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Positive solutions for a second-order p-Laplacian impulsive boundary value problem

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Abstract
In this work, we study the existence and multiplicity of positive solutions for a second-order p-Laplacian boundary value problem involving impulsive effects. We establish our main results *via* Jensen's inequality, the first eigenvalue of a relevant linear operator and the Krasnoselskii-Zabreiko fixed point theorem. Some examples are presented to illustrate the main results.

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1 Introduction

Second-order differential equations with the p-Laplacian operator arise in modeling some physical and natural phenomena and can occur, for example, in non-Newtonian mechanics, nonlinear elasticity, glaciology, population biology, combustion theory, and nonlinear flow laws, see [1, 2]. Recently, many cases of the existence and multiplicity of positive solutions for boundary value problems of differential equations with the p-Laplacian operator have appeared in the literature. For details, see, for example, [3–13] and the references therein.

In [3], Lian and Ge investigated the Sturm-Liouville-like boundary value problem

$$\begin{cases} (\varphi_p(u'(t)))' + f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) - \alpha u'(\xi) = 0, & u(1) + \beta u'(\eta) = 0, \end{cases} \tag{1.1}$$

and by virtue of Krasonsel'skii's fixed point theorem, they obtained the existence of positive solutions and multiple positive solutions under suitable conditions imposed on the nonlinear term $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

In [6], Xu *et al.* studied the existence of multiple positive solutions for the following boundary value problem with the p-Laplacian operator and impulsive effects

$$\begin{cases} (\phi_p(u'(t)))' + q(t)f(t,u(t)) = 0, & t \neq t_k, 0 < t < 1, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1,2,\ldots,m, \\ au(0) - bu'(0) = \sum_{i=1}^{l} \alpha_i u(\xi_i), & u'(1) = \sum_{i=1}^{l} \beta_i u'(\xi_i), \end{cases} \tag{1.2}$$

where the nonlinear term may be singular on $u = 0$. The main tools are fixed point index theorems for compact maps in Banach spaces. They stated the proofs by considering an approximating completely continuous operator.

In [7], Feng studied an integral boundary value problem of fourth-order p-Laplacian differential equations involving the impulsive effect $\Delta y'|_{t=t_k} = -I_k(y(t_k))$, $k = 1, 2, \dots, m$. Using a suitably constructed cone and fixed point theory for cones, the existence of multiple positive solutions was established. Furthermore, upper and lower bounds for these positive solutions were given.

Motivated by the above works, in this paper, we investigate the existence and multiplicity of positive solutions for the second-order p-Laplacian boundary value problems involving impulsive effects

$$\begin{cases} (\varphi_p(u'(t)))' = -f(t, u(t)), & 0 < t < 1, t \neq t_k, k = 1, 2, \dots, m, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \dots, m, \\ \Delta u'|_{t=t_k} = 0, & k = 1, 2, \dots, m, \\ u(0) = u'(1) = 0, \end{cases} \tag{1.3}$$

where $J = [0, 1]$, $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, t_k ($k = 1, 2, \dots, m$, where m is a fixed positive integer) are fixed points with $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < 1$; $\varphi_p(t)$ is the p-Laplacian operator, *i.e.*, $\varphi_p(t) = |t|^{p-2}t$, $p > 1$, $(\varphi_p)^{-1} = \varphi_q$, $p^{-1} + q^{-1} = 1$; $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, *i.e.*, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ at $t = t_k$, respectively. In addition, we suppose that $I_k \in C([0, +\infty), [0, +\infty))$, $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$.

The main features of this paper are as follows. Firstly, we convert the boundary value problem (1.3) into an equivalent integral equation so that we can construct a special cone. Next, we consider impulsive effect as a perturbation to the corresponding problem without the impulsive terms, so that we can construct an integral operator for an appropriate linear Robin boundary value problem and obtain its first eigenvalue and eigenfunction, which are used in the proofs of main theorems by Jensen's inequalities. Finally, employing the Krasnoselskii-Zabreiko fixed point theorem, we establish the existence and multiplicity of positive solutions of (1.3). Although our problem (1.3) merely involves Robin boundary conditions, our methods are different from those in [3, 6, 7], and our main results are optimal.

