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On the boundedness of the solutions in nonlinear discrete Volterra difference equations

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Abstract

In this article, we investigate the boundedness property of the solutions of linear and nonlinear discrete Volterra equations in both convolution and non-convolution case. Strong interest in these kind of discrete equations is motivated as because they represent a discrete analogue of some integral equations. The most important result of this article is a simple new criterion, which unifies and extends several earlier results in both discrete and continuous cases. Examples are also given to illustrate our main theorem.

1 Introduction

We consider the nonlinear system of Volterra difference equations

$$x(n+1) = \sum_{j=0}^{n} f(n,j,x(j)) + h(n), \quad n \geq 0, \tag{1.1}$$

with the initial condition

$$x(0) = x_0, \tag{1.2}$$

where

(**A**) The function $f(n,j,.) : \mathbb{R}^d \to \mathbb{R}^d$ is a mapping for any fixed $0 \leq j \leq n$, $x_0 \in \mathbb{R}^d$ and $h(n) \in \mathbb{R}^d$, $n \geq 0$.

(**B**) For any $0 \leq j \leq n$, there exists an $a(n, j) \in \mathbb{R}_+$, such that

$$||f(n,j,x)|| \leq a(n,j)\phi(||x||), \quad 0 \leq j \leq n, \quad x \in \mathbb{R}^d,$$

and

$$\sup_{n\geq 0} ||h(n)|| < \infty$$

hold. Here $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone non-decreasing mapping such that $\varphi(v) > 0$, $v > 0$, and $\varphi(0) = 0$ where $||.||$ is any fixed norm on $\mathbb{R}^d$.

In recent years, there has been an increasing interest in the study of the asymptotic behavior of the solutions of both convolution and non-convolution-type linear and nonlinear Volterra difference equations (see [1-17] and references therein). Appleby et al. [2], under appropriate assumptions, have proved that the solutions of the discrete linear Volterra equation converge to a finite limit, which in general is non-trivial. The main result on the boundedness of solutions of a linear Volterra difference system in

[2] was improved by Györi and Horváth [8]. In terms of the kernel of a linear system Györi and Reynolds [10] found necessary conditions for the solutions to be bounded. Also Györi and Reynolds [9] studied some connections between results obtained in [2,8]. Elaydi et al. [6] have shown that under certain conditions there is a one-to-one correspondence between bounded solutions of linear Volterra difference equations with infinite delay and its perturbation. Also Cuevas and Pinto [4] have shown that under certain conditions there is a one to one correspondence between weighted bounded solutions of a linear Volterra difference equation with unbounded delay and its perturbation. In most of our references linear and perturbed linear equations are investigated, moreover the boundedness and estimation of the solutions are founded by using the resolvent of the equations.

This article studies the boundedness of the solution of (1.1) under initial condition (1.2). As an illustration, we formulate the following statement which is an interesting consequence of our Corollary 5.7.

Consider the linear convolution-type Volterra equation

$$x(n+1) = \sum_{j=0}^{n} A(n-j)x(j) + h(n), \quad n \geq 0, \tag{1.3}$$

with the initial condition

$$x(0) = x_0. \tag{1.4}$$

Here $A(n) \in \mathbb{R}^{d \times d}$ are given matrices, $h(n) \in \mathbb{R}^d$ are given vectors and $x_0 \in \mathbb{R}^d$.

Proposition 1.1. *Assume that one of the following conditions is satisfied*:

$$(\alpha) \sum_{i=0}^{\infty} ||A(i)|| < 1 \textit{ and } \sup_{n \geq 0} ||h(n)|| < \infty;$$

$$(\beta) \sum_{i=0}^{\infty} ||A(i)|| = 1 \textit{ and } \sum_{i=0}^{n} ||A(i)|| < 1, n \geq 0, \textit{ moreover}$$

$$\sup_{n \geq 0} \frac{||h(n)||}{\sum_{i=n+1}^{\infty} ||A(i)||} < \infty.$$

Then the solution of (1.3) *and* (1.4) *is bounded for any* $x_0 \in \mathbb{R}^d$.

We remark that the above proposition gives sufficient conditions for the boundedness, but they are not necessary in general, see Remark 6.5 below.

To the best of our knowledge, this is the first article dealing with the boundedness property of the solutions of a linear inhomogeneous Volterra difference system with the critical case $\sum_{i=0}^{\infty} ||A(i)|| = 1$. For some recent literature on the boundedness of the solutions of linear Volterra difference equations, we refer the readers to [10-12]. We give some applications of our main result for sub-linear, linear, and super-linear Volterra difference equations. We study the boundedness of solutions of convolution cases and we get a result parallel to the corresponding result of Lipovan [18] for integral equation. Also we give some examples to illustrate our main results.

The rest of the article is organized as follows. In Section 2, we briefly explain some notation and two definitions which are used to state and to prove our results. In Section 3, we sate our main result with its proof. In Section 4, we give three applications based on our main result. In Section 5, some corollaries with convolution estimations and boundedness of convolution infinite delay equation are given. Examples are also given to illustrate our main theorem in Section 6.

2 Preliminaries

In this section, we give some notation and some definitions which are used in this article.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers, $\mathbb{Z}$ is the set of integer numbers, and $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$. Let d be a positive integer, $\mathbb{R}^d$ is the space of d-dimensional real column vectors with convenient norm $||.||$. Let $\mathbb{R}^{d \times d}$ be the space of all $d \times d$ real matrices. By the norm of a matrix $A \in \mathbb{R}^{d \times d}$, we mean its induced norm $||A|| = \sup\{||Ax|| \; |x \in \mathbb{R}^d, ||x|| = 1\}$. The zero matrix in $\mathbb{R}^{d \times d}$ is denoted by 0 and the identity matrix by I. The vector x and the matrix A are non-negative if $x_i \geq 0$ and $A_{ij} \geq 0, 1 \leq i,j \leq d$, respectively. Sequence $(x(n))_{n \geq 0}$ in $\mathbb{R}^d$ is denoted by $x : \mathbb{Z}_+ \to \mathbb{R}^d$.

The following definitions will be useful to prove the main results.

Definition 2.1. *Let the function* φ *and the sequence* $a(n,j) \in \mathbb{R}_+$, $0 \leq j \leq n$, *be given in condition (B). We say that the non-negative constant u has property* (P_N) *with an integer* $N \geq 0$ *if there is* $v > 0$, *such that*

$$\sum_{j=0}^{N} a(N,j)\phi(u) + ||h(N)|| \leq v, \tag{2.1}$$

and

$$\sum_{j=0}^{N} a(n,j)\phi(u) + \sum_{j=N+1}^{n} a(n,j)\phi(v) + ||h(n)|| \leq v, \quad n \geq N+1 \tag{2.2}$$

hold.

