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Anti-periodic solutions for nonlinear evolution equations

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Abstract

In this paper, we use the homotopy method to establish the existence and uniqueness of anti-periodic solutions for the nonlinear anti-periodic problem

$$\begin{cases} \dot{x} + A(t, x) + Bx = f(t) & \text{a.e. } t \in R, \\ x(t + T) = -x(t), \end{cases}$$

where $A(t, x)$ is a nonlinear map and B is a bounded linear operator from R^N to R^N . Sufficient conditions for the existence of the solution set are presented. Also, we consider the nonlinear evolution problems with a perturbation term which is multivalued. We show that, for this problem, the solution set is nonempty and weakly compact in $W^{1,2}(I, R^N)$ for the case of convex valued perturbation and prove the existence theorems of anti-periodic solutions for the nonconvex case. All illustrative examples are provided.

Keywords: anti-periodic solution; evolution equation; Leray-Schauder alternative theorem; measurable selection; continuous selection

1 Introduction

Anti-periodic problems have important applications in auto-control, partial differential equations and engineering, and they have been studied extensively in the past ten years. For example, anti-periodic trigonometric polynomials are important in the study of interpolation problems [1], and anti-periodic wavelets are discussed in [2]. Recently, anti-periodic boundary conditions have been considered for the Schrödinger and Hill differential operator [3, 4]. Also, anti-periodic boundary conditions appear in the study of difference equations [5, 6]. Moreover, anti-periodic boundary conditions appear in physics in a variety of situations, see [7–10].

The study of anti-periodic solutions for nonlinear evolution equations was initiated by Okochi [11]. Since then, many authors have been devoted to investigation of the existence of anti-periodic solutions to nonlinear evolution equations in Hilbert spaces. For the details, see [12–17] and the references therein. In [15], Chen studied the anti-periodic solution for the following first-order semilinear evolution equation:

$$\begin{cases} \dot{u} + Au(t) + f(t, u) = 0 & \text{a.e. } t \in R, \\ u(t + T) = -u(t), \end{cases} \quad (1.1)$$

where $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a matrix, $f : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a continuous function satisfying $f(t + T, u) = -f(t, -u)$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$. Here they assume that $f(t, u)$ is a uniform bound with respect to u and $\frac{T^2}{4} \|A^2\| < 1$. We do not need these assumptions and consider the following semilinear anti-periodic problem:

$$\begin{cases} \dot{x} + A(t, x) + Bx = f(t) & \text{a.e. } t \in \mathbb{R}, \\ x(t + T) = -x(t), \end{cases} \quad (1.2)$$

where $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a hemicontinuous function satisfying $A(t + T, x) = -A(t, -x)$, $f : \mathbb{R} \rightarrow \mathbb{R}^N$ is a measurable function satisfying $f(t + T) = -f(t)$ for all $t \in \mathbb{R}$ and B is a bounded linear operator from \mathbb{R}^N to \mathbb{R}^N . We will establish some sufficient conditions for the existence and uniqueness of anti-periodic solutions of Eq. (1.2) by the theory of topological degree.

In addition, we also consider the following nonlinear evolution inclusion problem:

$$\begin{cases} \dot{x} + A(t, x) + Bx \in F(t, x) & \text{a.e. } t \in I, \\ x(T) = -x(0), \end{cases} \quad (1.3)$$

where $I = [0, T]$. We refer the reader to the work of [18, 19]. These works focused on the problem in which the multivalued term $F(t, x)$ is an even lower semi-continuous convex function with a compact assumption. But, in this paper, we prove the existence theorems of anti-periodic solutions for the cases of a convex and of a nonconvex valued perturbation term which is multivalued based on the techniques and results of the theory of set-valued analysis and the Leray-Schauder fixed point theorem. As far as we know, there are few papers which deal with this type of anti-periodic problems. For recent developments involving the existence of anti-periodic solutions of differential equations, inequalities and other interesting results on anti-periodic boundary value problems, the reader is referred to [20–27] and the references therein.

On the one hand, it is well known that the neural networks have been successfully applied to signal and image processing, pattern recognition and optimization. However, many neural networks with discontinuous neuron activation functions appear in the theoretical study on dynamics of neural networks, see [28, 29]. In order to solve some practical engineering problems, people also need to present new neural networks with discontinuous activation functions. Therefore, developing a new class of neural networks with discontinuous neuron activation functions and giving the conditions of the stability are very valuable in both theory and practice. Motivated by the above discussions, in this paper, we present a class of neural networks with discontinuous neuron activation functions. Based on our results, the existence and uniqueness of the equilibrium point is investigated.

On the other hand, it has been well recognized that differential inclusions, which are certainly of their own interest, provide a useful generalization of control systems governed by differential/evolution equations with control parameters

$$\dot{x} = f(t, x, u), \quad u \in U(t, x), \quad (1.4)$$

where the control sets $U(\cdot, \cdot)$ may also depend on the state variable x . Let $F(t, x) = f(t, x, U(t, x))$. Then Eq. (1.4) is reduced to $\dot{x} \in F(t, x)$, which is a particular case of the

inclusion relation in Eq. (1.3). Hence, we present an example of a nonlinear anti-periodic distributed parameter control system with *a priori* feedback for our results.

This paper is organized as follows. In Section 2, we state some basic knowledge from multivalued analysis. In Section 3, we first establish the existence of anti-periodic solutions for an evolution equation by the theory of topological degree, and then, by applying the Leray-Schauder fixed point theorem, we prove the existence of anti-periodic solutions for convex and nonconvex cases. Finally, two examples for our results are presented in Section 4.