This paper is organized as follows. Section 2 contains some preliminary results. Section 3 is devoted to the existence and multiplicity of positive solutions for (1.3). Section 4 contains some illustrative examples.

2 Preliminaries

Let $PC[J, R] := \{u | u : J \to R$ is continuous at $t \neq t_k$, $u(t_k^-) = u(t_k)$ and $u(t_k^+)$ exist, $k = 1, 2, \dots, m\}$. Then $PC[J, R]$ is a Banach space with norm $\|u\| = \max_{t \in [0,1]} |u(t)|$. We denote $B_r := \{u \in PC[J, R] : \|u\| < r\}$ for $r > 0$ in the sequel.

A function $u \in PC[J, R] \cap C^2(J', R)$ is called a solution of (1.3) if it satisfies the boundary value problem (1.3).

Lemma 2.1 (see [4]) *Let f and I_k be as in (1.3). Then the problem (1.3) is equivalent to*

$$u(t) = \int_0^t \varphi_q\left(\int_s^1 f\bigl(\tau, u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{t_k<t} I_k\bigl(u(t_k)\bigr). \tag{2.1}$$

It is clear that $u'(t) = \varphi_q(\int_t^1 f(\tau, u(\tau))\,\mathrm{d}\tau) > 0$, $t \in J'$ and $I_k > 0$, which implies that $u(t)$ is increasing on $[0,1]$. Furthermore, for given $s_1, s_2 \in J'$ with $s_1 \le s_2$, we have $u'(s_2) \le u'(s_1)$. Hence, $u'(t)$ is nonincreasing on J', and thus

$$\frac{u(1) - u(0)}{1} \le \frac{u(t) - u(0)}{t}, \quad t \in (0,1],$$

i.e., $u(t) \ge tu(1) = t\|u\|$. Therefore,

$$u(t) \ge t\|u\|, \quad \forall t \in [0,1], \quad \textit{in particular}, \quad u(t) \ge t_1\|u\|, \quad \forall t \in [t_1, t_m]. \tag{2.2}$$

We denote P by

$$P := \bigl\{u \in PC[J,R] : u(t) \ge t\|u\|, t \in [0,1]\bigr\}. \tag{2.3}$$

Then P is a cone on $PC[J,R]$.

Define an operator $A : P \to PC[J,R]$

$$\begin{aligned} Au(t) &:= \int_0^t \varphi_q\left(\int_s^1 f\bigl(\tau, u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{t_k<t} I_k\bigl(u(t_k)\bigr) \\ &= \int_0^t \varphi_q\left(\int_s^1 f\bigl(\tau, u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{k=1}^m H(t,t_k) I_k\bigl(u(t_k)\bigr), \end{aligned}$$

where

$$H(t,t_k) = \begin{cases} 1, & t_k < t, \\ 0, & t_k \ge t. \end{cases}$$

Clearly, the operator A is a completely continuous operator, and the existence of positive solutions for (1.3) is equivalent to that of positive fixed points of A. Moreover, it is easy to see $A(P) \subset P$ by (2.2).

In what follows, we consider the following eigenvalue problem:

$$\begin{cases} -u'' = \lambda u, & t \in [0,1], \\ u(0) = u'(1) = 0, \end{cases} \tag{2.4}$$

where λ is a parameter. We easily know that (2.4) has a nontrivial solution if $\lambda > 0$. Furthermore, we have

$$u(t) = c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t,$$

where c_1, c_2 are constants and $c_1^2 + c_2^2 \neq 0$. $u(0) = u'(1) = 0$ implies that $\cos\sqrt{\lambda} = 0$, and thus $\lambda = (-\frac{\pi}{2} + k\pi)^2$, $k = 1, 2 \ldots$. $\lambda_1^{-1} := (-\frac{\pi}{2} + 1 \times \pi)^2 = \frac{\pi^2}{4}$ and $\sin(\frac{\pi t}{2})$ are called the first eigenvalue and the corresponding eigenfunction associated with λ_1, respectively. Consequently, it is easy to have the following result:

Lemma 2.2 *If $\psi(t) := \sin(\frac{\pi t}{2})$, then*

$$\int_0^1 G(t,s)\psi(t)\,\mathrm{d}t = \frac{4}{\pi^2}\psi(s), \tag{2.5}$$

where $G(t,s) = \min\{t,s\}$, $t, s \in [0,1]$.