Definition 2.2. *We say that the vector* $x_0 \in \mathbb{R}^d$ *belongs to the set S if there exist a non-negative constant u and an integer* $N \geq 0$ *such that u has property* (P_N) *and the solution* $x(n;x_0)$, $n \geq 0$, *of* (1.1) *and* (1.2) *satisfies*

$$\| x(n;x_0) \| \leq u, \quad 0 \leq n \leq N. \tag{2.3}$$

Remark 2.3. $x_0 \in \mathbb{R}^d$ *belongs to the set S if*

$$a(0,0)\phi(\| x_0 \|) + \| h(0) \| \leq v, \tag{2.4}$$

and

$$a(n,0)\phi(\| x_0 \|) + \sum_{j=1}^{n} a(n,j)\phi(v) + \| h(n) \| \leq v, \quad n \geq 1 \tag{2.5}$$

hold. In this case $N = 0$ *and* $u = ||x_0||$ *have property* (P_0).

Remark 2.4. *If there exists an* $N \geq 0$ *and two positive constants u and v such that* (2.2) *holds, then*

$$\alpha_N := \sup_{n \geq N+1} \sum_{j=0}^{N} a(n,j) < \infty, \tag{2.6}$$

$$\beta_N := \sup_{n\geq N+1} \sum_{j=N+1}^{n} a(n,j) < \infty, \tag{2.7}$$

$$\gamma := \sup_{n\geq 0} \parallel h(n) \parallel < \infty. \tag{2.8}$$

Conditions (2.6) *and* (2.7) *are equivalent to*

$$\sup_{n\geq 0} \sum_{j=0}^{n} a(n,j) < \infty. \tag{2.9}$$

3 Main result

Our main goal in this section is to establish the following result with the proof.

Theorem 3.1. *Let* (*A*) *and* (*B*) *be satisfied and assume that the initial vector* x_0 *belongs to the set S. Then the solution* $x(n;x_0)$, $n \geq 0$, *of* (1.1) *and* (1.2) *is bounded. More exactly the solution satisfies* (2.3) *with suitable u and N, such that*

$$\parallel x(n; x_0) \parallel \leq v, \quad n \geq N+1, \tag{3.1}$$

where v is defined in (2.1) *and* (2.2).

Proof. Let $x_0 \in S$ and consider the solution $x(n) = x(n; x_0)$, $n \geq 0$, of (1.1) with the condition (1.2) and let u and N be defined in (2.3). Then

$$\begin{aligned} \parallel x(N+1) \parallel &\leq \sum_{j=0}^{N} \parallel f(N,j,x(j)) \parallel + \parallel h(N) \parallel \\ &\leq \sum_{j=0}^{N} a(N,j)\phi(\parallel x(j) \parallel) + \parallel h(N) \parallel \\ &\leq \sum_{j=0}^{N} a(N,j)\phi(u) + \parallel h(N) \parallel \leq v, \end{aligned}$$

where we used the monotonicity of φ, and the definition of v. Thus (3.1) holds for $n = N + 1$.

Now we show that (3.1) holds for any $n \geq N + 1$. Assume, for the sake of contradiction, that (3.1) is not satisfied for all $n \geq N + 1$. Then there exists $n_0 \geq N + 1$ such that

$$\parallel x(n_0+1) \parallel = \parallel x(n_0+1; x_0) \parallel > v, \tag{3.2}$$

and

$$\parallel x(n) \parallel = \parallel x(n; x_0) \parallel \leq v, \quad N+1 \leq n \leq n_0. \tag{3.3}$$

Hence, from Equation (1.1), we get

$$\begin{aligned} \parallel x(n_0+1) \parallel &\leq \sum_{j=0}^{N} \parallel f(n_0,j,x(j)) \parallel + \sum_{j=N+1}^{n_0} \parallel f(n_0,j,x(j)) \parallel + \parallel h(n_0) \parallel \\ &\leq \sum_{j=0}^{N} a(n_0,j)\phi(\parallel x(j) \parallel) + \sum_{j=N+1}^{n_0} a(n_0,j)\phi(\parallel x(j) \parallel) + \parallel h(n_0) \parallel . \end{aligned}$$

Since φ is a monotone non-decreasing mapping, (2.3) and (3.3) yield

$$\| x(n_0 + 1) \| \le \sum_{j=0}^{N} a(n_0, j)\phi(u) + \sum_{j=N+1}^{n_0} a(n_0, j)\phi(v) + \| h(n_0) \|.$$

But $x_0 \in S$ and u has property (P_N), and hence $||x(n_0 + 1)|| \le v$. This contradicts the hypothesis that (3.1) does not hold for $n_0 \ge N + 1$. So inequality (3.1) holds.

4 Applications

In this section, we give some applications of our main result. Throughout this section we take $\varphi(t) = t^p$, $t > 0$ with $p > 0$. There are three cases:

1. Sub-linear case when $0 < p < 1$;
2. Linear case when $p = 1$;
3. Super-linear case when $p > 1$.

4.1 Sub-linear case

Our aim in this section is to establish a sufficient, as well as a necessary and sufficient, condition for the boundedness of all solutions of (1.1) and the scalar case of (1.1), respectively.

The next result provides a sufficient condition for the boundedness of solutions of (1.1).

Theorem 4.1. *Let* (*A*), (*B*) *be satisfied and* $\varphi(t) = t^p$, $t > 0$, *with fixed* $p \in (0,1)$. *If* (2.8) *and* (2.9) *hold, then for any* $x_0 \in \mathbb{R}^d$ *the solution* $x(n; x_0)$, $n \ge 0$ *of* (1.1) *and* (1.2) *is bounded.*

The next Lemma provides a necessary and sufficient condition for the condition (2.2) be satisfied, and will be useful in the proof of Theorem 4.1.

Lemma 4.2. *Assume* $\varphi(t) = t^p$, $t > 0$ *and* $0 < p < 1$. *Any positive constant* u *has property* (P_0) *if and only if* (2.8) *and* (2.9) *are satisfied.*

Proof. Let the non-negative constant u have property (P_0) ($N = 0$ in Definition 2.1). Then the condition (2.2) is satisfied for some positive v and for all $n \ge 1$, so

$$\alpha_0 := \sup_{n\ge 1} a(n, 0) < \infty, \quad \beta_0 := \sup_{n \ge 1} \sum_{j=1}^{n} a(n, j) < \infty, \tag{4.1}$$

$$\text{and} \quad \gamma_0 := \sup_{n\ge 1} \| h(n) \| < \infty, \tag{4.2}$$

this imply that conditions (2.8) and (2.9) are satisfied.

Conversely, we assume (2.8), (2.9) and we prove that any positive constant u has property (P_0). Clearly, (2.9) is equivalent to $\alpha_0 < \infty$, $\beta_0 < \infty$ and (2.8) implies $\gamma_0 < \infty$.

