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Controllability for Sobolev type fractional integro-differential systems in a Banach space

Hamdy M Ahmed*

*Correspondence:
Hamdy_17eg@yahoo.com
Higher Institute of Engineering, El-Shorouk Academy, P.O. Box 3 El-Shorouk City, Cairo, Egypt

Abstract

In this paper, by using compact semigroups and the Schauder fixed-point theorem, we study the sufficient conditions for controllability of Sobolev type fractional integro-differential systems in a Banach space. An example is provided to illustrate the obtained results.

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1 Introduction

A Sobolev-type equation appears in a variety of physical problems such as flow of fluids through fissured rocks, thermodynamics and propagation of long waves of small amplitude (see [1–3]). Recently, there has been an increasing interest in studying the problem of controllability of Sobolev type integro-differential systems. Balachandran and Dauer [4] studied the controllability of Sobolev type integro-differential systems in Banach spaces. Balachandran and Sakthivel [5] studied the controllability of Sobolev type semilinear integro-differential systems in Banach spaces. Balachandran, Anandhi and Dauer [6] studied the boundary controllability of Sobolev type abstract nonlinear integro-differential systems.

In this paper, we study the controllability of Sobolev type fractional integro-differential systems in Banach spaces in the following form:

$$\begin{aligned} &{}^{c}D^{\alpha}\big(Ex(t)\big) + Ax(t) = Bu(t) + f\big(t,x(t)\big) + \int_0^t g\left(t,s,x(s),\int_0^s H\big(s,\tau,x(\tau)\big)\,d\tau\right)ds,\\ &\quad t \in J = [0,a], a > 0, x(0) = x_0, \end{aligned} \tag{1.1}$$

where E and A are linear operators with domain contained in a Banach space X and ranges contained in a Banach space Y. The control function $u(\cdot)$ is in $L^2(J,U)$, a Banach space of admissible control functions, with U as a Banach space. B is a bounded linear operator from U into Y. The nonlinear operators $f \in C(J \times X, Y)$, $H \in C(J \times J \times X, X)$ and $g \in C(J \times J \times X \times X, Y)$ are all uniformly bounded continuous operators. The fractional derivative ${}^{c}D^{\alpha}$, $0 < \alpha < 1$ is understood in the Caputo sense.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Definition 2.1 (see [7–9]) The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2 (see [7–9]) The Caputo derivative of order α with the lower limit zero for a function f can be written as

$${}^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, 0 \le n-1 < \alpha < n.$$

If f is an abstract function with values in X, then the integrals appearing in the above definitions are taken in Bochner's sense.

The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the following hypotheses:

(H_1) A and E are closed linear operators,

(H_2) $D(E) \subset D(A)$ and E is bijective,

(H_3) $E^{-1} : Y \to D(E)$ is continuous.

The hypotheses H_1, H_2 and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \to Y$.

(H_4) For each $t \in [0, a]$ and for some $\lambda \in \rho(-AE^{-1})$, the resolvent set of $-AE^{-1}$, the resolvent $R(\lambda, -AE^{-1})$ is a compact operator.

Lemma 2.1 [10] *Let $S(t)$ be a uniformly continuous semigroup. If the resolvent set $R(\lambda; A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.*

From the above fact, $-AE^{-1}$ generates a compact semigroup $\{T(t), t \ge 0\}$ in Y, which means that there exists $M > 1$ such that

$$\max_{t \in J} \| T(t) \| \le M. \tag{2.1}$$

Definition 2.3 The system (1.1) is said to be controllable on the interval J if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1.1) satisfies $x(a) = x_1$.

(H_5) The linear operator W from U into X defined by

$$Wu = \int_0^a E^{-1}(a-s)^{\alpha-1} T_{\alpha}(a-s) Bu(s)\, ds$$

has an inverse bounded operator W^{-1} which takes values in $L^2(J, U)/\ker W$, where the kernel space of W is defined by $\ker W = \{x \in L^2(J, U) : Wx = 0\}$, B is a bounded linear operator and $T_{\alpha}(t)$ is defined later.