Lemma 2.3 (see [14]) *Let E be a real Banach space and W a cone of E. Suppose $A : (\overline{B}_R \setminus B_r) \cap W \to W$ is a completely continuous operator with $0 < r < R$. If either*

(1) $Au \nleq u$ for each $\partial B_r \cap W$ and $Au \ngeq u$ for each $\partial B_R \cap W$ or

(2) $Au \ngeq u$ for each $\partial B_r \cap W$ and $Au \nleq u$ for each $\partial B_R \cap W$,

then A has at least one fixed point in $(B_R \setminus \overline{B}_r) \cap W$.

Lemma 2.4 (Jensen's inequalities, see [10]) *Let $\theta > 0$, $n \geq 1$, $a_i \geq 0$ ($i = 1, 2, \ldots, n$), and $\varphi \in C([a,b],[0,+\infty))$. Then*

$$\begin{aligned}
&\left(\int_a^b \varphi(t)\,\mathrm{d}t\right)^\theta \leq (b-a)^{\theta-1}\int_a^b \big(\varphi(t)\big)^\theta\,\mathrm{d}t \quad \textit{and} \\
&\left(\sum_{i=1}^n a_i\right)^\theta \leq n^{\theta-1}\sum_{i=1}^n a_i^\theta, \quad \forall\theta \geq 1, \\
&\left(\int_a^b \varphi(t)\,\mathrm{d}t\right)^\theta \geq (b-a)^{\theta-1}\int_a^b \big(\varphi(t)\big)^\theta\,\mathrm{d}t \quad \textit{and} \\
&\left(\sum_{i=1}^n a_i\right)^\theta \geq n^{\theta-1}\sum_{i=1}^n a_i^\theta, \quad \forall 0 < \theta \leq 1.
\end{aligned}$$

3 Main results

Let $p^* := \max\{1, p-1\}$, $p_* := \min\{1, p-1\}$, $\kappa_1 := 2^{p_*-1}$, $\kappa_2 := (2m)^{p_*-1}$, $\kappa_3 := 2^{p^*-1}$, $\kappa_4 := (2m)^{p^*-1}$, $\kappa_5 := 2^{\frac{pp^*}{p-1}-2}$, $\kappa_6 := (4m)^{p^*-1}$. We now list our hypotheses.

(H1) There exist $r > 0$ and $a_1 \geq 0$, $a_2 \geq 0$ satisfying

$$a_1^{\frac{p_*}{p-1}}\kappa_1 + \frac{\pi^2}{4}a_2^{p_*}t_1^{p_*}\kappa_2\sum_{k=1}^m \cos\left(\frac{\pi t_k}{2}\right) > \frac{\pi^2}{4},$$

such that

$$f(t,y) \geq a_1 y^{p-1}, \qquad I_k(y) \geq a_2 y, \quad \forall t \in [0,1], 0 < y < r. \tag{3.1}$$

(H2) There exist $c > 0$ and $b_1 \geq 0$, $b_2 \geq 0$ satisfying

$$b_1^2 + b_2^2 \neq 0, \qquad b_1^{\frac{p^*}{p-1}}\kappa_5 + \frac{\pi b_2^{p^*}\kappa_6\sum_{k=1}^m \cos(\frac{\pi t_k}{2})}{2\int_0^1 t^{p^*}\sin(\frac{\pi t}{2})\mathrm{d}t} < \frac{\pi^2}{4},$$

such that

$$f(t,y) \le b_1 y^{p-1} + c, \qquad I_k(y) \le b_2 y + c, \quad \forall t \in [0,1], y \ge 0. \tag{3.2}$$