Since $p \in (0,1)$, it is clear that for an arbitrarily fixed $u > 0$, (2.1) and

$$\frac{\alpha_0 u^p}{v} + \beta_0 v^{p-1} + \frac{\gamma_0}{v} \le 1, \tag{4.3}$$

are satisfied for any v large enough. From (4.3) we get

$$\alpha_0 u^p + \beta_0 v^p + \gamma_0 \le v,$$

that (2.5) is satisfied for $||x_0|| = u$ and all $n \geq 1$. Then by Definition 2.1, u has property (P_0).

Now we prove Theorem 4.1.

Proof. Let (A) and (B) be satisfied. By Lemma 4.2, we have that for any $x_0 \in \mathbb{R}^d$, $u = ||x_0||$ has property (P_0) (see Remark 2.3). Thus, the conditions of Theorem 3.1 hold, and the initial vector x_0 belongs to S, and hence the solution $x(n; x_0)$ of (1.1) and (1.2) is bounded.

We consider the scalar case of Volterra difference equation

$$x(n+1) = \sum_{j=0}^{n} a(n,j)x^p(j) + h(n), \quad n \geq 0, \tag{4.4}$$

$$x(0) = x_0, \tag{4.5}$$

where $x_0 \in \mathbb{R}_+$, $a(n,j) \in \mathbb{R}_+$, $h(n) \in \mathbb{R}_+$, $0 \leq j \leq n$ and $p \in (0,1)$.

The following result provides a necessary and sufficient condition for the boundedness of the solution of (4.4) and (4.5). The necessary part of the next theorem was motivated by a similar result of Lipovan [18] proved for convolution-type integral equation.

Theorem 4.3. *Assume*

$$\liminf_{n\to\infty} \left(\sum_{j=0}^{n} a(n,j) \right) > 0, \tag{4.6}$$

moreover for any $n \geq 0$, *there exists an index* j_n *such that*

$$0 \leq j_n \leq n \quad \text{and} \quad a(n, j_n) + h(n) > 0. \tag{4.7}$$

For any $x_0 \in (0, \infty)$, *the solution of* (4.4) *is bounded, if and only if* (2.8) *and* (2.9) *are satisfied.*

Proof. Assume (2.8) and (2.9) are satisfied. Clearly, by Theorem 4.1 the solution of (4.4) is bounded. Conversely, let the solution $x(n) = x(n;x_0)$ of (4.4) be bounded on $\mathbb{R}_+$, with $x_0 > 0$. Under condition (4.7), by mathematical induction we show that $x(n) > 0$, $n \geq 0$. For $n = 0$ this is clear. Suppose that required inequality is not satisfied for all $n \geq 0$. Then there exists index $\ell \geq 0$ such that $x(0) > 0, \ldots, x(\ell) > 0$ and $x(\ell+1) \leq 0$. But by condition (4.7), we get

$$\begin{aligned} x(\ell+1) &= \sum_{j=0}^{\ell} a(\ell,j)x^p(j) + h(\ell) \\ &\geq a(\ell, j_\ell)x^p(j_\ell) + h(\ell) > 0, \quad 0 \leq j_\ell \leq \ell, \end{aligned}$$

which is a contradiction. So $x(n) > 0$ for all $n \geq 0$. On the other hand, for any $n \geq N^* \geq 1$

$$x(n+1) \geq \sum_{j=0}^{N^*-1} a(n,j)x^p(j) \geq \sum_{j=0}^{N^*-1} a(n,j) \min_{0\leq j\leq N^*} x^p(j),$$

hence

$$\sup_{n\geq 0}\sum_{j=0}^{N^*-1} a(n,j) < \infty. \tag{4.8}$$

Since $x(n + 1) \geq h(n)$ for all $n \geq 0$, clearly $\sup_{n\geq 0} h(n)$ is finite.

Define now

$$m = \liminf_{n\to\infty} x(n),$$

which is finite. First we show that $m > 0$.

Assume for the sake of contradiction that $m = 0$. In this case we can find a strictly increasing sequence $(N_k)_{k\geq 1}$, such that

$$x(N_k) = \min_{0\leq n\leq N_k} x(n) > 0, \quad \text{and } x(N_k) \to 0 \text{ as } k \to \infty.$$

From (4.4) if $n = N_k - 1$, we deduce

$$\begin{aligned} x(N_k) &= \sum_{j=0}^{N_k-1} a(N_k-1,j)x^p(j) + h(N_k-1) \\ &\geq \sum_{j=0}^{N_k-1} a(N_k-1,j) \min_{0\leq j\leq N_k} x^p(j) \\ &= x^p(N_k)\sum_{j=0}^{N_k-1} a(N_k-1,j). \end{aligned}$$

Since $x(N_k) > 0$, we have that

$$x^{1-p}(N_k) \geq \sum_{j=0}^{N_k-1} a(N_k-1,j), \quad k\geq 1.$$

Since $p \in (0,1)$ and $x(N_k) \to 0$, an $k \to \infty$, we get

$$\liminf_{k\to\infty}\left(\sum_{j=0}^{N_k-1} a(N_k-1,j)\right) = 0,$$

which contradicts (4.6). So $m > 0$ and for $\frac{1}{2}m$, there exists $N^* \geq 0$ such that

$$x(n) \geq \frac{1}{2}m, \quad n \geq N^*.$$

Hence,

$$\begin{aligned} x(n+1) &= \sum_{j=0}^{n} a(n,j)x^p(j) + h(n) \\ &\geq \sum_{j=N^*}^{n} a(n,j)x^p(j) \geq \frac{1}{2^p}m^p\sum_{j=N^*}^{n} a(n,j), \quad n\geq N^*. \end{aligned}$$

But the solution $x(n)$ is a bounded sequence, and hence

$$\sup_{n \geq N^*} \sum_{j=N^*}^{n} a(n,j) < \infty.$$

This and (4.8) imply condition (2.9).

Remark 4.4. *In general, without condition* (4.7) *the necessary part of Theorem* 4.3 *is not true. In fact if* $a(0, 0) = 0$, *and* $h(0) = 0$, *that is* (4.7) *does not hold for* $n = 0$, *then for any* $x_0 \in (0, \infty)$ *the solution of* (4.4) *satisfies* $x(1;x_0) = 0$, *and hence*

$$x(n+1) = \sum_{\substack{j=0 \\ j \neq 1}}^{n} a(n,j)x^p(j) + h(n), \quad n \geq 1.$$

Thus, the solution $x(n; x_0)$ *does not depend on the choice of the sequence* $(a(n, 1))_{n \geq 1}$. *This shows that the boundedness of the solutions does not imply* (2.9), *in general.*

4.2 Linear case

Our aim in this section is to obtain sufficient condition for the boundedness of the solution of (1.1) under the initial condition (1.2), but in the linear case of Volterra difference equation.