(H_6) The function f satisfies the following two conditions:

(i) For each $t \in J$, the function $f(t,\cdot) : X \to Y$ is continuous, and for each $x \in X$, the function $f(\cdot,x) : J \to Y$ is strongly measurable.

(ii) For each positive number $k \in N$, there is a positive function $g_k(\cdot) : [0,a] \to R^+$ such that

$$\sup_{|x|\le k} \left| f(t,x) \right| \le g_k(t),$$

the function $s \to (t-s)^{1-\alpha} g_k(s) \in L^1([0,t], R^+)$, and there exists a $\beta > 0$ such that

$$\lim_{k\to\infty} \inf \frac{\int_0^t (t-s)^{1-\alpha} g_k(s)\, ds}{k} = \beta < \infty, \quad t \in [0,a].$$

(H_7) For each $(t,s) \in J \times J$, the function $H(t,s,\cdot) : X \to X$ is continuous, and for each $x \in X$, the function $H(\cdot,\cdot,x) : J \times J \to X$ is strongly measurable.

(H_8) The function g satisfies the following two conditions:

(i) For each $(t,s,x) \in J \times J \times X$, the function $g(t,s,\cdot,\cdot) : X \times X \to Y$ is continuous, and for each $x \in X$, $H \in X$, the function $g(\cdot,x,y) : J \times J \to Y$ is strongly measurable.

(ii) For each positive number $k \in N$, there is a positive function $h_k(\cdot) : [0,a] \to R^+$ such that

$$\sup_{|x|\le k} \left| \int_0^t g\left(t,s,x, \int_0^s H(s,\tau,x)\, d\tau\right) ds \right| \le h_k(t),$$

the function $s \to (t-s)^{1-\alpha} h_k(s) \in L^1([0,t], R^+)$, and there exists a $\gamma > 0$ such that

$$\lim_{k\to\infty} \inf \frac{\int_0^t (t-s)^{1-\alpha} h_k(s)\, ds}{k} = \gamma < \infty, \quad t \in [0,a].$$

According to [11, 12], a solution of equation (1.1) can be represented by

$$\begin{aligned} x(t) = {} & E^{-1} S_\alpha(t) E x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) E^{-1} f\big(s, x(s)\big)\, ds \\ & + \int_0^t (t-s)^{\alpha-1} E^{-1} T_\alpha(t-s) B u(s)\, ds \\ & + \int_0^t (t-s)^{\alpha-1} E^{-1} T_\alpha(t-s) \left\{ \int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\, d\tau \right\} ds, \quad t \in J, \end{aligned} \tag{2.2}$$

where

$$R(\tau) = \int_0^\tau H\big(\tau,\eta,x(\eta)\big)\, d\eta, \qquad S_\alpha(t)x = \int_0^\infty \xi_\alpha(\theta) T\big(t^\alpha\theta\big) x\, d\theta,$$
$$T_\alpha(t)x = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T\big(t^\alpha\theta\big) x\, d\theta$$

with ξ_α being a probability density function defined on $(0,\infty)$, that is, $\xi_\alpha(\theta) \ge 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \xi_\alpha(\theta)\, d\theta = 1$.

Remark $\int_0^\infty \theta \xi_\alpha(\theta)\, d\theta = \frac{1}{\Gamma(1+\alpha)}$.

Definition 2.4 By a mild solution of the problem (1.1), we mean that the function $x \in C(J, X)$ satisfies the integral equation (2.2).

Lemma 2.2 (see [11]) *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties*:

(I) *For any fixed $x \in X$, $\|S_\alpha(t)x\| \le M\|x\|$, $\|T_\alpha(t)x\| \le \frac{\alpha M}{\Gamma(\alpha+1)}\|x\|$*;

(II) *$\{S_\alpha(t), t \ge 0\}$ and $\{T_\alpha(t), t \ge 0\}$ are strongly continuous*;

(III) *For every $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are also compact operators if $T(t)$, $t > 0$ is compact.*

3 Controllability result

In this section, we present and prove our main result.

