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Oscillation of a class of second-order linear impulsive differential equations

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Abstract

In this paper, we investigate the oscillation of a class of second-order linear impulsive differential equations of the form

$$\begin{cases} (a(t)[x'(t)+\lambda x(t)])' + p(t)x(t) = 0, & t \geq t_0, t \neq t_k, \\ x(t_k^+) = b_k x(t_k), \quad x'(t_k^+) = c_k x'(t_k), & k = 1,2,\ldots. \end{cases}$$

By using the equivalence transformation and the associated Riccati techniques, some interesting results are obtained.

MSC: 34A37; 34C10

Keywords: oscillation; nonoscillation; impulsive differential equation

1 Introduction

Impulsive differential equations are recognized as adequate mathematical models for studying evolution processes that are subject to abrupt changes in their states at certain moments. Many applications in physics, biology, control theory, economics, applied sciences and engineering exhibit impulse effects (see [1–4]). In recent years, the study of the oscillation of all solutions of impulsive differential equations have been the subject of many research works. See, for example, [5–11] and the references cited therein.

In this article, we consider the second-order linear impulsive differential equation of the form

$$\begin{cases} (a(t)[x'(t)+\lambda x(t)])' + p(t)x(t) = 0, & t \geq t_0, t \neq t_k, \\ x(t_k^+) = b_k x(t_k), \quad x'(t_k^+) = c_k x'(t_k), & k = 1,2,\ldots, \end{cases} \tag{1.1}$$

where $0 \leq t_0 < t_1 < \cdots < t_k \to \infty$, $\lim_{k\to\infty} t_k = +\infty$, $a(t) \in C([t_0,\infty),(0,\infty))$ and $p(t) \in C([t_0,\infty),\mathbb{R})$, $\{b_k\}$ and $\{c_k\}$ are two known sequences of positive real numbers, λ is a real number, and

$$\begin{aligned} x'(t_k) = x'(t_k^-) &= \lim_{h\to 0^-} \frac{x(t_k+h)-x(t_k)}{h}, \\ x'(t_k^+) &= \lim_{h\to 0^+} \frac{x(t_k+h)-x(t_k^+)}{h}. \end{aligned}$$

Let $J \subset \mathbb{R}$ be an interval and $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ be continuous everywhere except at some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)\}$.

A function $x \in PC([t_0,\infty),\mathbb{R})$ is said to be a solution of Eq. (1.1) if

(i) $x(t)$ satisfies $(a(t)[x'(t) + \lambda x(t)])' + p(t)x(t) = 0$ for $t \in [t_0,\infty)$ and $t \neq t_k$,

(ii) $x(t_k^+) = b_k x(t_k)$, $x'(t_k^+) = c_k x'(t_k)$ for each t_k, and $x(t)$ and $x'(t)$ are left continuous for each t_k, $k = 1,2,\ldots$.

Definition 1.1 A nontrivial solution of Eq. (1.1) is said to be nonoscillatory if the solution is eventually positive or eventually negative. Otherwise, it is said to be oscillatory. Eq. (1.1) is said to be oscillatory if all solutions are oscillatory.

If $\lambda = 0$, then Eq. (1.1) reduces to the second-order linear differential equation with impulses

$$\begin{cases} (a(t)x'(t))' + p(t)x(t) = 0, & t \geq t_0, t \neq t_k, \\ x(t_k^+) = b_k x(t_k), \qquad x'(t_k^+) = c_k x'(t_k), & k = 1,2,\ldots. \end{cases} \tag{1.2}$$

In [12] Luo *et al.* and [13] Guo *et al.* gave some excellent results on the oscillation and nonoscillation of Eq. (1.2) by using associated Riccati techniques and an equivalence transformation. Moreover, Luo *et al.* showed that the oscillation of Eq. (1.2) can be caused by impulsive perturbations, though the corresponding equation without impulses admits a nonoscillatory solution.

If $a(t) \equiv 1$ and $\lambda \neq 0$, then Eq. (1.1) reduces to the impulsive Langevin equation of the form

$$\begin{cases} \frac{d}{dt}(\frac{d}{dt} + \lambda)x(t) + p(t)x(t) = 0, & t \geq t_0, t \neq t_k, \\ x(t_k^+) = b_k x(t_k), \qquad x'(t_k^+) = c_k x'(t_k), & k = 1,2,\ldots. \end{cases} \tag{1.3}$$

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments. For more details of the Langevin equation without impulses and its applications, we refer the reader to [14].