The following result gives a sufficient condition for the boundedness.

Theorem 4.5. *Assume* (A), (B) *are satisfied and* $\varphi(t) = t$, $t \geq 0$. *Then the solution* $x(n; x_0)$, $x_0 \in S$, $n \geq 0$ *of* (1.1) *and* (1.2) *is bounded, if* (2.9) *is satisfied and there exists an* $N \geq 0$ *such that one of the following conditions holds:*

(i) *condition* (2.8) *holds and*

$$\beta_N = \sup_{n \geq N+1} \sum_{j=N+1}^{n} a(n,j) < 1; \tag{4.9}$$

(ii) $\beta_N = \sup_{n \geq N+1} \sum_{j=N+1}^{n} a(n,j) = 1,$

and for any $n \in \Gamma_N^{(1)}$,

$$\sum_{j=0}^{N} a(n,j) = 0, \quad h(n) = 0 \tag{4.10}$$

hold, moreover

$$\sup_{n \in \Gamma_N^{(2)}} \left(1 - \sum_{j=N+1}^{n} a(n,j)\right)^{-1} \sum_{j=0}^{N} a(n,j) < \infty, \tag{4.11}$$

and

$$\sup_{n \in \Gamma_N^{(2)}} \left(1 - \sum_{j=N+1}^{n} a(n,j)\right)^{-1} \| h(n) \| < \infty, \tag{4.12}$$

where

$$\Gamma_N^{(1)} = \left\{ n \geq N+1 : \sum_{j=N+1}^{n} a(n,j) = 1 \right\},$$

and

$$\Gamma_N^{(2)} = \left\{ n \geq N+1 := \sum_{j=N+1}^{n} a(n,j) < 1 \right\}.$$

For the proof of Theorem 4.4, we need the following lemma.

Lemma 4.6. *Assume* $\varphi(t) = t$, $t \geq 0$. *A positive constant* u *has property* (P_N) *with an integer* $N \geq 0$ *if and only if the condition* (2.9) *and either* (i) *or* (ii) *are satisfied.*

Proof. Necessity. We show that (P_N) implies (2.9) and either (i) or (ii). Suppose a positive constant u has property (P_N) with an integer $N \geq 0$, hence (2.1) and (2.2) are satisfied for $v > 0$ and for any $n \geq N + 1$.

From (2.1) and (2.2), it is clear that (2.8), (2.9) are satisfied and

$$\sum_{j=0}^{N} a(n,j)u + \| h(n) \| \leq \left(1 - \sum_{j=N+1}^{n} a(n,j) \right) v, \quad n \geq N+1. \tag{4.13}$$

Therefore

$$1 - \sum_{j=N+1}^{n} a(n,j) \geq 0, \ \ n \geq N+1,$$

and hence

$$\beta_N = \sup_{n \geq N+1} \sum_{j=N+1}^{n} a(n,j) \leq 1.$$

The latest inequality implies two cases with respect the value of β_N.

- The first case $\beta_N < 1$. In this case the condition (i) is satisfied.
- Consider now the second case where $\beta_N = 1$. Clearly if $\sum_{j=N+1}^{n} a(n,j) = 1$, then from (4.13), we get

$$\sum_{j=0}^{N} a(n,j)u + h(n) = 0, \quad n \in \Gamma_N^{(1)},$$

or equivalently (4.10).

But if $\sum_{j=N+1}^{n} a(n,j) < 1$, then

$$1 - \sum_{j=N+1}^{n} a(n,j) > 0, \quad n \in \Gamma_N^{(2)},$$

and (4.13) yields

$$\left(1-\sum_{j=N+1}^{n} a(n,j)\right)^{-1}\left(\sum_{j=0}^{N} a(n,j)u+\parallel h(n)\parallel\right)\leq v,\quad n\in\Gamma_N^{(2)}$$

and hence (4.11) and (4.12) are satisfied. Then condition (ii) holds.

Sufficiency. We show that if (2.9) and one of the conditions (i) and (ii) is satisfied with some $u \geq 0$ and $N \geq 0$, then u has property (P_N). It is easy to observe that (2.9) yields

$$\sup_{n\geq N+1}\sum_{j=0}^{N} a(n,j)<\infty,\quad \text{and}\quad \sup_{n\geq N+1}\sum_{j=N+1}^{n} a(n,j)<\infty.$$

Let (i) of Theorem 4.5 be satisfied, that is $\beta_N < 1$. Then for $u \geq 0$ and $n \geq N + 1$, $N \geq 0$, there exists $v > 0$ such that

$$\sum_{j=0}^{N} a(n,j)u+\parallel h(n)\parallel\leq(1-\beta_N)v,\quad n\geq N+1.$$

It implies

$$\sum_{j=0}^{N} a(n,j)u+\parallel h(n)\parallel\leq\left(1-\sum_{j=N+1}^{n} a(n,j)\right)v,$$

i.e. (2.2) is satisfied and (2.1) also is satisfied for all v large enough, hence u has property P_N.

Now suppose $\beta_N = 1$, and (4.10) holds. Then, clearly (2.1) and (2.2) are satisfied for any $v \geq 0$ and $n \in \Gamma_N^{(1)}$.

If $n \in \Gamma_N^{(2)}$ and (4.11) and (4.12) are satisfied, then for $u \geq 0$, we have

$$\sup_{n\in\Gamma_N^{(2)}}\left(1-\sum_{j=N+1}^{n} a(n,j)\right)^{-1}\left[\sum_{j=0}^{N} a(n,j)u+\parallel h(n)\parallel\right]<\infty.$$

Then there exists $v > 0$ large enough such that

$$\left(1-\sum_{j=N+1}^{n} a(n,j)\right)^{-1}\left[\sum_{j=0}^{N} a(n,j)u+\parallel h(n)\parallel\right]\leq v,\quad n\in\Gamma_N^{(2)},$$

since $1-\sum_{j=N+1}^{n} a(n,j)>0$, for all $n\in\Gamma_N^{(2)}$.

Therefore

$$\sum_{j=0}^{N} a(n,j)u+\parallel h(n)\parallel\leq\left(1-\sum_{j=N+1}^{n} a(n,j)\right)v,$$

i.e. for all v large enough the conditions (2.2) and (2.1) are satisfied. Hence, u has property (P_N).

The following lemma is extracted from [2] (Lemma 5.3) and will be needed in this section.

Lemma 4.7. *Assume* (*A*), (*B*) *are satisfied and* $\varphi(t) = t$, $t \geq 0$. *For every integer* $N > 0$, *there exists a non-negative constant* $K_1(N)$ *independent of the sequence* $(h(n))_{n\geq 0}$ *and* x_0, *such that the solution* $(x(n))_{n\geq 0}$ *of* (1.1) *and (1.2), satisfies*

$$\| x(n) \| \leq K_1(N) \left(\max_{0\leq m\leq N} \| h(m) \| + \| x_0 \| \right), \quad 0 \leq n \leq N. \tag{4.14}$$

Now we give the proof of Theorem 4.5.