2 Preliminaries

For convenience, we introduce some notations as follows. In Euclidean space, (\cdot, \cdot) expresses an inner product, while $|\cdot|$ expresses the Euclidean norm. Let $L^2([0, T]; R^N)$ denote the set of the map $x : [0, T] \rightarrow R^N$ which satisfies $\int_0^T |x|^2 dt < \infty$, and the norm in $L^2([0, T]; R^N)$ is denoted by $\|x\|_2 = (\int_0^T |x|^2 dt)^{\frac{1}{2}}$. We recall some basic definitions and facts from multivalued analysis which we shall need in what follows. For details, we refer to the book of Hu and Papageorgiou [30]. Let $I = [0, T]$, (I, Σ) be the Lebesgue measurable space and X be a separable Banach space. Denote

$$P_{(w)kc}(X) = \{A \subset X : \text{nonempty, (weakly) compact and convex}\}.$$

Let $A \subset P_f(X)$, $x \in X$, then the distance from x to A is given by $d(x, A) = \inf\{|x - a| : a \in A\}$. A multifunction $F : I \rightarrow P_f(X)$ is said to be measurable if and only if, for every $z \in X$, the function $t \rightarrow d(z, F(t)) = \inf\{\|z - x\| : x \in F(t)\}$ is measurable. A multifunction $G : I \rightarrow 2^X \setminus \{\emptyset\}$ is said to be graph measurable if $GrG = \{(t, x) : x \in G(t)\} \in \Sigma \times B(X)$ with $B(X)$ being the Borel σ -field of X . On $P_f(X)$ we can define a generalized metric known in the literature as the 'Hausdorff metric', by setting

$$h(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

for all $A, B \in P_f(X)$. It is well known that $(P_f(X), h)$ is a complete metric space and $P_{fc}(X)$ is a closed subset of it. When Z is a Hausdorff topological space, a multifunction $G : Z \rightarrow P_f(X)$ is said to be h -continuous if it is continuous as a function from Z into $(P_f(X), h)$.

Let Y, Z be Hausdorff topological spaces and $G : Y \rightarrow 2^Z \setminus \{\emptyset\}$. We say that $G(\cdot)$ is 'upper semicontinuous (USC)' (resp. 'lower semicontinuous (LSC)'), if for all $C \subseteq Z$ nonempty closed, $G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$ (resp. $G^+(C) = \{y \in Y : G(y) \subseteq C\}$) is closed in Y . A USC multifunction has a closed graph in $Y \times Z$, while the converse is true if G is locally compact (*i.e.*, for every $y \in Y$, there exists a neighborhood U of y such that $\overline{F(U)}$ is compact in Z). A multifunction which is both USC and LSC is said to be 'continuous'. If Y, Z are both metric spaces, then the above definition of LSC is equivalent to saying that for all $z \in Z$, $y \rightarrow d_Z(z, G(y)) = \inf\{d_Z(z, v) : v \in G(y)\}$ is upper semicontinuity as R_+ -valued function. Also, lower semicontinuity is equivalent to saying that if $y_n \rightarrow y$ in Y as $n \rightarrow \infty$, then

$$\begin{aligned} G(y) &\subseteq \varliminf G(y_n) = \{z \in Z : \lim d_Z(z, G(y_n)) = 0\} \\ &= \{z \in Z : z = \lim z_n, z_n \in G(y_n), n \geq 1\}. \end{aligned}$$

A set $D \subseteq L^2(I, X)$ is said to be 'decomposable', if for every $g_1, g_2 \in D$ and for every $J \subseteq I$ measurable, we have $\chi_J g_1 + \chi_{J^c} g_2 \in D$. The following lemmas are still needed in the proof of our main theorems.

Lemma 2.1 (see [31]) *If X is a Banach space, $C \subset X$ is nonempty, closed and convex with $0 \in C$, and $G : C \rightarrow P_{kc}(C)$ is an upper semicontinuous multifunction which maps bounded sets into relatively compact sets, then one of the following statements is true:*

- (i) *the set $\Gamma = \{x \in C : x \in \lambda G(x), \lambda \in (0, 1)\}$ is unbounded;*
- (ii) *the $G(\cdot)$ has a fixed point, i.e., there exists $x \in C$ such that $x \in G(x)$.*

Let X be a Banach space and let $L^2(I, X)$ be the Banach space of all functions $u : I \rightarrow X$ which are Bochner integrable. $D(L^2(I, X))$ denotes the collection of nonempty decomposable subsets of $L^2(I, X)$. Now, let us state the Bressan-Colombo continuous selection theorem.

Lemma 2.2 (see [32]) *Let X be a separable metric space and let $F : X \rightarrow D(L^2(I, X))$ be a lower semicontinuous multifunction with closed decomposable values. Then F has a continuous selection.*

3 Main results

3.1 The evolution equation

In this section, let

$$C_T = \{x | R \mapsto R^N \text{ is continuous, } x(t+T) = -x(t)\},$$

$$W^{1,2} = \left\{x \in C_T : \int_0^T (|x|^2 + |\dot{x}(t)|^2) dt < \infty\right\},$$

where \dot{x} is the weak derivative of x . C_T is a Banach space under the norm $\|x\|_c = \max_{t \in R} |x|$. Equipped with the norm

$$\|x\|_{1,2} = \left(\int_0^T (|x|^2 + |\dot{x}(t)|^2) dt \right)^{\frac{1}{2}},$$

$W^{1,2}$ becomes a separable Banach space. The following is our main result of this part.