Theorem 3.1 *If the assumptions (H_1)-(H_8) are satisfied, then the system* (1.1) *is controllable on J provided that $\frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}(\beta + \gamma)[1 + \frac{a^\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\|B\|\|W^{-1}\|] < 1$.*

Proof Using the assumption (H_5), for an arbitrary function $x(\cdot)$, define the control

$$u(t) = W^{-1}\Bigg[x_1 - E^{-1}S_\alpha(t)Ex_0 - \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)f\big(s, x(s)\big)\,ds - \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)\left\{\int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\,d\tau\right\}ds\Bigg](t).$$

It shall now be shown that when using this control, the operator Q defined by

$$\begin{aligned}(Qx)(t) = {} & E^{-1}S_\alpha(t)Ex_0 + \int_0^t (t-s)^{\alpha-1}E^{-1}T_\alpha(t-s)f\big(s,x(s)\big)\,ds \\ & + \int_0^t (t-s)^{\alpha-1}E^{-1}T_\alpha(t-s)Bu(s)\,ds \\ & + \int_0^t (t-s)^{\alpha-1}E^{-1}T_\alpha(t-s)\left\{\int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\,d\tau\right\}ds\end{aligned}$$

from $C(J, X)$ into itself for each $x \in C = C(J, X)$ has a fixed point. This fixed point is then a solution of equation (2.2).

$$\begin{aligned}(Qx)(a) = {} & E^{-1}S_\alpha(a)Ex_0 + \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)f\big(s,x(s)\big)\,ds \\ & + \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)BW^{-1} \\ & \times \Bigg[x_1 - E^{-1}S_\alpha(a)Ex_0 - \int_0^a (a-\tau)^{\alpha-1}E^{-1}T_\alpha(a-\tau)f\big(\tau,x(\tau)\big)\,d\tau \\ & - \int_0^a (a-\tau)^{\alpha-1}E^{-1}T_\alpha(a-\tau)\left\{\int_0^\tau g\big(\tau,\eta,x(\eta),R(\eta)\big)\,d\eta\right\}d\tau\Bigg](s)\,ds \\ & + \alpha\int_0^a (a-s)^{\alpha-1}\,E^{-1}T_\alpha(a-s)\left\{\int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\,d\tau\right\}ds = x_1.\end{aligned}$$

It can be easily verified that Q maps C into itself continuously.

For each positive number $k > 0$, let $B_k = \{x \in C : x(0) = x_0, \|x(t)\| \le k, t \in J\}$. Obviously, B_k is clearly a bounded, closed, convex subset in C. We claim that there exists a positive number k such that $QB_k \subset B_k$. If this is not true, then for each positive number k, there exists a function $x_k \in B_k$ with $Qx_k \notin B_k$, that is, $\|Qx_k\| \geq k$, then $1 \leq \frac{1}{k}\|Qx_k\|$, and so