Proof. Let (A), (B), (2.9) and either (i) or (ii) in Theorem 4.5 be satisfied. By Lemma 4.7 the solution of (1.1) and (1.2) satisfies (4.14) for all $0 \leq n \leq N$. This means that, there exists a non-negative constant u such that

$$\| x(n) \| \leq u, \quad 0 \leq n \leq N.$$

By Lemma 4.6 we have u has property (P_N). Then the conditions of Theorem 3.1 hold for the initial vector x_0 belonging to S, and hence the solution $x(n; x_0)$ of (1.1) and (1.2) is bounded.

4.3 Super-linear case

Our aim in this section is to obtain sufficient condition for the boundedness in the super-linear case.

Theorem 4.8. *Assume that conditions* (*A*) *and* (*B*) *are satisfied with the function* $\varphi(t) = t^p$, $t > 0$, *where* $p > 1$. *Suppose also* (2.8) *and* (2.9) *hold. Then the solution* $x(n;x_0)$ *of* (1.1) *and* (1.2) *is bounded for some* $x_0 \in \mathbb{R}^d$ *if there exists* $v \in \left[\left(\frac{1}{p\beta_0}\right)^{\frac{1}{p-1}}, \left(\frac{1}{\beta_0}\right)^{\frac{1}{p-1}}\right]$ *such that*

$$a(0,0) \| x_0 \|^p + \| h(0) \| \leq v, \tag{4.15}$$

and

$$\alpha_0 \| x_0 \|^p + \gamma_0 \leq v - \beta_0 v^p, \tag{4.16}$$

where α_0, β_0 *and* γ_0 *are defined in* (4.1) *and* (4.2).

Proof. Assume (2.8), (2.9), (4.15), and (4.16) are satisfied. The case $\beta_0 = 0$ is clear. So we assume that $\beta_0 > 0$. In this case, clearly, $v - \beta_0 v^p \geq 0$ if

$$v \in \left[0, \left(\frac{1}{\beta_0}\right)^{\frac{1}{p-1}}\right],$$

and the maximum value of the function $g(v) = v - \beta_0 v^p$ is $\left(\frac{1}{p\beta_0}\right)^{1/(p-1)}$.

Then there exists v such that

$$v \in \left[\left(\frac{1}{p\beta_0}\right)^{\frac{1}{p-1}}, \left(\frac{1}{\beta_0}\right)^{\frac{1}{p-1}}\right],$$

and the conditions (2.4) and (2.5) hold. By Remark 2.3 we get that under conditions (4.15) and (4.16), $u = ||x_0||$ has property (P_0) and x_0 belongs to S. Then the conditions of Theorem 3.1 hold, so the solution of (1.1) with the initial condition (1.2) is bounded.

5 Some corollaries with convolution estimations

In this section we give some corollaries on the boundedness of the solutions of (1.1) and (1.2) but in the convolution-type. Through out in this section we take $a(n, i) = \alpha(n - i)$, $n \geq 0$, and $0 \leq i \leq n$, and the following condition

(C) For any $n \geq 0$, there exists an $\alpha(n) \in \mathbb{R}_+$, such that

$$\| f(n, j, x) \| \leq \alpha(n - j)\phi(\| x \|),$$

with a monotone non-decreasing mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $||.||$ is any norm on $\mathbb{R}^d$.

Remark 5.1. If $a(n,j) = \alpha(n - j)$, $\alpha(n) \in \mathbb{R}_+$, $n \geq 0$, and $\varphi(t) > 0$, $t > 0$, then the non-negative constant u has property (P_N) with an integer $N \geq 0$ if and only if

$$\sum_{j=0}^{N} \alpha(j)\phi(u) + \| h(N) \| \leq v, \tag{5.1}$$

and

$$\sum_{j=n-N}^{n} \alpha(j)\phi(u) + \sum_{j=0}^{n-N-1} \alpha(j)\phi(v) + \| h(n) \| \leq v, \quad n \geq N + 1 \tag{5.2}$$

are satisfied.

Remark 5.2. *$x_0 \in \mathbb{R}^d$ belongs to the set S if*

$$\alpha(0)\phi(\| x_0 \|) + \| h(0) \| \leq v, \tag{5.3}$$

and

$$\alpha(n)\phi(\| x_0 \|) + \sum_{j=0}^{n-1} \alpha(j)\phi(v) + \| h(n) \| \leq v, \quad n \geq 1 \tag{5.4}$$

hold. In this case $N = 0$ and $u = ||x_0||$ has property (P_0).

By our main result, we have the following corollary.

Corollary 5.3. *Let (A), (C) and $\sup_{n\geq 0} ||h(n)|| < \infty$ be satisfied, and assume that the initial vector x_0 belongs to the set S. Then the solution $x(n;x_0)$, $n \geq 0$, of (1.1) and (1.2) is bounded.*

Proof. Assume (A), (C) are satisfied. By Theorem 3.1 and Remark 5.1, it is easy to prove that the solution of (1.1) and (1.2) is bounded.

The following two corollaries are immediate consequence of Theorems 4.1 and 4.3 of the sub-linear convolution case, respectively.

Corollary 5.4. *Assume (A), (C) are satisfied and $\varphi(t) = t^p$, $t > 0$, with fixed $p \in (0,1)$. If*

$$\sup_{n\geq 0} \| h(n) \| < \infty, \quad \text{and} \quad \sum_{j=0}^{\infty} \alpha(j) < \infty, \tag{5.5}$$

then for any $x_0 \in \mathbb{R}^d$ the solution $x(n; x_0)$, $n \geq 0$ of (1.1) and (1.2) is bounded.

Corollary 5.5. *Consider Equation* (4.4) *with* $p \in (0, 1)$ *and non-negative coefficients. Assume*

$$\sum_{n=0}^{\infty} \alpha(n) > 0,$$

and for any $n \geq 0$ *one has* $h(n) > 0$ *if* $\alpha(j) = 0,\ 0 \leq j \leq n$. *Then the solution of* (4.4) *and* (4.5) *is bounded if and only if the condition* (5.5) *is satisfied.*

Proof. The proof is immediate consequence from proof of Theorem 4.3 with Remark 5.1.