If $\lambda = 0$ and $b_k = c_k = 1$ for all $k = 1,2,\ldots$, then Eq. (1.1) reduces to the self-adjoint second-order differential equation

$$\big(a(t)x(t)\big)' + p(t)x(t) = 0, \quad t > t_0. \tag{1.4}$$

There are many good results on the oscillation and nonoscillation of Eq. (1.4); see, for example, [15–18].

2 Main results

Now we are in the position to establish the main result.

Lemma 2.1 *If the second-order differential equation*

$$\left[\left(\prod_{T\leq t_k<t} d_k^{-1}\right)a(t)\left\{y'(t) + \lambda\left(1 - \frac{2}{a(t)}\sum_{T\leq t_k<t}\prod_{t_k<t_j<t} d_j(1-d_k)a(t_k)\right)y(t)\right\}\right]'$$

$$+\left(\prod_{T\le t_k<t} d_k^{-1}\right)\left[p(t)+\frac{\lambda^2}{a(t)}\left(\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j(1-d_k)a(t_k)\right)^2 - \lambda^2\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j(1-d_k)a(t_k)\right]y(t)=0,\quad t>T, \tag{2.1}$$

is oscillatory, then Eq. (1.1) *is oscillatory, where* $d_k = c_k/b_k$, $k = 1, 2, \ldots$.

Proof For the sake of contradiction, suppose that Eq. (1.1) has an eventually positive solution $x(t)$. Then there exists a constant $T \ge t_0$ such that $x(t) > 0$ for $t \ge T$.

We let

$$u(t)=\frac{a(t)(e^{\lambda t}x(t))'}{e^{\lambda t}x(t)}=\frac{a(t)(x'(t)+\lambda x(t))}{x(t)},\quad t\ge T.$$

Then

$$\begin{aligned}
u'(t) &= \frac{e^{\lambda t}x(t)[a(t)(e^{\lambda t}x(t))']' - a(t)[(e^{\lambda t}x(t))']^2}{(e^{\lambda t}x(t))^2}\\
&= \frac{[a(t)(e^{\lambda t}x(t))']'}{e^{\lambda t}x(t)} - \frac{a(t)[(e^{\lambda t}x(t))']^2}{(e^{\lambda t}x(t))^2}\\
&= \frac{(e^{\lambda t}[a(t)(x'(t)+\lambda x(t))])'}{e^{\lambda t}x(t)} - \frac{a(t)[(e^{\lambda t}x(t))']^2}{(e^{\lambda t}x(t))^2}\\
&= \frac{(a(t)(x'(t)+\lambda x(t)))'}{x(t)} + \lambda\frac{a(t)(x'(t)+\lambda x(t))}{x(t)}\\
&\quad - \frac{a(t)[(e^{\lambda t}x(t))']^2}{(e^{\lambda t}x(t))^2}\\
&= -p(t)+\lambda u(t) - \frac{u^2(t)}{a(t)}
\end{aligned}$$

can be obtained. Therefore,

$$u'(t)-\lambda u(t)+\frac{u^2(t)}{a(t)}+p(t)=0,\quad t\ge T, t\ne t_k. \tag{2.2}$$

On the other hand, we have

$$u(t_k^+)=\frac{a(t_k^+)(x'(t_k^+)+\lambda x(t_k^+))}{x(t_k^+)}=\frac{a(t_k)(c_k x'(t_k)+\lambda b_k x(t_k))}{b_k x(t_k)}.$$

Let $e_k = \lambda(1-d_k)a(t_k)$, then we get

$$u(t_k^+)=d_k u(t_k)+e_k,\quad t_k\ge T, k=1,2,\ldots. \tag{2.3}$$

Now, we define

$$v(t)=\prod_{T\le t_k<t} d_k^{-1}\left[u(t)-\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j e_k\right],\quad t>T. \tag{2.4}$$

Then, for $t_n > T$, we get that

$$\begin{aligned} v(t_n^+) &= \prod_{T \le t_k \le t_n} d_k^{-1}\left[u(t_n^+) - \sum_{T \le t_k \le t_n}\prod_{t_k < t_j \le t_n} d_j e_k\right] \\ &= \prod_{T \le t_k \le t_n} d_k^{-1}\left[d_n u(t_n) + e_n - \sum_{T \le t_k < t_n}\prod_{t_k < t_j < t_n} d_j d_n e_k - e_n\right] \\ &= \prod_{T \le t_k < t_n} d_k^{-1}\left[u(t_n) - \sum_{T \le t_k < t_n}\prod_{t_k < t_j < t_n} d_j e_k\right] \\ &= v(t_n). \end{aligned}$$

Therefore, $v(t)$ is continuous on (T, ∞).