(H3) There exist $c > 0$ and $a_3 \ge 0$, $a_4 \ge 0$ satisfying

$$a_3^{\frac{p^*}{p-1}} \kappa_1 + \frac{\pi^2}{4} a_4^{p^*} t_1^{p^*} \kappa_2 \sum_{k=1}^{m} \cos\left(\frac{\pi t_k}{2}\right) > \frac{\pi^2}{4},$$

such that

$$f(t,y) \ge a_3 y^{p-1} - c, \qquad I_k(y) \ge a_4 y - c, \quad \forall t \in [0,1], y \ge 0. \tag{3.3}$$

(H4) There exist $r > 0$ and $b_3 \ge 0$, $b_4 \ge 0$ satisfying

$$b_3^2 + b_4^2 \ne 0, \qquad b_3^{\frac{p^*}{p-1}} \kappa_5 + \frac{\pi b_4^{p^*} \kappa_6 \sum_{k=1}^{m} \cos(\frac{\pi t_k}{2})}{2 \int_0^1 t^{p^*} \sin(\frac{\pi t}{2}) \mathrm{d}t} < \frac{\pi^2}{4},$$

such that

$$f(t,y) \le b_3 y^{p-1}, \qquad I_k(y) \le b_4 y, \quad \forall t \in [0,1], 0 < y < r. \tag{3.4}$$

(H5) There exists $\rho > 0$ such that $0 \le y \le \rho$, and $t \in [0,1]$ implies

$$f(t,y) \le \eta^{p-1} \rho^{p-1}, \qquad I_k(y) \le \eta_k \rho,$$

where $\eta, \eta_k \ge 0$ and $0 < \eta + \sum_{k=1}^{m} \eta_k \le 1$.
(H6) There exists $\rho > 0$ such that $t_1 \rho \le y \le \rho$, and $t \in [0,1]$ implies

$$f(t,y) \ge \eta^{p-1} \rho^{p-1}, \qquad I_k(y) \ge \eta_k \rho,$$

where $\eta, \eta_k \ge 0$, $(\eta p^{-1}(p-1)(1-t_1)^{p/(p-1)} + \sum_{k=1}^{m} \eta_k) > 1$.

Theorem 3.1 *Suppose that (H1)-(H2) are satisfied. Then (1.3) has at least one positive solution.*

Proof If (H1) is satisfied, then we obtain $u \ngeq Au$ for all $u \in P \cap \partial B_r$. Indeed, if the claim is false, there is a $u \in P \cap \partial B_r$ such that $u \ge Au$, *i.e.*,

$$u(t) \ge \int_0^t \varphi_q\left(\int_s^1 f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\right) \mathrm{d}s + \sum_{k=1}^{m} H(t,t_k) I_k\big(u(t_k)\big).$$

Now apply Lemma 2.4 to obtain

$$\begin{aligned} u^{p^*}(t) &\ge \left[\int_0^t \varphi_q\left(\int_s^1 f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\right) \mathrm{d}s + \sum_{k=1}^{m} H(t,t_k) I_k\big(u(t_k)\big)\right]^{p^*} \\ &\ge \kappa_1 \left[\int_0^t \varphi_q\left(\int_s^1 f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\right) \mathrm{d}s\right]^{p^*} + \kappa_1 \left[\sum_{k=1}^{m} H(t,t_k) I_k\big(u(t_k)\big)\right]^{p^*} \end{aligned}$$

$$\begin{aligned}
&\geq \kappa_1 \int_0^t \int_s^1 f^{\frac{p_*}{p-1}}\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s + \kappa_2 \sum_{k=1}^m H^{p_*}(t,t_k) I_k^{p_*}\big(u(t_k)\big) \\
&\geq \kappa_1 \int_0^1 G(t,s) f^{\frac{p_*}{p-1}}\big(s, u(s)\big)\,\mathrm{d}s + \kappa_2 \sum_{k=1}^m H(t,t_k) I_k^{p_*}\big(u(t_k)\big).
\end{aligned} \tag{3.5}$$