Remark 5.6. *The Corollary 5.5 is analogous to the corresponding result of Lipovan* (Theorem 3.1, [18]) *for integral equation.*

In the next corollary we assume that

$$\sum_{n=0}^{\infty} \alpha(n) \leq 1. \tag{5.6}$$

Under condition (5.6) we have three cases

Case 1. $\sum_{n=0}^{\infty} \alpha(n) < 1$;

Case 2. $\sum_{j=0}^{\infty} \alpha(j) = 1$ and for any $n \geq 1$,

$$\sum_{j=0}^{n-1} \alpha(j) < 1, \quad \text{or equivalently} \quad \sum_{j=n}^{\infty} \alpha(n) > 0;$$

Case 3. There exists an index $M \geq 0$ such that $\sum_{n=0}^{M} \alpha(n) = 1$, moreover $\alpha(M) \neq 0$ and $\alpha(n) = 0,\ n \geq M + 1$.

Corollary 5.7. *Assume* (*A*), (*C*) *and* (5.6), *and let* $\varphi(t) = t,\ t \geq 0$. *Then the solution of* (1.1) *and* (1.2) *is bounded for any* $x_0 \in \mathbb{R}$, *if one of the following conditions holds*

(a) *Case 1. holds and* $\sup_{n \geq 0} ||h(n)|| < \infty$.

(b) *Case 2. holds and*

$$\sup_{n \geq 1} \frac{\| h(n) \|}{\sum_{j=n}^{\infty} \alpha(j)} < \infty.$$

(c) *Case 3. holds and* $h(n) = 0,\ n \geq M + 1$.

Proof. (a) This part is an immediate consequence of (i) from Theorem 4.5 with Remark 5.1.

(b) Assume that (b) is satisfied. For a fixed $N \geq 0$ and $u \in \mathbb{R}_+$, there exists a positive constant ν such that

$$\frac{\sum_{j=n-N}^{n}\alpha(j)}{\sum_{j=n-N}^{\infty}\alpha(j)}u+\frac{\| h(n) \|}{\sum_{j=n-N}^{\infty}\alpha(j)}\le u+\frac{\| h(n) \|}{\sum_{j=n}^{\infty}\alpha(j)}\le v, \quad n\ge N+1,$$

and

$$\sum_{j=0}^{N}\alpha(j)u+\| h(N) \|\le v$$

hold. Thus

$$\sum_{j=n-N}^{n}\alpha(j)u+\| h(n) \|\le \sum_{j=n-N}^{\infty}\alpha(j)v\le \left(1-\sum_{j=0}^{n-N-1}\alpha(j)\right)v,$$

i.e.

$$\sum_{j=n-N}^{n}\alpha(j)u+\sum_{j=0}^{n-N-1}\alpha(j)v+\| h(n) \|\le v, \quad n\ge N+1. \tag{5.7}$$

By Remark 5.1, u has property (P_N). For the initial value $x_0 \in \mathbb{R}^d$ applying Corollary 5.3, we get the boundedness of the solution of (1.1) and (1.2).

(c) Assume the condition (c). Clearly, for all $v \ge 0$ the conditions (2.1) and (2.2) are satisfied and u has property (P_N). Then for any $x_0 \in S$, the solution of (1.1) is bounded according to Corollary 5.3.

The proof of the following corollary is an immediate consequence of Theorem 4.8 and Remark 5.1 and it is therefore omitted.

Corollary 5.8. *Assume that conditions (A) and (C) are satisfied with the function* $\varphi(t) = t^p$, $t > 0, p > 1$, *and (5.5) holds. Then for an* $x_0 \in \mathbb{R}^d$ *the solution* $x(n;x_0)$ *of (1.1) and (1.2) is bounded, if there exists* $v \in \left[\left(\frac{1}{p\beta_c}\right)^{\frac{1}{p-1}}, \left(\frac{1}{\beta_c}\right)^{\frac{1}{p-1}}\right]$ *such that*

$$\alpha(0)\| x_0\|^p+\| h(0) \|\le v,$$

and

$$\alpha_c \| x_0\|^p+\gamma_0\le v-\beta_c v^p,$$

where

$$\alpha_c=\sup_{n\ge 1}\alpha(n)<\infty, \quad \beta_c=\sum_{j=0}^{\infty}\alpha(j)<\infty, \quad \text{and} \quad \gamma_0=\sup_{n\ge 1}\| h(n) \| \infty.$$

Now we show the application of Corollary 5.7 to the linear convolution Volterra difference equation with infinite delay

$$x(n+1)=\sum_{i=-\infty}^{n}Q(n-i)x(i), \quad n\ge 0, \tag{5.8}$$

with initial condition

$$x(n) = \varphi(n), \quad n \le 0, \tag{5.9}$$

where $Q(n) \in \mathbb{R}^{d \times d}$, $n \geq 0$ and $\phi(m) \in \mathbb{R}^d$, $m \leq 0$.

From (5.8), we have

$$x(n+1) = \sum_{i=0}^{n} Q(n-i)x(i) + \sum_{i=-\infty}^{-1} Q(n-i)\varphi(i) \quad n \ge 0.$$

If we compare the latest equation with Equation (1.1), we have $f(n, i, x) = Q(n - i)x$ and

$$h(n) = \sum_{i=-\infty}^{-1} Q(n-i)\varphi(i) \quad n \ge 0.$$

If $M_\varphi = \sup_{m \le -1} \| \varphi(m) \|$, then we get

$$\| h(n) \| \le \sum_{j=n+1}^{\infty} \| Q(j) \| M_\varphi.$$

Corollary 5.9. *Assume* $Q(n) \in \mathbb{R}^{d \times d}$, $n \geq 0$, *and*

$$\sum_{n=0}^{\infty} \| Q(n) \| \le 1. \tag{5.10}$$

Then the solution of (5.8) *with the initial condition* (5.9) *is bounded, if*

$$M_\varphi = \sup_{m \le -1} \| \varphi(m) \| < \infty. \tag{5.11}$$

Proof. To prove that the solution of (5.8) with the initial condition (5.9) is bounded, it is enough to show that one of the hypotheses (a), (b), (c) of Corollary 5.7 holds. Let (5.10) and (5.11) be satisfied. There are three cases.

First consider the case when

$$\sum_{n=0}^{\infty} \| Q(n) \| < 1,$$

and (5.11) are satisfied. This implies (a) of Corollary 5.7.

Second consider the case when

$$\sum_{j=0}^{\infty} \| Q(j) \| = 1, \quad \text{and} \quad \sum_{j=n}^{\infty} \| Q(j) \| > 0, \quad n \ge 1.$$

This yields

$$\sup_{n \ge 1} \frac{\| h(n) \|}{\sum_{j=n}^{\infty} \| Q(j) \|} \le \sup_{n \ge 1} \frac{\sum_{j=n+1}^{\infty} \| Q(j) \| M_\varphi}{\sum_{j=n}^{\infty} \| Q(j) \|} \le M_\varphi < \infty,$$

and hence condition (b) of Corollary 5.7 is satisfied.