We have

$$\left(\prod_{T \le t_k < t} d_k^{-1}\right) \times \left(\sum_{T \le t_k < t}\prod_{t_k < t_j < t} d_j e_k\right) = \sum_{T \le t_k < t}\prod_{T \le t_j \le t_k} d_j^{-1} e_k.$$

Then, for $t > T$, $t \neq t_n$ and from (2.2), we get that

$$\begin{aligned} v'(t) &= \lim_{h \to 0}\left(\prod_{T \le t_k < t+h} d_k^{-1}\left[u(t+h) - \sum_{T \le t_k < t+h}\prod_{t_k < t_j < t+h} d_j e_k\right]\right. \\ &\quad \left. - \prod_{T \le t_k < t} d_k^{-1}\left[u(t) - \sum_{T \le t_k < t}\prod_{t_k < t_j < t} d_j e_k\right]\right) \Big/ h \\ &= \lim_{h \to 0} \frac{\prod_{T \le t_k < t+h} d_k^{-1} u(t+h) - \prod_{T \le t_k < t} d_k^{-1} u(t)}{h} \\ &= \left(\prod_{T \le t_k < t} d_k^{-1}\right) \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} \\ &= \left(\prod_{T \le t_k < t} d_k^{-1}\right) u'(t) = \left(\prod_{T \le t_k < t} d_k^{-1}\right)\left[\lambda u(t) - \frac{u^2(t)}{a(t)} - p(t)\right] \\ &= \left(\prod_{T \le t_k < t} d_k^{-1}\right)\left\{\lambda\left[v(t)\left(\prod_{T \le t_k < t} d_k\right) + \sum_{T \le t_k < t}\prod_{t_k < t_j < t} d_j e_k\right]\right. \\ &\quad \left. - \frac{1}{a(t)}\left[v(t)\left(\prod_{T \le t_k < t} d_k\right) + \sum_{T \le t_k < t}\prod_{t_k < t_j < t} d_j e_k\right]^2 - p(t)\right\} \\ &= \lambda v(t) + \lambda \sum_{T \le t_k < t}\prod_{T \le t_j \le t_k} d_j^{-1} e_k - \left(\prod_{T \le t_k < t} d_k\right)\frac{v^2(t)}{a(t)} \\ &\quad - 2\frac{v(t)}{a(t)} \sum_{T \le t_k < t}\prod_{t_k < t_j < t} d_j e_k - \frac{1}{a(t)}\left(\prod_{T \le t_k < t} d_k^{-1}\right) \\ &\quad \times \left(\sum_{T \le t_k < t}\prod_{t_k < t_j < t} d_j e_k\right)^2 - \left(\prod_{T \le t_k < t} d_k^{-1}\right) p(t). \end{aligned}$$

The left-hand and the right-hand derivatives of $v(t)$ at $t = t_n$ are given by

$$v'(t_n^-) = \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) u'(t_n^-) = \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) u'(t_n),$$

$$v'(t_n^+) = \left(\prod_{T\le t_k\le t_n} d_k^{-1}\right) u'(t_n^+) = \left(\prod_{T\le t_k\le t_n} d_k^{-1}\right) d_n u'(t_n) = \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) u'(t_n).$$