$$1 \leq \lim_{k\to\infty} k^{-1}\|Qx_k\|.$$

However,

$$\begin{aligned}
&\lim_{k\to\infty} k^{-1}\|Qx_k\| \\
&\quad\leq \lim_{k\to\infty} k^{-1}\biggl\{M\bigl\|E^{-1}\bigr\|\|E\|\|x_0\| + \frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\int_0^a (a-s)^{\alpha-1}g_k(s)\,ds \\
&\qquad + \frac{\alpha M\|E^{-1}\|\|B\|\|W^{-1}\|}{\Gamma(\alpha+1)}\int_0^a (a-s)^{\alpha-1}\biggl[\|x_1\| + M\bigl\|E^{-1}\bigr\|\|E\|\|x_0\| \\
&\qquad + \frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\int_0^a (a-\tau)^{\alpha-1}g_k(\tau)\,d\tau \\
&\qquad + \frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\int_0^a (a-\tau)^{\alpha-1}h_k(\tau)\,d\tau\biggr]\,ds + \frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\int_0^a (a-s)^{\alpha-1}h_k(s)\,ds\biggr\} \\
&\quad\leq \frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\beta + \frac{\alpha a^\alpha M^2(\|E^{-1}\|)^2}{(\Gamma(\alpha+1))^2}\|B\|\bigl\|W^{-1}\bigr\|(\beta+\gamma) + \frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\gamma \\
&\quad= \frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}(\beta+\gamma)\biggl[1 + \frac{a^\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\|B\|\bigl\|W^{-1}\bigr\|\biggr] < 1,
\end{aligned}$$

a contradiction. Hence, $QB_k \subset B_k$ for some positive number k. In fact, the operator Q maps B_k into a compact subset of B_k. To prove this, we first show that the set $V_k(t) = \{(Qx)(t) : x \in B_k\}$ is a precompact in X; for every $t \in J$: This is trivial for $t = 0$, since $V_k(0) = \{x_0\}$. Let t, $0 < t \leq a$; be fixed. For $0 < \epsilon < t$ and arbitrary $\delta > 0$; take

$$\begin{aligned}
\bigl(Q^{\epsilon,\delta}x\bigr)(t) &= \int_\delta^\infty \xi_\alpha(\theta)E^{-1}T\bigl(t^\alpha\theta\bigr)Ex_0\,d\theta \\
&\quad + \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\bigl((t-s)^\alpha\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds \\
&\quad + \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\bigl((t-s)^\alpha\theta\bigr) \\
&\quad \times BW^{-1}\biggl[x_1 - \int_0^\infty \xi_\alpha(\theta)E^{-1}T\bigl(a^\alpha\theta\bigr)Ex_0\,d\theta \\
&\quad - \alpha\int_0^a\int_0^\infty \theta(a-\tau)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\bigl((a-\tau)^\alpha\theta\bigr)f\bigl(\tau,x(\tau)\bigr)\,d\theta\,d\tau \\
&\quad - \alpha\int_0^a\int_0^\infty \theta(a-\tau)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\bigl((a-\tau)^\alpha\theta\bigr) \\
&\quad \times \biggl\{\int_0^\tau g\bigl(\tau,\eta,x(\eta),R(\eta)\bigr)\,d\eta\biggr\}\,d\theta\,d\tau\biggr](s)\,d\theta\,ds \\
&\quad + \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\bigl((t-s)^\alpha\theta\bigr) \\
&\quad \times \biggl\{\int_0^s g\bigl(s,\tau,x(\tau),R(\tau)\bigr)\,d\tau\biggr\}\,d\theta\,ds
\end{aligned}$$

$$\begin{aligned}
&= T\big(\epsilon^\alpha\delta\big)\int_\delta^\infty \xi_\alpha(\theta)E^{-1}T\big(t^\alpha\theta-\epsilon^\alpha\delta\big)Ex_0\,d\theta \\
&\quad + T\big(\epsilon^\alpha\delta\big)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta-\epsilon^\alpha\delta\big)f\big(s,x(s)\big)\,d\theta\,ds \\
&\quad + T\big(\epsilon^\alpha\delta\big)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta-\epsilon^\alpha\delta\big) \\
&\quad \times BW^{-1}\bigg[x_1-\int_0^\infty \xi_\alpha(\theta)E^{-1}T\big(a^\alpha\theta\big)Ex_0\,d\theta \\
&\quad -\alpha\int_0^a\int_0^\infty \theta(a-\tau)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((a-\tau)^\alpha\theta\big)f\big(\tau,x(\tau)\big)\,d\theta\,d\tau \\
&\quad -\int_0^a\int_0^\infty \theta(a-\tau)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((a-\tau)^\alpha\theta\big) \\
&\quad \times\left\{\int_0^\tau g\big(\tau,\eta,x(\eta),R(\eta)\big)\,d\eta\right\}d\theta\,d\tau\bigg](s)\,d\theta\,ds \\
&\quad + T\big(\epsilon^\alpha\delta\big)\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta-\epsilon^\alpha\delta\big) \\
&\quad \times\left\{\int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\,d\tau\right\}d\theta\,ds.
\end{aligned}$$