The last case is when there exists $N \geq 0$, such that

$$\sum_{j=0}^{N} \| Q(j) \|= 1, \quad \text{and} \quad \| Q(n) \|= 0, \quad \text{for all} \quad n \geq N + 1.$$

In this case, we get $||h(n)|| = 0$ for all $n \geq N + 1$, hence (c) of Corollary 5.7 holds.

6 Examples

In this section we give some examples to illustrate our results.

Example 6.1. Let us consider the case when

$$a(n, j) = \frac{2n + 1}{(n + j + 1)(n + j + 2)} \quad \text{for all} \quad 0 \leq j \leq n.$$

We have $a(0, 0) = \frac{1}{2}$, $a(n, 0) = \frac{2n+1}{(n+1)(n+2)}$, $\sup_{n\geq 1} a(n, 0) = \frac{1}{2}$,

$$\sum_{j=1}^{n} \frac{2n + 1}{(n + j + 1)(n + j + 2)} = \frac{n(2n + 1)}{2(n + 1)(n + 2)},$$

and

$$\sup_{n\geq 0} \sum_{j=0}^{n} \frac{2n + 1}{(n + j + 1)(n + j + 2)} = 1. \tag{6.1}$$

One can easily see that conditions (2.4) and (2.5) are equivalent to the inequalities

$$\frac{1}{2}\phi(\| x_0 \|)+ \| h(0) \| \leq v, \tag{6.2}$$

and

$$\frac{2n + 1}{(n + 1)(n + 2)}\phi(\| x_0 \|) + \frac{n(2n + 1)}{2(n + 1)(n + 2)}\phi(v)+ \| h(n) \| \leq v, \quad n \geq 1. \tag{6.3}$$

Consider the scalar equation

$$x(n + 1) = \sum_{j=0}^{n} \frac{(2n + 1)}{(n + j + 1)(n + j + 2)} x^p(j) + h(n), \quad n \geq 0, \tag{6.4}$$

with the initial condition

$$x(0) = x_0 \tag{6.5}$$

where $x_0 \in \mathbb{R}_+$, $h(n) \in \mathbb{R}_+$, $n \geq 0$ and $p > 0$. In fact there are three cases with respect to the value *of p*.

(a1) $p \in (0,1)$ and $\varphi(t) = t^p$, $t > 0$. If $\sup_{n\geq 0} h(n) < \infty$ and (6.1) are satisfied, then for any $x_0 \geq 0$, the solution of (6.4) and (6.5) is bounded by Theorem 4.1.

(a2) $p = 1$ and $\varphi(t) = t$, $t > 0$. It is not difficult to see that for any $v > 0$ large enough the inequalities (6.2) and (6.3) are equivalent to the inequalities

$$\frac{1}{2}x_0 + h(0) \leq v,$$

and

$$\frac{2(2n+1)}{5n+4}x_0 + \frac{2(n+1)(n+2)}{n(5n+4)}nh(n) \le v, \quad n \ge 1. \tag{6.6}$$

Let $k = \sup_{n\ge1} nh(n) < \infty$, it is easily to see that the inequality (6.6) is satisfied if

$$\frac{4}{5}x_0 + \frac{4}{3}k \le v.$$

Hence (2.4) and (2.5) are satisfied, by Lemma 4.6, x_0 has property (P_0). It is worth to note that in this case our Theorem 4.5 is applicable for any $x_0 \in \mathbb{R}_+$, but the results in [2,8-12,14] are not applicable in this case.

(a3) $p > 1$ and $\varphi(\mathrm{t}) = t^p$, $t > 0$. Assume $k = \sup_{n\ge1} h(n)$, and for $x_0 \ge 0$, there exists $v \in \left[\left(\frac{1}{p}\right)^{\frac{1}{p-1}}, 1\right]$, such that

$$\frac{1}{2}x_0^p + h(0) \le v$$

and

$$\frac{1}{2}x_0^p + k \le v - v^p$$

hold. Hence x_0 has property (P_0), and by Theorem 4.8, for small x_0, the solution of (6.4) and (6.5) is bounded.

Summarizing the observations and applying Theorem 4.5, we get the next new result.

Proposition 6.2. *If Equation* (6.4) *is linear, that is* $p = 1$, *and* $\sup_{n\ \ge\ 1} n\ h(n) < \infty$, *then every positive solution of* (6.4) *with initial condition* (6.5) *is bounded.*

Proposition 6.3. *Assume Equation* (6.4) *is super-linear, that is* $p > 1$. *If for some* $x_0 \ge 0$ *there exists* $v \in \left[\left(\frac{1}{p}\right)^{\frac{1}{p-1}}, 1\right]$, *such that*

$$\frac{1}{2}x_0^p + h(0) \le v,$$

and

$$\frac{1}{2}x_0^p + k \le v - v^p$$

hold, then the positive solution of (6.4) *with initial condition* (6.5) *is bounded.*

The following example shows that applicability of our result in the critical case.

Example 6.4. Consider the equation

$$x(n+1) = \sum_{i=0}^{n} cq^{n-i}x(i) + h(n), \quad n \ge 0, \tag{6.7}$$

$$x(0) = x_0, \tag{6.8}$$

where $x_0 \in \mathbb{R}_+$, $q \in (0,1)$, $c \in (0,1)$ and $h(n) \in \mathbb{R}_+$, $n \ge 0$.

If $q + c < 1$ and $\sup_{n\ge0} h(n) < \infty$, then our condition (a) in Corollary 5.7 holds. Relation $q + c = 1$ implies

$$\sum_{n=0}^{\infty} cq^n = 1.$$

Note that the results in [2,8-12] are not applicable. But our Corollary 5.7 is applicable under condition

$$\sup_{n\geq 1} \frac{h(n)}{q^n} < \infty,$$

since it implies condition (b) in Corollary 5.7:

$$\sup_{n\geq 1} \frac{h(n)}{\sum_{j=n}^{\infty} cq^j} = \sup_{n\geq 1} \frac{(1-q)h(n)}{cq^n} < \infty.$$

Remark 6.5. *By mathematical induction, it is easy to see the solution of (6.7) and (6.8) is given in the form*

$$x(n) = c(q+c)^{n-1}x_0 + \sum_{j=0}^{n-2} c(q+c)^{n-j-2}h(j) + h(n-1), \quad n \geq 2,$$

where $x(0) = x_0$ *and* $x(1) = cx_0 + h(0)$. *Let* $\sum_{j=0}^{\infty} cq^j = 1$ *or equivalently* $q + c = 1$ *moreover* $h(n) = k^{n+1}$, $0 < q < k < 1$. *In this case the above solution is bounded for any* $x_0 \in \mathbb{R}_+$. *At the same time condition (b) in Corollary 5.7 does not hold, and hence condition (β) in Proposition 1.1 is not necessary.*

Therefore by statements (a) and (b) of Corollary 5.7 we get the following.