Hence, for $t = t_n$, we have

$$\begin{aligned}
v'(t_n) &= \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) u'(t_n) = \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) \lim_{t\to t_n^-} u'(t) \\
&= \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) \lim_{t\to t_n^-} \left[\lambda u(t) - \frac{u^2(t)}{a(t)} - p(t)\right] \\
&= \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) \left[\lambda u(t_n) - \frac{u^2(t_n)}{a(t_n)} - p(t_n)\right] \\
&= \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) \left\{\lambda \left[v(t_n)\left(\prod_{T\le t_k<t_n} d_k\right) + \sum_{T\le t_k<t_n} \prod_{t_k<t_j<t_n} d_j e_k\right]\right. \\
&\quad \left. - \frac{1}{a(t_n)}\left[v(t_n)\left(\prod_{T\le t_k<t_n} d_k\right) + \sum_{T\le t_k<t_n} \prod_{t_k<t_j<t_n} d_j e_k\right]^2 - p(t_n)\right\} \\
&= \lambda v(t_n) + \lambda \sum_{T\le t_k<t_n} \prod_{T\le t_j\le t_k} d_j^{-1} e_k - \left(\prod_{T\le t_k<t_n} d_k\right) \frac{v^2(t_n)}{a(t_n)} \\
&\quad - 2\frac{v(t_n)}{a(t_n)} \sum_{T\le t_k<t_n} \prod_{t_k<t_j<t_n} d_j e_k - \frac{1}{a(t_n)}\left(\prod_{T\le t_k<t_n} d_k^{-1}\right) \\
&\quad \times \left(\sum_{T\le t_k<t_n} \prod_{t_k<t_j<t_n} d_j e_k\right)^2 - \left(\prod_{T\le t_k<t_n} d_k^{-1}\right) p(t_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
&v'(t) + \left(\prod_{T\le t_k<t} d_k\right) \frac{v^2(t)}{a(t)} - \lambda v(t) + 2\frac{v(t)}{a(t)} \sum_{T\le t_k<t} \prod_{t_k<t_j<t} d_j e_k \\
&\quad - \lambda \sum_{T\le t_k<t} \prod_{T\le t_j\le t_k} d_j^{-1} e_k + \frac{1}{a(t)}\left(\prod_{T\le t_k<t} d_k^{-1}\right)\left(\sum_{T\le t_k<t} \prod_{t_k<t_j<t} d_j e_k\right)^2 \\
&\quad + \left(\prod_{T\le t_k<t} d_k^{-1}\right) p(t) = 0, \quad t > T.
\end{aligned} \tag{2.5}$$

We set

$$w(t) = \exp\left\{\int_T^t \left[\left(\prod_{T\le t_k<s} d_k\right)\frac{v(s)}{a(s)} - \lambda + \frac{2}{a(s)} \sum_{T\le t_k<s} \prod_{t_k<t_j<s} d_j e_k\right] ds\right\}, \quad t > T.$$

Then, $w(t) > 0$ for $t > T$ and

$$w'(t) = w(t)\Bigg[\Bigg(\prod_{T\le t_k<t} d_k\Bigg)\frac{v(t)}{a(t)} - \lambda + \frac{2}{a(t)}\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j e_k\Bigg].$$

From (2.5), we obtain

$$\begin{aligned}
&\Bigg[\Bigg(\prod_{T\le t_k<t} d_k^{-1}\Bigg)a(t)w'(t) + \lambda\Bigg(\prod_{T\le t_k<t} d_k^{-1}\Bigg)a(t)w(t) - 2\sum_{T\le t_k<t}\prod_{T\le t_j\le t_k} d_j^{-1}e_k w(t)\Bigg]' \\
&\quad = w(t)v'(t) + v(t)w'(t) \\
&\quad = w(t)\Bigg[v'(t) + \Bigg(\prod_{T\le t_k<t} d_k\Bigg)\frac{v^2(t)}{a(t)} - \lambda v(t) + 2\frac{v(t)}{a(t)}\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j e_k\Bigg] \\
&\quad = w(t)\Bigg[\lambda\sum_{T\le t_k<t}\prod_{T\le t_j\le t_k} d_j^{-1}e_k - \frac{1}{a(t)}\Bigg(\prod_{T\le t_k<t} d_k^{-1}\Bigg) \\
&\qquad \times\Bigg(\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j e_k\Bigg)^2 - \Bigg(\prod_{T\le t_k<t} d_k^{-1}\Bigg)p(t)\Bigg].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Bigg[\Bigg(\prod_{T\le t_k<t} d_k^{-1}\Bigg)a(t)\Bigg\{w'(t) + \lambda\Bigg(1 - \frac{2}{a(t)}\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j(1-d_k)a(t_k)\Bigg)w(t)\Bigg\}\Bigg]' \\
&\quad + \Bigg(\prod_{T\le t_k<t} d_k^{-1}\Bigg)\Bigg[p(t) + \frac{\lambda^2}{a(t)}\Bigg(\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j(1-d_k)a(t_k)\Bigg)^2 \\
&\quad - \lambda^2\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j(1-d_k)a(t_k)\Bigg]w(t) = 0, \quad t > T.
\end{aligned}$$