Theorem 3.1 *Assume the following hold:*

- (i) *$f(t+T) = -f(t)$ and $A(t+T, -x) = -A(t, x)$ for all $(t, x) \in R \times R^N$;*
- (ii) *$t \rightarrow A(t, x)$ is measurable and $f \in L^2([0, T]; R^N)$;*
- (iii) *for each $t \in R$, the operator $A(t, \cdot) : R^N \rightarrow R^N$ is uniformly monotone and hemicontinuous, that is, there exists a constant $p > 0$ such that*

$$(A(t, x_1) - A(t, x_2), x_1 - x_2) \geq p|x_1 - x_2|^2$$
for all $x_1, x_2 \in R^N$, and the map

$$s \rightarrow (A(t, x + sz), y)$$
is continuous on $[0, 1]$ for all $x, y, z \in R^N$;
- (iv) *$B : R^N \rightarrow R^N$ is a bounded linear operator and there exists $c \in R^+$ such that*

$$(Bx, x) \geq c|x|^2, \quad \forall x \in R^N,$$

then the problem (1.2) has a unique T -anti-periodic solution.

In order to complete the proof of Theorem 3.1, we need the following lemmas.

Lemma 3.1 (see [33]) *Suppose Γ is a bounded open set of a normal space X , f is compact in $\bar{\Gamma}$ and $p \in X \setminus f(\partial\Gamma)$. Then the equation $f(x) = p$ has at least one solution in Γ , provided with $\deg(f, \Gamma, p) \neq 0$.*

Lemma 3.2 *Consider the equation*

$$\dot{x} + Bx = f(t) \quad \text{a.e. } t \in R, \quad (3.1)$$

where $B: R^N \rightarrow R^N$ is a bounded linear operator, and there exists $c \in R^+$ such that $(Bx, x) \geq c|x|^2$ for all $x \in R^N$, $f(t+T) = -f(t)$ and $f \in L^2([0, T]; R^N)$. Then the problem (3.1) has a unique T -anti-periodic solution.

Proof Let x be a solution of (3.1) satisfying the boundary value condition $x(0) = -x(T)$. Then x is a T -anti-periodic solution of (3.1). Denote $Lx = \dot{x} + Bx$ for all $x \in W^{1,2}$, then $L: W^{1,2} \rightarrow L^2([0, T]; R^N)$ is a linear operator.

Firstly, we show $L: W^{1,2} \rightarrow L^2([0, T]; R^N)$ is one to one. Suppose $L(x_1) = L(x_2)$, then $\dot{x}_1 + Bx_1 = \dot{x}_2 + Bx_2$ a.e. $t \in R$, and so $\dot{x}_1 - \dot{x}_2 + Bx_1 - Bx_2 = 0$ a.e. $t \in R$. Take an inner product above with $x_1 - x_2$ and note that

$$(\dot{x}_1 - \dot{x}_2, x_1 - x_2) + (Bx_1 - Bx_2, x_1 - x_2) = 0.$$

By using integration from 0 to T and the relation $x(0) = -x(T)$, one can see that

$$\int_0^T (\dot{x}_1 - \dot{x}_2, x_1 - x_2) dt = 0.$$

Since B is a linear bounded operator, then

$$(Bx_1 - Bx_2, x_1 - x_2) \geq c|x_1 - x_2|^2$$

for some constant $c > 0$. It follows that

$$\begin{aligned} 0 &= \int_0^T (Bx_1 - Bx_2, x_1 - x_2) dt \\ &\geq \int_0^T c|x_1 - x_2|^2 dt \\ &= c\|x_1 - x_2\|_2^2 \\ &\geq 0. \end{aligned}$$

Hence, $x_1 = x_2$ a.e. $t \in R$.

Next, we claim that $L: W^{1,2} \rightarrow L^2([0, T]; R^N)$ is surjective. For this purpose, consider the Cauchy problem

$$\begin{cases} \dot{x} + Bx = f(t), \\ x(0) = \eta. \end{cases} \quad (3.2)$$

It is well known that the above problem has a unique solution which can be written as follows:

$$x(t) = e^{Bt}\eta + \int_0^t e^{B(t-s)}f(s)ds.$$

Since $-x(0) = x(T)$, then we have that

$$-\eta = e^{BT}\eta + \int_0^T e^{B(T-s)}f(s)ds.$$

By hypothesis (iv), one has that $(-I - e^{BT})^{-1}$ exists; therefore, when we take

$$\eta = (-I - e^{BT})^{-1} \times \int_0^T e^{B(T-s)}f(s)ds,$$

the solution of the problem (3.2) is an anti-periodic solution of the problem (3.1). This completes the proof. \square

Proof of Theorem 3.1 Consider the homotopic systems of (1.2),

$$\dot{x} + Bx = \lambda f(t) - \lambda A(t, x), \quad (3.3)$$

where $\lambda \in [0, 1]$. Obviously, $\lambda f(t) - \lambda A(t, x)$ is hemicontinuous.