Since $u(s)$ is bounded and $T(\epsilon^\alpha\delta)$, $\epsilon^\alpha\delta>0$ is a compact operator, then the set $V_k^{\epsilon,\delta}(t)=\{(Q^{\epsilon,\delta}x)(t):x\in B_k\}$ is a precompact set in X for every ϵ, $0<\epsilon<t$, and for all $\delta>0$. Also, for $x\in B_k$, using the defined control $u(t)$ yields

$$\begin{aligned}
&\big\|(Qx)(t)-\big(Q^{\epsilon,\delta}x\big)(t)\big\| \\
&\quad\le\left\|\int_0^\delta \xi_\alpha(\theta)E^{-1}T\big(t^\alpha\theta\big)Ex_0\,d\theta\right\| \\
&\qquad+\alpha\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta\big)f\big(s,x(s)\big)\,d\theta\,ds\right\| \\
&\qquad+\alpha\bigg\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta\big) \\
&\qquad\times BW^{-1}\bigg[x_1-\int_0^\infty \xi_\alpha(\theta)E^{-1}T\big(a^\alpha\theta\big)Ex_0\,d\theta \\
&\qquad-\alpha\int_0^a\int_0^\infty \theta(a-\tau)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((a-\tau)^\alpha\theta\big)f\big(\tau,x(\tau)\big)\,d\theta\,d\tau \\
&\qquad-\alpha\int_0^a\int_0^\infty \theta(a-\tau)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((a-\tau)^\alpha\theta\big) \\
&\qquad\times\left\{\int_0^\tau g\big(\tau,\eta,x(\eta),R(\eta)\big)\,d\eta\right\}d\theta\,d\tau\bigg](s)\,d\theta\,ds\bigg\| \\
&\qquad+\alpha\left\|\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta\big)\left\{\int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\,d\tau\right\}d\theta\,ds\right\| \\
&\qquad+\alpha\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta\big)f\big(s,x(s)\big)\,d\theta\,ds\right\| \\
&\qquad+\alpha\bigg\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta\big)
\end{aligned}$$