This implies that $w(t)$ is an eventually positive solution of Eq. (2.1) which is a contradiction. A similar argument can be used to prove that Eq. (2.1) cannot have an eventually negative solution. Therefore, from Definition 1.1, the solution of Eq. (2.1) is oscillatory. The proof is complete. □

Lemma 2.2 (Leighton type oscillation criteria) *Assume that the functions* $g(t), q(t) \in PC([t_0,\infty),\mathbb{R})$ *and* $h(t) \in PC([t_0,\infty),(0,\infty))$.

If

$$\int_T^\infty g(s)e^{-\int_T^s q(\sigma)\,d\sigma}\,ds = \infty \quad \textit{and} \quad \int_T^\infty \frac{1}{h(s)}e^{\int_T^s q(\sigma)\,d\sigma}\,ds = \infty,$$

then

$$\big(h(t)\big[y'(t) + q(t)y(t)\big]\big)' + g(t)y(t) = 0 \tag{2.6}$$

is oscillatory.

Proof Let $y(t)$ be a nonoscillatory solution of the Eq. (2.6). Without loss of generality, we assume that there exists a $T \geq t_0$ such that $y(t) > 0$ for $t \geq T$. We define

$$u(t) = \frac{h(t)(e^{\int_T^t q(\sigma)\,d\sigma} y(t))'}{e^{\int_T^t q(\sigma)\,d\sigma} y(t)}, \quad t \geq T.$$

Then the equation

$$u'(t) - q(t)u(t) + \frac{u^2(t)}{h(t)} + g(t) = 0 \tag{2.7}$$

has a solution $u(t)$ on $[T, \infty)$. It is easy to see that the solution of Eq. (2.7) satisfies the following equation:

$$\begin{aligned} u(t) &= e^{\int_T^t q(\sigma)\,d\sigma} u(T) - e^{\int_T^t q(\sigma)\,d\sigma} \int_T^t \frac{u^2(s)}{h(s)} e^{-\int_T^s q(\sigma)\,d\sigma}\,ds \\ &\quad - e^{\int_T^t q(\sigma)\,d\sigma} \int_T^t g(s) e^{-\int_T^s q(\sigma)\,d\sigma}\,ds. \end{aligned} \tag{2.8}$$

Since $\int_T^\infty g(s) e^{-\int_T^s q(\sigma)\,d\sigma}\,ds = \infty$, then there exists $\tau > T$ such that

$$u(T) - \int_T^t g(s) e^{-\int_T^s q(\sigma)\,d\sigma}\,ds < 0$$

for all t in $[\tau, \infty)$. Hence, from (2.8), it follows that

$$u(t) < -e^{\int_T^t q(s)\,ds} \int_T^t \frac{u^2(s)}{h(s)} e^{-\int_T^s q(\sigma)\,d\sigma}\,ds, \quad t \in [\tau, \infty).$$

Let

$$r(t) = \int_T^t \frac{u^2(s)}{h(s)} e^{-\int_T^s q(\sigma)\,d\sigma}\,ds, \quad t \in [\tau, \infty).$$

Then $u(t) < -r(t) e^{\int_T^t q(\sigma)\,d\sigma}$ and

$$r'(t) = \frac{u^2(t)}{h(t)} e^{-\int_T^t q(\sigma)\,d\sigma} > \frac{r^2(t)}{h(t)} e^{\int_T^t q(\sigma)\,d\sigma}, \quad t \in [\tau, \infty). \tag{2.9}$$

Integrating (2.9) from $\tau > T$ to ∞, we obtain

$$-\frac{1}{r(\infty)} + \frac{1}{r(\tau)} > \int_\tau^\infty \frac{1}{h(s)} e^{\int_T^s q(\sigma)\,d\sigma}\,ds.$$

Hence,

$$\int_\tau^\infty \frac{1}{h(s)} e^{\int_T^s q(\sigma)\,d\sigma}\,ds < \frac{1}{r(\tau)} < \infty,$$

which is a contradiction. Thus, the solution $y(t)$ is oscillatory. The proof is complete. □