First, *a priori* bound of the solution set is derived. We claim that there is *a priori* bound in $W^{1,2}$ for the possible solution $x(t)$ of (3.3). Take the inner product with $x(t)$, and then integrate from 0 to T . It follows that

$$\int_0^T (\dot{x}, x) dt + \int_0^T (Bx, x) dt = \lambda \int_0^T (f, x) dt - \lambda \int_0^T (A(t, x), x) dt.$$

Without loss of generality, we assume $A(t, 0) = 0$. Since $\int_0^T (\dot{x}, x) dt = 0$, and then

$$\int_0^T (Bx, x) dt = \lambda \int_0^T (f, x) dt - \lambda \int_0^T (A(t, x), x) dt.$$

By hypothesis (iii), we deduce that

$$c\|x\|_2^2 \leq \lambda\|f\|_2\|x\|_2,$$

which implies

$$\|x\|_2 \leq M_1 \quad (3.4)$$

for some constant $M_1 > 0$. Hence, there is a constant $\tau \in [0, T]$ such that

$$|x(\tau)|^2 \leq M_2 \quad (3.5)$$

for some constant $M_2 > 0$. By (3.3), one has that

$$\begin{aligned}(\dot{x}, x) &= \lambda(f, x) - \lambda(A(t, x), x) - (Bx, x) \\ &\leq \lambda|f| \cdot |x| - \lambda p|x|^2.\end{aligned}$$

Integrating above from τ to t , we have that

$$\left| \int_{\tau}^t (\dot{x}, x) dt \right| \leq \int_0^T |f| \cdot |x| dt \leq \|f\|_2 \|x\|_2.$$

From $\int_{\tau}^t (\dot{x}, x) dt = |x(t)|^2 - |x(\tau)|^2$, we know

$$||x(t)|^2 - |x(\tau)|^2| \leq \|f\|_2 \|x\|_2. \quad (3.6)$$

By (3.4) and (3.5), we obtain that there is some constant $M_3 > 0$ (independent of λ) such that

$$|x(t)| \leq (|x(\tau)|^2 + \|f\|_2 \|x\|_2)^{\frac{1}{2}} \leq M_3$$

for any $t \in R$. Thus,

$$\|x\|_c = \max_{t \in R} |x(t)| \leq M_3. \quad (3.7)$$

Since the operator A is hemicontinuous, and B is a bounded linear operator, we show that

$$\begin{aligned}\|\dot{x}\|_2 &\leq M_4, \\ \|x\|_{1,2} &\leq M_5,\end{aligned}$$

where the constants $M_4, M_5 > 0$. The claim is proved.

Secondly, we can prove the existence of anti-periodic solutions for Eq. (1.2). Set

$$\Gamma = \{x \in W^{1,2} : \|x\|_{1,2} < M_5 + 1\}.$$

Then Γ is a bounded open set in $W^{1,2}$. By Lemma 3.2, it is easy to see that

$$L^{-1} : L^2([0, T]; R^N) \rightarrow W^{1,2}$$

is well defined. We define the operator $N : \Gamma \rightarrow W^{1,2}$, $N(x) = L^{-1}(f - A(t, x))$. Obviously, N is compact. Hence, the fixed point of N in $\bar{\Gamma}$ is the anti-periodic solutions of Eq. (1.2). Let $h_{\lambda}(x) : \bar{\Gamma} \times [0, 1] \rightarrow W^{1,2}$

$$h_{\lambda}(x) = x - \lambda N(x).$$

By (3.7), we obtain $\theta \in h(\partial\Gamma)$. So for each $\lambda \in [0, 1]$, then we have that

$$\begin{aligned}\deg(h_{\lambda}, \Gamma, \theta) &= \deg(h_1, \Gamma, \theta) \\ &= \deg(id - N, \Gamma, \theta)\end{aligned}$$

$$\begin{aligned} &= \deg(h_0, \Gamma, \theta) \\ &= \deg(id, \Gamma, \theta) = 1, \end{aligned}$$

where id is the identity. Consequently, N has a fixed point in Γ by Lemma 3.1. Namely, Eq. (1.2) has an anti-periodic solution.

Next, we prove the uniqueness. Suppose that x_1, x_2 are two solutions of Eq. (1.2). Then

$$\dot{x}_1 + A(t, x_1) + Bx_1 = \dot{x}_2 + A(t, x_2) + Bx_2.$$

So,

$$\dot{x}_1 - \dot{x}_2 + A(t, x_1) - A(t, x_2) + Bx_1 - Bx_2 = 0.$$

Take an inner product above with $x_1 - x_2$ and note that

$$(\dot{x}_1 - \dot{x}_2, x_1 - x_2) + (A(t, x_1) - A(t, x_2), x_1 - x_2) + (Bx_1 - Bx_2, x_1 - x_2) = 0.$$

By using integration from 0 to T and the relation $x(0) = -x(T)$, one can see that

$$\begin{aligned} 0 &= \int_0^T (A(t, x_1) - A(t, x_2), x_1 - x_2) dt + \int_0^T (Bx_1 - Bx_2, x_1 - x_2) dt \\ &\geq \int_0^T (c + p)|x_1 - x_2|^2 dt \\ &= (c + p)\|x_1 - x_2\|_2^2 \\ &\geq 0. \end{aligned}$$

Hence, $x_1 = x_2$ a.e. $t \in R$. This ends the proof. \square

3.2 The evolution inclusions

Let $I = [0, T]$ and $C(I; R^N)$ be all the continuous functions from I to R^N with the max norm. Let $C_\beta = \{\nu(\cdot) \in C(I; R^N) : \nu(0) = -\nu(T)\}$, and $W^{1,2}(I, R^N) = \{u(\cdot) \in C_\beta : \dot{u}(\cdot) \in L^2(I; R^N)\}$. $W^{1,2}(I, R^N)$ is a separable Banach space under the norm $\|\cdot\|_{1,2}$.