Multiply both sides of (3.5) by $\sin(\frac{\pi t}{2})$ and then integrate over $[0,1]$ and use (2.5) to obtain

$$\begin{aligned}
\int_0^1 u^{p_*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t &\geq \kappa_1 \int_0^1 \sin\left(\frac{\pi t}{2}\right) \int_0^1 G(t,s) f^{\frac{p_*}{p-1}}\big(s,u(s)\big)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \kappa_2 \sum_{k=1}^m \int_0^1 \sin\left(\frac{\pi t}{2}\right) H(t,t_k) I_k^{p_*}\big(u(t_k)\big)\,\mathrm{d}t \\
&\geq \frac{4\kappa_1}{\pi^2} \int_0^1 f^{\frac{p_*}{p-1}}\big(t,u(t)\big)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\
&\quad + \frac{2\kappa_2}{\pi} \sum_{k=1}^m I_k^{p_*}\big(u(t_k)\big)\cos\left(\frac{\pi t_k}{2}\right).
\end{aligned} \tag{3.6}$$

The above and (H1) imply that

$$\begin{aligned}
\int_0^1 u^{p_*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t &\geq \frac{4a_1^{\frac{p_*}{p-1}}\kappa_1}{\pi^2} \int_0^1 u^{p_*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\
&\quad + \frac{2a_2^{p_*}\kappa_2}{\pi} \sum_{k=1}^m u^{p_*}(t_k)\cos\left(\frac{\pi t_k}{2}\right).
\end{aligned} \tag{3.7}$$

By (3.7), we have $\frac{4a_1^{\frac{p_*}{p-1}}\kappa_1}{\pi^2} \leq 1$. If $\frac{4a_1^{\frac{p_*}{p-1}}\kappa_1}{\pi^2} = 1$, then $u(t_k) \equiv 0$, $k = 1,2,\ldots,m$, and in view of the concavity and the nondecreasing nature of u, we find $u(t) \equiv 0$, $0 \leq t \leq 1$, contradicting $u \in P \cap \partial B_r$. So, $\frac{4a_1^{\frac{p_*}{p-1}}\kappa_1}{\pi^2} < 1$.

Since $u \in P \cap \partial B_r$, $u^{p_*}(t) \leq \|u\|^{p_*} = r^{p_*}$. Therefore,

$$\int_0^1 u^{p_*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \leq r^{p_*} \int_0^1 \sin\left(\frac{\pi t}{2}\right)\mathrm{d}t = \frac{2r^{p_*}}{\pi}.$$

Combining (3.7) and (2.2), we obtain

$$\frac{2(\pi^2 - 4a_1^{\frac{p_*}{p-1}}\kappa_1)r^{p_*}}{\pi^3} \geq \frac{2a_2^{p_*}\kappa_2 r^{p_*} t_1^{p_*}}{\pi} \sum_{k=1}^m \cos\left(\frac{\pi t_k}{2}\right).$$

Therefore, $4a_1^{\frac{p_*}{p-1}}\kappa_1 + \pi^2 a_2^{p_*} t_1^{p_*}\kappa_2 \sum_{k=1}^m \cos(\frac{\pi t_k}{2}) \leq \pi^2$, which contradicts (H1). Thus we have

$$u \ngeq Au, \quad \text{for any } u \in P \cap \partial B_r. \tag{3.8}$$

On the other hand, by (H2), we shall prove that there exists a sufficiently large number $R > 0$ such that $u \nleq Au$, $\forall u \in P \cap \partial B_R$. Suppose there exists $u \in P \cap \partial B_R$ such that $u \leq Au$.