Theorem 2.3 *Assume that*

$$\begin{aligned}&\int_T^\infty \Bigl(\prod_{T\le t_k<t} d_k^{-1}\Bigr)\Bigl[p(t)+\frac{\lambda^2}{a(t)}\Bigl(\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j(1-d_k)a(t_k)\Bigr)^2\\&\quad-\lambda^2\sum_{T\le t_k<t}\prod_{t_k<t_j<t} d_j(1-d_k)a(t_k)\Bigr]\exp\Bigl\{-\lambda\int_T^t\Bigl(1-\frac{2}{a(s)}\\&\quad\times\sum_{T\le t_k<s}\prod_{t_k<t_j<s} d_j(1-d_k)a(t_k)\Bigr)\,ds\Bigr\}\,dt=\infty \end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\int_T^\infty \Bigl(\prod_{T\le t_k<t} d_k\Bigr)\frac{1}{a(t)}\\&\quad\times\exp\Bigl\{\lambda\int_T^t\Bigl(1-\frac{2}{a(s)}\times\sum_{T\le t_k<s}\prod_{t_k<t_j<s} d_j(1-d_k)a(t_k)\Bigr)\,ds\Bigr\}\,dt=\infty,\end{aligned}\tag{2.11}$$

where $d_k = c_k/b_k$, $k = 1,2,\ldots$. *Then Eq.* (1.1) *is oscillatory.*

If $b_k = c_k$, $k = 1,2,\ldots$, then $d_k = 1$, $k = 1,2,\ldots$ and (1.1) becomes

$$\begin{cases}(a(t)[x'(t)+\lambda x(t)])' + p(t)x(t) = 0, & t\ge t_0, t\ne t_k,\\ x(t_k^+) = b_k x(t_k), \qquad x'(t_k^+) = b_k x'(t_k), & k = 1,2,\ldots.\end{cases}\tag{2.12}$$

Theorem 2.4 *Eq.* (2.12) *is oscillatory if and only if*

$$\bigl(a(t)\bigl[y'(t)+\lambda y(t)\bigr]\bigr)' + p(t)y(t) = 0, \quad t\ge t_0,\tag{2.13}$$

is oscillatory.

Proof From Lemma 2.1, we only need to prove that if Eq. (2.12) is oscillatory, then Eq. (2.13) is oscillatory.

Without loss of generality, we suppose that $y(t)$ is an eventually positive solution of (2.13) such that $y(t) > 0$ for $t \ge T \ge t_0$. Set

$$x(t) = \Bigl(\prod_{T\le t_k<t} b_k\Bigr)y(t), \quad t > T.$$

Then, for $t > T$, we have $x(t) > 0$, and for $t_n > T$,

$$x(t_n^+) = \Bigl(\prod_{T\le t_k\le t_n} b_k\Bigr)y(t_n^+) = b_n\Bigl(\prod_{T\le t_k<t_n} b_k\Bigr)y(t_n) = b_n x(t_n).$$

Moreover, for $t \ne t_n > T$, we have

$$x'(t) = \Bigl(\prod_{T\le t_k<t} b_k\Bigr)y'(t),$$

and

$$x'(t_n^+) = \left(\prod_{T\le t_k\le t_n} b_k\right) y'(t_n^+) = b_n\left(\prod_{T\le t_k< t_n} b_k\right) y'(t_n) = b_n x'(t_n) = c_n x'(t_n).$$

Now we have for $t \neq t_n$

$$\begin{aligned}
\bigl(a(t)\bigl[x'(t)+\lambda x(t)\bigr]\bigr)' &= \left(a(t)\left[\left(\prod_{T\le t_k<t} b_k\right) y'(t) + \lambda\left(\prod_{T\le t_k<t} b_k\right) y(t)\right]\right)' \\
&= \left(\prod_{T\le t_k<t} b_k\right)\bigl(a(t)\bigl[y'(t)+\lambda y(t)\bigr]\bigr)' \\
&= -\left(\prod_{T\le t_k<t} b_k\right) p(t)y(t) = -p(t)x(t).
\end{aligned}$$

Therefore,

$$\bigl(a(t)\bigl[x'(t)+\lambda x(t)\bigr]\bigr)' + p(t)x(t) = 0, \quad t \neq t_n, t > T.$$

We get that $x(t)$ is an eventually positive solution of (2.12), a contradiction, and so the proof is complete. □

Corollary 2.5 *Assume that*

$$\int_T^\infty p(t)e^{-\lambda t}\,dt = \infty \tag{2.14}$$

and

$$\int_T^\infty \frac{e^{\lambda t}}{a(t)}\,dt = \infty. \tag{2.15}$$

Then Eq. (2.12) *is oscillatory.*

3 Some examples

In this section, we illustrate our results with two examples.