Consider the following anti-periodic problem:

$$\begin{cases} \dot{x} + A(t, x) + Bx \in F(t, x) & \text{a.e. } t \in I, \\ x(T) = -x(0), \end{cases} \quad (3.8)$$

where $A : R^N \rightarrow R^N$ is a hemicontinuous function, B is a bounded linear operator from R^N to R^N , and $F : R \times R^N \rightarrow 2^{R^N}$ is a multifunction. By a solution x of the problem (3.8), we mean a function $x \in W^{1,2}(I, R^N)$, and there exists a function $f(t) \in F(t, x(t))$ such that

$$\langle \dot{x}(t), v \rangle + \langle A(t, x(t)), v \rangle + \langle Bx, v \rangle = \langle f(t), v \rangle$$

for all $v \in R^N$ and almost all $t \in I$.

In this section, we prove two existence theorems under the hypothesis that the multi-valued nonlinearity F is convex-valued ('convex existence theorem') or nonconvex-valued ('nonconvex existence theorem'). The precise hypotheses on the data of the problem (3.8) are as follows:

$H(A)$: $A : I \times R^N \rightarrow R^N$ is a nonlinear function such that

- (i) $t \rightarrow A(t, x)$ is measurable;
- (ii) for each $t \in I$, the operator $A(t, \cdot) : R^N \rightarrow R^N$ is uniformly monotone and hemicontinuous, that is, there exists a constant $p > 0$ such that $(A(t, x_1) - A(t, x_2), x_1 - x_2) \geq p|x_1 - x_2|^2$ for all $x_1, x_2 \in R^N$, and the map $s \mapsto (A(t, x + sz), y)$ is continuous on $[0, 1]$ for all $x, y, z \in R^N$.

$H(B)$: $B : R^N \rightarrow R^N$ is a bounded linear operator, and there exists $c \in R^+$ such that

$$(Bx, x) \geq c|x|^2, \quad \forall x \in R^N.$$

$H(F)_1$: $F : R \times R^N \rightarrow P_k(R^N)$ is a multifunction such that

- (i) $(t, x) \rightarrow F(t, x)$ is graph measurable;
- (ii) for almost all $t \in I$, $x \rightarrow F(t, x)$ is LSC;
- (iii) there exists a nonnegative function $b(\cdot) \in L_+^2(I)$ and a constant $c_1 > 0$ such that

$$|F(t, x)| = \sup\{\|f\| : f \in F(t, x)\} \leq b(t) + c_1|x|^\alpha,$$

for all $x \in R^N$, $t \in T$, where $\alpha < 1$ or $\alpha = 1$ with $c_1 < c$ (c in $H(B)$).

$H(F)_2$: $F : I \times R^N \rightarrow P_{kc}(R^N)$ is a multifunction such that

- (i) $(t, x) \rightarrow F(t, x)$ is graph measurable;
- (ii) for almost all $t \in I$, $x \rightarrow F(t, x)$ has a closed graph; and $H(F)_1$ (iii) holds.

Theorem 3.2 *If hypotheses $H(A)$, $H(B)$ and $H(F)_1$ hold, then the problem (3.8) has a solution $x \in W^{1,2}(I, R^N)$.*

Proof Let $Lx = \dot{x} + A(t, x) + Bx$ for all $x \in W^{1,2}(I, R^N)$. By Theorem 3.1, we have $L : W^{1,2}(I, R^N) \rightarrow L^2([0, T]; R^N)$ is one to one and surjective, and so $L^{-1} : L^2([0, T]; R^N) \rightarrow W^{1,2}(I, R^N)$ is well defined. So, we prove that

$$L^{-1} : L^2([0, T]; R^N) \rightarrow L^2([0, T]; R^N) (W^{1,2}(I, R^N) \subset L^2([0, T]; R^N))$$

is completely continuous (i.e., it is continuous and maps bounded sets into relatively compact sets). To this end, let $K \subset L^2([0, T]; R^N)$ be bounded. We shall show that $L^{-1}(K)$ is relatively compact in $L^2([0, T]; R^N)$. For this purpose, let $x \in L^{-1}(K)$, then $x = L^{-1}(u)$ with $u \in K$. By (3.7), we have $\|x\|_2 \leq c\|Lx\|_2 = \|u\|_2 \leq c|K| = c \sup\{\|u\|_2 : u \in K\} < +\infty$ and $\|\dot{x}\|_2 \leq \|u\|_2 + \|A(x)\|_2 + \|Bx\|_2 \leq M$ for some constant $M > 0$. From these bounds we infer that $L^{-1}(K)$ is bounded in $W^{1,2}(I, R^N)$. But $W^{1,2}(I, R^N)$ is compactly embedded in $L^2([0, T]; R^N)$. Therefore, $L^{-1}(K)$ is relatively compact in $L^2([0, T]; R^N)$. Also, from the fact that L^{-1} is a compact operator, $L^{-1} : L^2([0, T]; R^N) \rightarrow L^2([0, T]; R^N)$ is continuous.

Next, let $N : L^2([0, T]; R^N) \rightarrow 2^{L^2([0, T]; R^N)}$ be the multivalued Nemitsky operator corresponding to F and N be defined by $N(x) = \{v \in L^2([0, T]; R^N) : v(t) \in F(t, x(t))\}$ a.e. on I .