Proposition 6.6. *The solution of* (6.7) *and* (6.8) *is bounded if either* $q + c < 1$ *and* $\sup_{n\geq 0} h(n) < \infty$, *or* $q + c = 1$, *and*

$$\sup_{n\geq 1} \frac{h(n)}{q^n} < \infty.$$

The next example shows the sharpness of our Corollary 5.8.

Example 6.7. Consider the equation

$$x(n+1) = \sum_{i=0}^{n} q^{n-i}x^p(i), \quad n \geq 0, \tag{6.9}$$

where $x(0) = x_0 > 0$, $p > 1$, $q \in (0,1)$.

Since $x_0 > 0$, implies $x(n) > 0$ for all $n \geq 0$, therefore from (6.9), we have $x(n + 1) \geq x^p(n)$, $n \geq 0$. For fixed $p > 1$, inequality $x_0^p > x_0$ holds if and only if $x_0 > 1$. In the latest, by mathematical induction it is easy to prove that the sequence $(x(n))_{n\geq 0}$ is strictly increasing. Now for $x_0 > 1$, we prove that $x(n) \to \infty$ as $n \to \infty$. Assume, for the sake of contradiction, that the sequence $(x(n))_{n\geq 0}$ is bounded. Since it is strictly increasing, $x^* = \lim_{n\to\infty} x(n)$ is finite and $x^* > x_0$. On the other hand $x(n+ 1) \geq x^p(n)$, and hence we get $x^* \geq (x^*)\ p > x^*$, which is a contradiction. Hence for all $x_0 > 1$ the solution of (6.9) is unbounded. Now applying Corollary 5.8 to (6.9), there exists

$$v \in \left[\left(\frac{1-q}{p} \right)^{\frac{1}{p-1}}, (1-q)^{\frac{1}{p-1}} \right],$$

such that $x_0^p \le v$ and

$$qx_0^p \le v - \frac{1}{1-q} v^p.$$

Hence

$$v = \frac{1}{q} \left(v - \frac{1}{1-q} v^p \right) \quad \text{implies} \quad v = (1-q)^{\frac{2}{p-1}},$$

i.e.

$$x_0 \le (1-q)^{\frac{2}{p(p-1)}}.$$

Then the solution of (6.9) with initial value $x_0 \le (1-q)^{\frac{2}{p(p-1)}}$ is bounded, but the solution of (6.9) with initial value $x_0 > 1$ is unbounded. Our results do not give any information about the boundedness of the solutions, whenever $x_0 \in \left((1-q)^{\frac{2}{p(p-1)}}, 1 \right]$ but this gap tends to zero if either p is large enough or q is very close to zero. Hence, our results for the super-linear case are sharp in some sense. As a special case let $p = 2$. In this case, the solution of (6.9) with initial condition $x(0) = x_0$ is bounded whenever $x_0 \in [0,1 - q]$ and it is unbounded whenever $x_0 > 1$.

Based on our results we state the following conjecture as an open problem.

Conjecture 6.8. *Let $p > 1$ and $0 < q < 1$. Then there exists a constant $\kappa > 1$ such that the solution of (6.9) with initial condition $x(0) = x_0$ is bounded whenever $x_0 \in [0, \kappa)$ and it is unbounded whenever $x_0 > \kappa$.*

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This was a joint work in every aspect. All the authors have read and approved the final manuscript.

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References

1. Agarwal, RP: Difference Equations and Inequalities, Theory, Methods, and Application. Marcel Dekker, New York, 2 (2000)
2. Appleby, JAD, Gyori, I, Reynolds, D: On exact convergence rates for solutions of linear systems of Volterra difference equations. J Diff Equ Appl. **12**(12):1257–1275 (2006). doi:10.1080/10236190600986594
3. Choi, SK, Koo Nam, Jip: Asymptotic property of linear Volterra difference systems. J Math Anal Appl. **321**, 260–272 (2006). doi:10.1016/j.jmaa.2005.08.028

4. Cuevas, C, Pinto, M: Asymptotic behavior in Volterra difference systems with unbounded delay. J Comput Appl Math. **113**, 217–225 (2000). doi:10.1016/S0377-0427(99)00257-5
5. Elaydi, S: An Introduction to Difference Equations. Springer-Verlag, New York, 3 (2005)
6. Elaydi, S, Murakami, S, Kamiyama, E: Asymptotic equivalence for difference equations with infinite delay. J Diff Equ Appl. **5**, 1–23 (1999). doi:10.1080/10236199908808167
7. Furumochi, T, Murakami, S, Nagabuchi, Y: A generalization of Wiener's lemma and its application to Volterra difference equations on a Banach space. J Diff Eq Appl. **10**(13-15):1201–1214 (2004). doi:10.1080/10236190410001652748
8. Györi, I, Horváth, L: Asymptotic representation of the solutions of linear Volterra difference equations. Adv Diff Equ **22** (2008). (Art. ID 932831), (2008)
9. Györi, I, Reynolds, DW: On admissimility of the resolvent of discrete Volterra equations. J Diff Equ Appl. **16**(12):1393–1412 (2010). doi:10.1080/10236190902824196
10. Györi, I, Reynolds, DW: Sharp conditions for boundedness in linear discrete Volterra equations. J Diff Equ Appl. **15**, 1151–1164 (2009). doi:10.1080/10236190902932726
11. Kolmanovskii, VB, Castellanos-Velasco, E, Torres-Muñoz, JA: A survey: stability and boundedness of Volterra difference equations. Nonlinear Anal. **53**, 861–928 (2003). doi:10.1016/S0362-546X(03)00021-X
12. Kolmanovskii, V, Shaikhet, L: Some conditions for boundedness of solutions of difference Volterra equations. Appl Math Lett. **16**, 857–862 (2003). doi:10.1016/S0893-9659(03)90008-5
13. Medina, M: Asymptotic properties of solutions of nonlinear difference equations. J Comput Appl Math. **70**, 57–66 (1996). doi:10.1016/0377-0427(95)00139-5
14. Medina, R, Pinto, M: Asymptotic equivalence and asymptotic behavior of difference systems. Commun Appl Anal. **1**, 511–523 (1997)
15. Pachpatte, BG: Qualitative properties of solutions of certain Volterra type difference equations. Tamsui Oxf J Math Sci. **26**(3):273–285 (2010)
16. Yihong, Song, Baker, CTH: Admissibility for discrete Volterra equations. J Diff Equ Appl. **12**(5):433–457 (2006). doi:10.1080/10236190600563260
17. Yihong, Song, Baker, CTH: Linearized stability analysis of discrete Volterra equations. J Math Anal Appl. **294**, 310–333 (2004). doi:10.1016/j.jmaa.2004.02.019
18. Lipovan, O: On the asymptotic behavior of solutions to some nonlinear integral equations of convolution type. Dyn Contin Discrete Impuls Syst. **16**, 147–154 (2009)

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