Example 3.1 Consider the following impulsive Langevin equation:

$$\begin{cases} \frac{d}{dt}(\frac{d}{dt}+\frac{2}{3})x(t) + 5^{t^3}x(t) = 0, & t > 0, t \neq k, \\ x(k^+) = \frac{k}{k+1}x(k), \qquad x'(k^+) = x'(k), & k = 1, 2, \dots. \end{cases} \tag{3.1}$$

Set $d_k = \frac{k+1}{k}$, $\lambda = \frac{2}{3}$, $a(t) = a(t_k) \equiv 1$ and $p(t) = 5^{t^3}$. If $T \in (m, m+1]$ for some integer $m \ge 0$, then we get

$$\left(\prod_{T\le t_k<[t]+1} \frac{k}{k+1}\right) = \frac{m+1}{m+2}\cdot\frac{m+2}{m+3}\cdots\frac{[t]}{[t]+1} = \frac{m+1}{[t]+1},$$

and

$$\begin{aligned}
&\sum_{T\le t_k<[t]}\prod_{t_k<t_j<[t]}\left(\frac{j+1}{j}\right)\left(\frac{1}{k}\right)\\
&\quad=\frac{1}{m+1}\cdot\frac{m+3}{m+2}\cdot\frac{m+4}{m+3}\cdots\frac{[t]}{[t]-1}\\
&\qquad+\frac{1}{m+2}\cdot\frac{m+4}{m+3}\cdot\frac{m+5}{m+4}\cdots\frac{[t]}{[t]-1}+\cdots+\frac{1}{[t]-2}\cdot\frac{[t]}{[t]-1}+\frac{1}{[t]-1}\\
&\quad=[t]\left(\frac{1}{m+1}\cdot\frac{1}{m+2}+\frac{1}{m+2}\cdot\frac{1}{m+3}+\cdots+\frac{1}{[t]-2}\cdot\frac{1}{[t]-1}\right)+\frac{1}{[t]-1}\\
&\quad=[t]\left(\frac{1}{m+1}-\frac{1}{[t]-1}\right)+\frac{1}{[t]-1}\\
&\quad=\frac{[t]}{m+1}-1,
\end{aligned}$$

where $[\cdot]$ denotes the greatest integer function.

Hence,

$$\begin{aligned}
&\int_T^\infty\left(\prod_{T\le t_k<t}d_k^{-1}\right)\left[p(t)+\frac{\lambda^2}{a(t)}\left(\sum_{T\le t_k<t}\prod_{t_k<t_j<t}d_j(1-d_k)a(t_k)\right)^2\right.\\
&\qquad\left.-\lambda^2\sum_{T\le t_k<t}\prod_{t_k<t_j<t}d_j(1-d_k)a(t_k)\right]\exp\left\{-\lambda\int_T^t\left(1-\frac{2}{a(s)}\right.\right.\\
&\qquad\left.\left.\times\sum_{T\le t_k<s}\prod_{t_k<t_j<s}d_j(1-d_k)a(t_k)\right)ds\right\}dt\\
&\quad=\int_T^\infty\left(\prod_{T\le t_k<t}\frac{k}{k+1}\right)\left[5^{t^3}+\left(\frac{2}{3}\right)^2\left(\sum_{T\le t_k<t}\prod_{t_k<t_j<t}\left(\frac{j+1}{j}\right)\left(-\frac{1}{k}\right)\right)^2\right.\\
&\qquad\left.-\left(\frac{2}{3}\right)^2\sum_{T\le t_k<t}\prod_{t_k<t_j<t}\left(\frac{j+1}{j}\right)\left(-\frac{1}{k}\right)\right]\exp\left\{-\frac{2}{3}\int_T^t\left(1-2\right.\right.\\
&\qquad\left.\left.\times\sum_{T\le t_k<s}\prod_{t_k<t_j<s}\left(\frac{j+1}{j}\right)\left(-\frac{1}{k}\right)\right)ds\right\}dt\\
&\quad\ge\int_T^\infty\left(\prod_{T\le t_k<[t]+1}\frac{k}{k+1}\right)\left[5^{t^3}+\left(\frac{2}{3}\right)^2\left(\sum_{T\le t_k<[t]}\prod_{t_k<t_j<[t]}\left(\frac{j+1}{j}\right)\left(\frac{1}{k}\right)\right)\right.\\
&\qquad\left.\times\left(\sum_{T\le t_k<[t]}\prod_{t_k<t_j<[t]}\left(\frac{j+1}{j}\right)\left(\frac{1}{k}\right)+1\right)\right]\exp\left\{-\frac{2}{3}\int_T^t\left(1+2\right.\right.\\
&\qquad\left.\left.\times\sum_{T\le t_k<[s]+1}\prod_{t_k<t_j<[s]+1}\left(\frac{j+1}{j}\right)\left(\frac{1}{k}\right)\right)ds\right\}dt\\
&\quad\ge\int_T^\infty\left(\frac{m+1}{[t]+1}\right)\left[5^{t^3}+\left(\frac{2}{3}\right)^2\left(\frac{[t]}{m+1}-1\right)\right.\\
&\qquad\left.\times\left(\frac{[t]}{m+1}\right)\right]\exp\left\{-\frac{2}{3}\int_T^t\left(1+2\left(\frac{[s]+1}{m+1}-1\right)\right)ds\right\}dt
\end{aligned}$$