We claim that $N(\cdot)$ has nonempty, closed, decomposable values and is LSC. The closedness and decomposability of the values of $N(\cdot)$ are easy to check. For the nonemptiness, note that if $x \in L^2([0, T]; \mathbb{R}^N)$, by hypothesis $H(F)_1(i)$, $(t, x) \rightarrow F(t, x)$ is graph measurable, so we apply Aumann's selection theorem and obtain a measurable map $v: I \rightarrow \mathbb{R}^N$ such that $v(t) \in F(t, x(t))$ a.e. on I . By hypothesis $H(F)_1(iii)$, $v \in L^2([0, T]; \mathbb{R}^N)$. Thus for every $x \in \mathbb{R}^N$, $N(x) \neq \emptyset$. To prove the lower semicontinuity of $N(\cdot)$, we only show that every $u \in L^2([0, T]; \mathbb{R}^N)$, $x \rightarrow d(u, N(x))$ is a USC \mathbb{R}_+ -valued function. Note that

$$\begin{aligned} d(u, N(x)) &= \inf\{\|u - v\|_2 : v \in N(x)\} \\ &= \inf\left\{\left(\int_0^T |u(t) - v(t)|^2 dt\right)^{\frac{1}{2}} : v \in N(x)\right\} \\ &= \left(\int_0^T \inf\{|u(t) - v(t)|_2 : v \in F(t, x)\} dt\right)^{\frac{1}{2}} \\ &= \left(\int_0^T d(u(t), F(t, x))^2 dt\right)^{\frac{1}{2}} \end{aligned}$$

(see Hiai and Umegaki [34] Th. 2.2). We shall show that for every $\lambda \geq 0$, the superlevel set $U_\lambda = \{x \in L^2([0, T]; \mathbb{R}^N) : d(u, N(x)) \geq \lambda\}$ is closed in $L^2([0, T]; \mathbb{R}^N)$. Let $\{x_n\}_{n \geq 1} \subseteq U_\lambda$ and assume that $x_n \rightarrow x$ in $L^2([0, T]; \mathbb{R}^N)$. By passing to a subsequence, if necessary, we may assume that $x_n(t) \rightarrow x(t)$ a.e. on I as $n \rightarrow \infty$. By hypothesis $H(F)_1(ii)$, $x \rightarrow d(u, F(t, x))$ is an upper semicontinuous \mathbb{R}_+ -valued function. So, via Fatou's lemma, we have

$$\begin{aligned} \lambda^2 &\leq \overline{\lim}[d(u, N(x_n))]^2 = \overline{\lim} \int_0^T [d(u(t), F(t, x_n))]^2 dx \\ &\leq \int_0^T \overline{\lim}[d(u(t), F(t, x_n))]^2 dx \\ &\leq \int_0^T [d(u, F(t, x))]^2 dt = [d(u, N(x))]^2. \end{aligned}$$

Therefore, $x \in U_\lambda$ and this proves the LSC of $N(\cdot)$.

We apply Lemma 2.2 and obtain a continuous map $f: L^2([0, T]; \mathbb{R}^N) \rightarrow L^2([0, T]; \mathbb{R}^N)$ such that $f(x) \in N(x)$. To finish our proof, we only need to solve the fixed point problem: $x = L^{-1}f(x)$.

We claim that the set $\Gamma = \{x \in L^2([0, T]; \mathbb{R}^N) : x = \sigma L^{-1}f(x), \sigma \in (0, 1)\}$ is bounded. Let $x \in \Gamma$, then $x = \sigma L^{-1}f(x)$. By hypothesis $H(F)_1(iii)$, we can derive

$$|f(x)| \leq b(t) + c_1|x|^\alpha,$$

then

$$\begin{aligned} \|f(x)\|_2 &\leq \|b\|_2 + \|c_1|x|^\alpha\|_2 \\ &\leq \|b\|_2 + \left(\|c_1^2\|_{\frac{1}{1-\alpha}} \| |x|^{2\alpha} \|_{\frac{1}{\alpha}}\right)^{1/2} \\ &= \|b\|_2 + \|c_1\|_{\frac{2}{1-\alpha}} \|x\|_2^\alpha \\ &= \|b\|_2 + c_1 T^{\frac{2}{1-\alpha}} \|x\|_2^\alpha \end{aligned}$$

with $\alpha < 1$. By (3.6), we get that

$$\|x\|_2 \leq c_2 \|f\|_2,$$

for some constant $c_2 > 0$. So, we have that

$$\|x\|_2 \leq c_2 \|b\|_2 + c_1 c_2 T^{\frac{2}{1-\alpha}} \|x\|_2^\alpha.$$

Thus, we can find a constant $c_3 > 0$ such that $\|x\|_2 \leq c_3$. If $\alpha = 1$, we can also find a constant $\bar{c}_3 = \frac{\|b\|_2}{c-c_1} > 0$ such that $\|x\|_2 \leq \bar{c}_3$. Similar to the estimation of (3.7), we have that

$$\|x\|_c = \max_{t \in R} |x(t)| \leq c_4, \quad (3.9)$$

for some constant $c_4 > 0$. So, Γ is bounded in $L^2([0, T]; R^N)$. Invoking Leray-Schauder's alternative theorem, we obtain there exists $x \in W^{1,2}(I, R^N)$ such that $x = L^{-1}f(x)$, x is a solution of the problem (3.8). This ends the proof. \square

Theorem 3.3 *If hypotheses $H(A)$, $H(B)$ and $H(F)_2$ hold, then the problem (3.8) has a solution $x \in W^{1,2}(I, R^N)$. Moreover, the solution set is weakly compact in $W^{1,2}(I, R^N)$.*

Proof The proof is as that of Theorem 3.2. So, we only present those particular points where the two proofs differ.