$$
\begin{aligned}
&\times BW^{-1}\Bigg[x_1 - \int_0^\infty \xi_\alpha(\theta)E^{-1}T\big(a^\alpha\theta\big)Ex_0\,d\theta \\
&-\alpha\int_0^a\int_0^\infty \theta(a-\tau)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((a-\tau)^\alpha\theta\big)f\big(\tau,x(\tau)\big)\,d\theta\,d\tau \\
&-\alpha\int_0^a\int_0^\infty \theta(a-\tau)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((a-\tau)^\alpha\theta\big) \\
&\times\left\{\int_0^\tau g\big(\tau,\eta,x(\eta),R(\eta)\big)\,d\eta\right\}d\theta\,d\tau\Bigg](s)\,d\theta\,ds\Bigg\| \\
&+\alpha\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)E^{-1}T\big((t-s)^\alpha\theta\big)\left\{\int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\,d\tau\right\}d\theta\,ds\right\| \\
\le{}& M\|E^{-1}\|\|E\|\|x_0\|\int_0^\delta \xi_\alpha(\theta)\,d\theta \\
&+\alpha M\|E^{-1}\|\left(\int_{t-\epsilon}^t (t-s)^{\alpha-1}g_k(s)\,ds\right)\left(\int_\delta^\infty \theta\xi_\alpha(\theta)\,d\theta\right) \\
&+\alpha M\|E^{-1}\|\|B\|\|W^{-1}\|\int_{t-\epsilon}^t (t-s)^{\alpha-1}\Bigg[\|x_1\| + M\|E^{-1}\|\|x_0\| \\
&+\frac{\alpha}{\Gamma(\alpha+1)}M\|E^{-1}\|\int_0^a (a-\tau)^{\alpha-1}g_k(\tau)\,d\tau \\
&+\frac{\alpha}{\Gamma(\alpha+1)}M\|E^{-1}\|\int_0^a (a-\tau)^{\alpha-1}h_k(\tau)\,d\tau\Bigg](s)\,ds\left(\int_\delta^\infty \theta\xi_\alpha(\theta)\,d\theta\right) \\
&+\alpha M\|E^{-1}\|\left(\int_{t-\epsilon}^t (t-s)^{\alpha-1}h_k(s)\,ds\right)\left(\int_\delta^\infty \theta\xi_\alpha(\theta)\,d\theta\right) \\
&+\alpha M\|E^{-1}\|\left(\int_0^t (t-s)^{\alpha-1}g_k(s)\,ds\right)\left(\int_0^\delta \theta\xi_\alpha(\theta)\,d\theta\right) \\
&+\alpha M\|E^{-1}\|\|B\|\|W^{-1}\|\int_0^t (t-s)^{\alpha-1}\Bigg[\|x_1\| + M\|E^{-1}\|\|x_0\| \\
&+\frac{\alpha}{\Gamma(\alpha+1)}M\|E^{-1}\|\int_0^a (a-\tau)^{\alpha-1}g_k(\tau)\,d\tau \\
&+\frac{\alpha}{\Gamma(\alpha+1)}M\|E^{-1}\|\int_0^a (a-\tau)^{\alpha-1}h_k(\tau)\,d\tau\Bigg](s)\,ds\left(\int_0^\delta \theta\xi_\alpha(\theta)\,d\theta\right) \\
&+\alpha M\|E^{-1}\|\left(\int_0^t (t-s)^{\alpha-1}h_k(s)\,ds\right)\left(\int_0^\delta \theta\xi_\alpha(\theta)\,d\theta\right).
\end{aligned}
$$

Therefore, as $\epsilon \to 0^+$ and $\delta \to 0^+$, there are precompact sets arbitrary close to the set $V_k(t)$ and so $V_k(t)$ is precompact in X.

Next, we show that $QB_k = \{Qx : x \in B_k\}$ is an equicontinuous family of functions.