$$
\begin{aligned}
&\ge \int_T^\infty \left(\frac{m+1}{t+1}\right)(5^{t^3}) \exp\left\{-\frac{2}{3}\int_T^t \left(1+2\left(\frac{s+1}{T}\right)\right) ds\right\} dt \\
&= \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_T^\infty \left(\prod_{T\le t_k<t} d_k\right)\frac{1}{a(t)} \exp\left\{\lambda \int_T^t \left(1-\frac{2}{a(s)}\sum_{T\le t_k<s}\prod_{t_k<t_j<s} d_j(1-d_k)a(t_k)\right) ds\right\} dt \\
&= \int_T^\infty \left(\prod_{T\le t_k<t} \frac{k+1}{k}\right) \\
&\quad \times \exp\left\{\frac{2}{3}\int_T^t \left(1+2\sum_{T\le t_k<s}\prod_{t_k<t_j<s}\frac{j+1}{j}\cdot\frac{1}{k}\right) ds\right\} dt \\
&\ge \int_T^\infty \left(\prod_{T\le t_k<[t]} \frac{k+1}{k}\right) \exp\left\{\frac{2}{3}\int_T^t (1)\, ds\right\} dt \\
&= \int_T^\infty \left(\frac{m+2}{m+1}\cdot\frac{m+3}{m+2}\cdots\frac{[t]}{[t]-1}\right) \exp\left\{\frac{2}{3}\int_T^t (1)\, ds\right\} dt \\
&= \int_T^\infty \left(\frac{[t]}{m+1}\right) \exp\left\{\frac{2}{3}(t-T)\right\} dt \\
&\ge \int_T^\infty \left(\frac{t-1}{m+1}\right) \exp\left\{\frac{2}{3}(t-T)\right\} dt \\
&= \infty.
\end{aligned}
$$

By Theorem 2.3, Eq. (3.1) is oscillatory.

Example 3.2 Consider the equation

$$
\begin{cases}
\left(\frac{1}{(t^2+1)^{\frac{1}{2}}}[x'(t)+\frac{3}{4}x(t)]\right)' + \pi^t x(t) = 0, & t\ge 0, t\ne t_k, \\
x(t_k^+) = b_k x(t_k), \qquad x'(t_k^+) = b_k x'(t_k), & k=1,2,\ldots,
\end{cases}
\tag{3.2}
$$

where $\{b_k\}$ is a known sequence of positive real numbers. It is easy to see that

$$
\int_0^\infty \pi^t e^{-\frac{3}{4}t}\, dt = \infty
$$

and

$$
\int_0^\infty (t^2+1)^{\frac{1}{2}} e^{\frac{3}{4}t}\, dt = \infty.
$$

By Corollary 2.5, Eq. (3.2) is oscillatory.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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