$= \left(q^{x\alpha} \tilde{E}_{q,w}^{(\alpha)}(k|0) + [x]_{q^\alpha} \right)^n, n \in \mathbb{Z}_+,$$

with the usual convention about replacing $(\tilde{E}_{q,w}^{(\alpha)}(k|0))^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}(k|0)$.

Put $x = 0$ in (25), we get

$$w q^k \tilde{E}_{n,q,w}^{(\alpha)}(k|1) + \tilde{E}_{n,q,w}^{(\alpha)}(k|0) = [2]_q \tilde{E}_{n,q}^{(\alpha)}(k-1|0).$$

Thus, we have

$$w q^k (q^\alpha \tilde{E}_{q,w}^{(\alpha)}(k|0) + 1)^n + \tilde{E}_{n,q,w}^{(\alpha)}(k|0) = [2]_q \tilde{E}_{n,q,w}^{(\alpha)}(k-1|0),$$

with the usual convention about replacing $(\tilde{E}_{q,w}^{(\alpha)}(k|0))^n$ by $\tilde{E}_{n,q,w}^{(\alpha)}(k|0)$.

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