In this case, the multivalued Nemitsky operator $N : L^2([0, T]; R^N) \rightarrow 2^{L^2([0, T]; R^N)}$ has nonempty closed, convex values in $L^2([0, T]; R^N)$ and is USC. The closedness and convexity of the values of $N(\cdot)$ are clear. To prove the nonemptiness, let $x \in W^{1,2}(I, R^N)$ and $\{s_n\}_{n \geq 1}$ be a sequence of step functions such that

$$s_n(t) \rightarrow x \quad \text{and} \quad |s_n(t)| \leq |x(t)| \quad \text{a.e. on } I.$$

Then by virtue of hypothesis $H(F)_2(i)$, for every $n \geq 1$, $t \rightarrow F(t, s_n)$ admits a measurable selector $f_n(t)$. From hypothesis $H(F)_2(iii)$, we have that there exists a constant $c_5 > 0$ such that

$$\sup_{f_n \in F(t, s_n)} \|f_n\|_2 \leq \|b\|_2 + c_5 \|x\|_2^\alpha.$$

So $\{f_n\}_{n \geq 1}$ is uniformly integrable. By the Dunford-Pettis theorem, and by passing to a subsequence if necessary, we may assume that $f_n \rightarrow f$ weakly in $L^2([0, T]; R^N)$. Then from Theorem 3.1 in [35], we have

$$f(t) \in \overline{\text{conv}} \lim_{n \geq 1} \{f_n(t)\} \subseteq \overline{\text{conv}} \lim F(t, s_n) \subseteq F(t, x) \quad \text{a.e. on } I \quad (3.10)$$

the last inclusion being a consequence of hypothesis $H(F)_2(ii)$. So $f \in N(x)$. Thus we prove the nonemptiness of $N(\cdot)$.

Next, we show that $N(\cdot)$ is USC from $W^{1,2}(I, R^N)$ into $L^2([0, T]; R^N)_w$. Let C be a nonempty and weakly closed subset of $L^2([0, T]; R^N)$. We need to show that the set

$$N^{-1}(C) = \{x \in D(L) : N(x) \cap C \neq \emptyset\}$$

is closed. Let $\{x_n\}_{n \geq 1} \subseteq N^{-1}(C)$ and assume $x_n \rightarrow x$ in $W^{1,2}(I, R^N)$. Passing to a subsequence, we can get that $x_n(t) \rightarrow x(t)$ a.e. on I . Let $f_n \in N(x_n) \cap C$, $n \geq 1$. Then by virtue of hypothesis $H(F)_2$ (iii) and the Dunford-Pettis theorem, we may assume that $f_n \rightarrow f \in C$ weakly in $L^2([0, T]; R^N)$. As before, we have

$$f(t) \in \overline{\text{conv}} \overline{\lim} \{f_n(t)\}_{n \geq 1} \subseteq \overline{\text{conv}} \overline{\lim} F(t, x_n) \subseteq F(t, x) \quad \text{a.e. on } I,$$

then $f \in N(x) \cap C$, i.e., $N^{-1}(C)$ is closed in $W^{1,2}(I, R^N)$. This proves the upper semicontinuity of $N(\cdot)$ from $W^{1,2}(I, R^N)$ into $L^2([0, T]; R^N)_w$.

We consider the following fixed point problem:

$$x \in L^{-1}N(x).$$

Recalling that $L^{-1} : L^2([0, T]; R^N) \rightarrow L^2([0, T]; R^N)$ is completely continuous, we see that $L^{-1}N : L^2([0, T]; R^N) \rightarrow P_{kc}(L^2([0, T]; R^N))$ is USC and maps bounded sets into relatively compact sets. We easily check that the set

$$\Gamma_1 = \{u \in L^2([0, T]; R^N) : u \in \sigma L^{-1}N(u), \sigma \in (0, 1)\}$$

is bounded, as in the proof of Theorem 3.2. Invoking Lemma 2.1, there exists $u \in W^{1,2}(I, R^N)$ such that $u \in L^{-1}N(u)$. Evidently, this is a solution of the problem (3.8).

Let S denote the solution set of the problem (3.8). As in the proof of Theorem 3.2, we have that $|S| = \sup\{\|u\|_{1,2} : u \in S\} \leq M$, where $M > 0$. By virtue of hypothesis $H(F)_2$ (iii) and the Dunford-Pettis theorem, we may assume that $u_n \rightarrow u$ weakly in $W^{1,2}(I, R^N)$. As before, we have

$$Lu \in \overline{\text{conv}} \overline{\lim} \{Lu_n\}_{n \geq 1} \subseteq \overline{\text{conv}} \overline{\lim} F(t, u_n) \subseteq F(t, u) \quad \text{a.e. on } I,$$

then $u \in S$, hence S is weakly compact in $W^{1,2}(I, R^N)$. □

4 Examples

As an application of the previous results, we introduce two examples. Consider a class of neural networks described by the system of differential equations

$$\dot{x} = -Bx(t) + Ag(x) + I(t), \quad (4.1)$$

where $x = (x_1, x_2, \dots, x_N)^T \in R^N$ is the vector of neuron state, $A = \text{diag}(a_1, a_2, \dots, a_N)$ is an $N \times N$ diagonal matrix, where $a_i < 0$, $i = 1, 2, \dots, N$, are the neuron self-inhibitions; $B = (b_{ij})$ is an $N \times N$ positive definite matrix, which represents the neuron interconnection matrix. Moreover, $g(x) = (g_1(x_1), g_2(x_2), \dots, g_N(x_N))^T : R^N \rightarrow R^N$ is a mapping where $g_i : i = 1, 2, \dots, N$, represents the neuron input-output activation and $I(t) = (I_1(t), I_2(t), \dots, I_N(t))^T : R \rightarrow R^N$ is the mapping of neuron inputs.