Let $x \in B_k$ and $t, \tau \in J$ such that $0 < t < \tau$, then

$$
\begin{aligned}
&\big\|(Qx)(t) - (Qx)(\tau)\big\| \\
&\le \big\|T\big(t^\alpha\theta\big) - T\big(\tau^\alpha\theta\big)\big\|\|E^{-1}\|\|E\|\|x_0\| \\
&+\frac{\alpha\|E^{-1}\|}{\Gamma(\alpha+1)}\int_0^t \big\|(t-s)^{1-\alpha}T\big((t-s)^\alpha\theta\big) - (\tau-s)^{1-\alpha}T\big((\tau-s)^\alpha\theta\big)\big\|g_k(s)\,ds \\
&+\frac{\alpha M\|E^{-1}\|}{\Gamma(\alpha+1)}\int_t^\tau (\tau-s)^{1-\alpha}g_k(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\alpha}{\Gamma(\alpha+1)} \int_0^t \big\| (t-s)^{1-\alpha} T\big((t-s)^\alpha \theta\big) - (\tau-s)^{1-\alpha} T\big((\tau-s)^\alpha\theta\big) \big\| \big\|E^{-1}\big\| \|B\| \big\| W^{-1} \big\| \\
&\times \bigg[\|x_1\| + \big\|E^{-1}\big\| M \| \|E\| \|x_0\| + \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha+1)} \int_0^a (a-\tau)^{1-\alpha} g_k(\tau)\, d\tau \\
&+ \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha+1)} \int_0^a (a-\tau)^{1-\alpha} h_k(\tau)\, d\tau \bigg](s)\, ds \\
&+ \frac{\alpha M}{\Gamma(\alpha+1)} \int_t^\tau (\tau-s)^{1-\alpha} \big\|E^{-1}\big\| \|B\| \big\| W^{-1} \big\| \bigg[\|x_1\| + \big\|E^{-1}\big\| M \|E\| \|x_0\| \\
&+ \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha+1)} \int_0^a (a-\tau)^{1-\alpha} g_k(\tau)\, d\tau \\
&+ \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha+1)} \int_0^a (a-\tau)^{1-\alpha} h_k(\tau)\, d\tau \bigg](s)\, ds \\
&+ \frac{\alpha \|E^{-1}\|}{\Gamma(\alpha+1)} \int_0^t \big\| (t-s)^{1-\alpha} T\big((t-s)^\alpha \theta\big) - (\tau-s)^{1-\alpha} T\big((\tau-s)^\alpha\theta\big) \big\| h_k(s)\, ds \\
&+ \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha+1)} \int_t^\tau (\tau-s)^{1-\alpha} h_k(s)\, ds.
\end{aligned}
$$

Now, $T(t)$ is continuous in the uniform operator topology for $t > 0$ since $T(t)$ is compact, and the right-hand side of the above inequality tends to zero as $t \to \tau$. Thus, QB_k is both equicontinuous and bounded. By the Arzela-Ascoli theorem, QB_k is precompact in $C(J, X)$. Hence, Q is a completely continuous operator on $C(J, X)$.

From the Schauder fixed-point theorem, Q has a fixed point in B_k. Any fixed point of Q is a mild solution of (1.1) on J satisfying $(Qx)(t) = x(t) \in X$. Thus, the system (1.1) is controllable on J. □

4 Example

In this section, we present an example to our abstract results.

We consider the fractional integro-partial differential equation in the form

$$
\begin{aligned}
&{}^c\partial_t^\alpha \big(z(t,x) - z_{xx}(t,x)\big) - z_{xx}(t,x) \\
&\quad = Bu + \mu_1\big(t, z_{xx}(t,x)\big) \\
&\qquad + \int_0^t \mu_3\left(t, s, z_{xx}(s,x), \int_0^s \mu_2\big(s, \tau, z_{xx}(\tau,x)\big)\, d\tau\right) ds, \quad 0 \le x \le \pi, t \in J, \qquad (4.1) \\
&z(t,0) = z(t,\pi) = 0, \quad t \in J, \\
&z(0,x) = z_0(x), \quad x \in [0,\pi],
\end{aligned}
$$

where ${}^c\partial_t^\alpha$ is the Caputo fractional partial derivative of order $0 < \alpha < 1$.

Take $X = Y = L^2[0,\pi]$ and define the operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ by $Az = -z_{xx}$ and $Ez = z - z_{xx}$, where each domain $D(A)$ and $D(E)$ is given by $\{z \in X : z, z_x$ are absolutely continuous, $z_{xx} \in X, z(0) = z(\pi) = 0\}$.

Then A and E can be written respectively as [13]

$$
Az = \sum_{n=1}^\infty n^2 (z, z_n) z_n, \quad z \in D(A),
$$

$$Ez = \sum_{n=1}^{\infty} (1 + n^2)(z, z_n)z_n, \quad z \in D(E),$$

where $z_n(x) = \sqrt{2/\pi} \sin nx$, $n = 1, 2, \ldots$, is the orthonormal set of eigenvectors of A and (z, z_n) is the L^2 inner product. Moreover, for $z \in X$, we get

$$E^{-1}z = \sum_{n=1}^{\infty} \frac{1}{1 + n^2}(z, z_n)z_n,$$

$$-AE^{-1}z = \sum_{n=1}^{\infty} \frac{-n^2}{1 + n^2}(z, z_n)z_n,$$

$$T(t)z = \sum_{n=1}^{\infty} e^{\frac{-n^2}{(1+n^2)}t}(z, z_n)z_n.$$

We assume that

(A_1): The operator $B : U \to Y$, with $U \subset J$, is a bounded linear operator.

(A_2): The linear operator $W : U \to X$ defined by

$$Wu = \int_0^a E^{-1}(a - s)^{\alpha-1} T_\alpha(a - s)Bu(s)\, ds$$

has an inverse bounded operator W^{-1} which takes values in $L^2(J, U)/\ker W$, where the kernel space of W is defined by $\ker W = \{x \in L^2(J, U) : Wx = 0\}$, B is a bounded linear operator.

(A_3): The nonlinear operator $\mu_1 : J \times X \to Y$ satisfies the following three conditions:

(i) For each $t \in J$, $\mu_1(t, z)$ is continuous.

(ii) For each $z \in X$, $\mu_1(t, z)$ is measurable.

(iii) There is a constant ν $(0 < \nu < 1)$ and a function $h(\cdot) : [0, a] \to R^+$ such that for all $(t, z) \in J \times X$,

$$\big\| \mu_1(t, z) \big\| \le h(t)|z|^\nu.$$

(A_4): The nonlinear operator $\mu_2 : J \times J \times X \to X$ satisfies the following two conditions:

(i) For each $(t, s) \in J \times J$, $\mu_2(t, s, z)$ is continuous.

(ii) For each $z \in X$, $\mu_2(t, s, z)$ is measurable.

(A_5): The nonlinear operator $\mu_3 : J \times J \times X \times X \to Y$ satisfies the following three conditions:

(i) For each $(t, s, z) \in J \times J \times X$, $\mu_3(t, s, z)$ is continuous.

(ii) For each $z \in X$, $\mu_3(t, s, z)$ is measurable.

(iii) There is a constant ν $(0 < \nu < 1)$ and a function $g(\cdot) : [0, a] \to R^+$ such that for all $(t, s, z, y) \in J \times J \times X \times X$,

$$\left\| \int_0^t \mu_3\left(t, s, z, \int_0^s \mu_2(s, \tau, z)\, d\tau\right) ds \right\| \le g(t)|z|^\nu.$$

Define an operator $f : J \times X \to Y$ by

$$f(t, z)(x) = \mu_1\big(t, z_{xx}(x)\big)$$

and let

$$H(t,s,z)(x) = \mu_2\big(t,s,z_{xx}(x)\big), \quad (t,s,z) \in J \times J \times X,$$

$$g\left(t,s,z,\int_0^s H(s,\tau,z)\,d\tau\right)(x) = \mu_3\left(t,s,z_{xx},\int_0^s \mu_2\big(s,\tau,z_{xx}(x)\big)\,d\tau\right), \quad x \in [0,\pi].$$

Then the problem (4.1) can be formulated abstractly as:

$$\begin{aligned} &{}^cD^\alpha\big(Ez(t)\big) + Az(t) \\ &\quad = Bu(t) + f\big(t,z(t)\big) + \int_0^t g\left(t,s,z,\int_0^s H\big(s,\tau,z(\tau)\big)\,d\tau\right)ds, \quad t \in J, z(0) = z_0. \end{aligned}$$

It is easy to see that $-AE^{-1}$ generates a uniformly continuous semigroup $\{S(t)\}_{t\geq 0}$ on Y which is compact, and (2.1) is satisfied. Also, the operator f satisfies condition (H_6) and the operator H and g satisfy (H_7) and (H_8). Also all the conditions of Theorem 3.1 are satisfied. Hence, the equation (4.1) is controllable on J.

Competing interests
The author declare that he has no competing interests.

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