We set $A(t, x) = -Ag(x)$. It is easy to check $A(t, x)$ satisfies the condition of Theorem 3.1. Moreover, $I(t)$ is bounded and B is a positive definite matrix. Thus, by Theorem 3.1 we easily obtain the following theorem.

Theorem 4.1 *If for any $x, \bar{x} \in R^N$, there exists a constant $\alpha \in R_+$ such that $(g(x) - g(\bar{x}), x - \bar{x}) \geq \alpha |x - \bar{x}|^2$, and $g(-x) = -g(x)$, $I(t + T) = -I(t)$ for all $t \in R$, $x \in R^N$, then the problem (4.1) has a unique anti-periodic solution.*

Discontinuous dynamical systems, particularly neural networks with discontinuous activation functions, arise in a number of applications. Further, we need the following assumptions.

$H(C)$: We have $I_i \in \Omega$, for any $i = 1, 2, \dots, N$, where Ω denotes the class of functions from R^N to R which are monotone nondecreasing bounded and have at most a finite number of jump discontinuities in every compact interval.

We note that if I satisfies $H(C)$, then any I_i , $i = 1, 2, \dots, N$, possesses only isolated jump discontinuities where I_i is not necessary defined. Hence for all $x \in R^N$, we have

$$\Phi[I(x)] = ([I_1(x_1^-), I_1(x_1^+)], [I_2(x_2^-), I_2(x_2^+)], \dots, [I_N(x_N^-), I_N(x_N^+)])$$

where $I_i(x_i^-) = \lim_{\varepsilon \rightarrow x_i^-} I_i(\varepsilon)$, $I_i(x_i^+) = \lim_{\varepsilon \rightarrow x_i^+} I_i(\varepsilon)$. Thus the differential equations (4.1) become the following differential inclusions:

$$\dot{x} \in Ag(x) - Bx(t) + \Phi[I(x)]. \quad (4.2)$$

The existence and the stability of the equilibrium point of (4.1) were first discussed in [29] ($I(t)$ is constant). In [28], the authors proved the existence of periodic solutions of (4.1) when $I(t)$ is the continuous periodic input and $g(x)$ is discontinuous.

We set $F(t, x) = \Phi[I(x)]$, it is easy to check $F(t, x)$ satisfies $H(F)_2$. Thus, by Theorem 3.3, we obtain the following theorem.

Theorem 4.2 *If for any $x, \bar{x} \in R^N$, there exists a constant $\alpha \in R_+$ such that $(g(x) - g(\bar{x}), x - \bar{x}) \geq \alpha |x - \bar{x}|^2$, and $H(C)$ hold, then the problem (4.1) has a nonempty set of solutions $x \in W^{1,2}(I, R^N)$.*

Next, we present an example of a nonlinear anti-periodic distributed parameter control system, with *a priori* feedback (i.e., state dependent control constraint set). Let $T = [0, b]$, $\dot{x} = (\dot{x}_1, \dot{x}_2, \dots, \dot{x}_N)$. We consider the following control system:

$$\begin{cases} \dot{x} + a(t, x)x + Bx = g(t, x)u(t) & \text{a.e. } t \in T, \\ x(0) = -x(b) \\ u(t) \in U(t, x(t)) & \text{a.e. } t \in T, \end{cases} \quad (4.3)$$

where B is a positive definite matrix. The hypotheses on the data (4.3) are as follows:

$H(a)$: $a : T \times R^N \rightarrow R^+$, $g : T \times R^N \rightarrow R$ are Carathéodory functions such that, for almost all $t \in T$,

$$\begin{aligned} 0 < \theta_1 &\leq a(t, x) \leq \theta_2, \\ |g(t, x)| &\leq \eta_1(t) + \eta_2(t)|x|^\alpha, \end{aligned}$$

with $\theta_1, \theta_2 > 0$, $0 < \alpha < 1$, $\eta_1(t) \in L^2_+(T)$, $\eta_2(t) \in L^\infty(T)$.

$H(U)$: $U : T \times R^N \rightarrow P_k(R^N)$ is a multifunction such that

- (i) for all $x \in R^N$, $t \rightarrow U(t, x)$ is measurable;
- (ii) for all $t \in T$, $x \rightarrow U(t, x)$ is h -continuous;
- (iii) for almost all $t \in T$ and all $x \in R^N$, $|U(t, x)| \leq \gamma$, with $\gamma > 0$.

Let $A : T \times R^N \rightarrow R^N$ be the operator defined by $A(t, x) = a(t, x)x$. Evidently, using hypothesis $H(a)$, it is straightforward to check that A satisfies hypothesis $H(A)$, B satisfies hypothesis $H(B)$. Also, let $F : T \times R^N \rightarrow P_k(R^N)$ be defined by

$$F(t, x) = \{y \in R^N : y(t) = g(t, x(t))u(t), u(t) \in U(t, x(t)), \text{ a.e. on } T\}.$$

Using hypotheses $H(a)$ and $H(U)$, it is straightforward to check that F satisfies hypothesis $H(F)_1$.

Rewrite the problem (4.3) in the following equivalent evolution inclusion form:

$$\begin{cases} \dot{x} + A(t, x(t)) + Bx \in F(t, x) & \text{a.e. } t \in T, \\ x(0) = -x(b). \end{cases} \quad (4.4)$$

We can apply Theorem 3.2 on the problem (4.3) and obtain:

Theorem 4.3 *If hypotheses $H(a)$ and $H(U)$ hold, then the problem (4.3) has a solution $x \in W^{1,2}(I, R^N)$.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YC carried out the main part of this manuscript. FC participated in the discussion and corrected the main theorem. HH provided all examples for our results. All authors read and approved the final manuscript.

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