This, together with Lemma 2.4, yields

$$
\begin{aligned}
u^{p^*}(t) &\le \left[\int_0^t \varphi_q\left(\int_s^1 f\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{k=1}^m H(t,t_k)I_k\big(u(t_k)\big)\right]^{p^*} \\
&\le \kappa_3\left[\int_0^t \varphi_q\left(\int_s^1 f\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right]^{p^*} + \kappa_3\left[\sum_{k=1}^m H(t,t_k)I_k\big(u(t_k)\big)\right]^{p^*} \\
&\le \kappa_3 \int_0^t \int_s^1 f^{\frac{p^*}{p-1}}\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s + \kappa_4 \sum_{k=1}^m H^{p^*}(t,t_k)I_k^{p^*}\big(u(t_k)\big) \\
&\le \kappa_3 \int_0^1 G(t,s) f^{\frac{p^*}{p-1}}\big(s,u(s)\big)\,\mathrm{d}s + \kappa_4 \sum_{k=1}^m H(t,t_k)I_k^{p^*}\big(u(t_k)\big).
\end{aligned}
\tag{3.9}
$$

Multiply both sides of the above by $\sin(\frac{\pi t}{2})$ and integrate over $[0,1]$ and use (2.5) to obtain

$$
\begin{aligned}
\int_0^1 u^{p^*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t &\le \kappa_3 \int_0^1 \sin\left(\frac{\pi t}{2}\right)\int_0^1 G(t,s) f^{\frac{p^*}{p-1}}\big(s,u(s)\big)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \kappa_4 \sum_{k=1}^m \int_0^1 \sin\left(\frac{\pi t}{2}\right) H(t,t_k) I_k^{p^*}\big(u(t_k)\big)\,\mathrm{d}t \\
&\le \frac{4\kappa_3}{\pi^2}\int_0^1 f^{\frac{p^*}{p-1}}\big(t,u(t)\big)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\
&\quad + \frac{2\kappa_4}{\pi}\sum_{k=1}^m I_k^{p^*}\big(u(t_k)\big)\cos\left(\frac{\pi t_k}{2}\right).
\end{aligned}
\tag{3.10}
$$

Combining this and (H2), we get

$$
\begin{aligned}
\int_0^1 u^{p^*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t &\le \frac{4\kappa_3}{\pi^2}\int_0^1 \big(b_1 u^{p-1}(t) + c\big)^{\frac{p^*}{p-1}}\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\
&\quad + \frac{2\kappa_4}{\pi}\sum_{k=1}^m \big(b_2 u(t_k) + c\big)^{p^*}\cos\left(\frac{\pi t_k}{2}\right) \\
&\le \frac{4b_1^{\frac{p^*}{p-1}}\kappa_5}{\pi^2}\int_0^1 u^{p^*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\
&\quad + \frac{2b_2^{p^*}\kappa_6}{\pi}\sum_{k=1}^m u^{p^*}(t_k)\cos\left(\frac{\pi t_k}{2}\right) + c_0,
\end{aligned}
\tag{3.11}
$$

where $c_0 = \frac{8c^{\frac{p^*}{p-1}}\kappa_5}{\pi^3} + \frac{2c^{p^*}\kappa_6}{\pi}\sum_{k=1}^m \cos(\frac{\pi t_k}{2})$. Consequently, by (2.2) we have

$$
\begin{aligned}
\|u\|^{p^*}\big(\pi^2 - 4b_1^{\frac{p^*}{p-1}}\kappa_5\big)\int_0^1 t^{p^*}\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t &\le \big(\pi^2 - 4b_1^{\frac{p^*}{p-1}}\kappa_5\big)\int_0^1 u^{p^*}\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\
&\le 2\pi b_2^{p^*}\kappa_6\|u\|^{p^*}\sum_{k=1}^m \cos\left(\frac{\pi t_k}{2}\right) + \pi^2 c_0.
\end{aligned}
$$

Therefore,

$$\|u\|^{p^*} \le \frac{\pi^2 c_0}{(\pi^2 - 4b_1^{\frac{p^*}{p-1}}\kappa_5)\int_0^1 t^{p^*}\sin(\frac{\pi t}{2})\,\mathrm{d}t - 2\pi b_2^{p^*}\kappa_6\sum_{k=1}^m \cos(\frac{\pi t_k}{2})} := K_1 > 0.$$

Choosing $R > \sqrt[p^*]{K_1}$ and $R > r$, we have

$$u \nleq Au, \quad \text{for all } u \in P \cap \partial B_R. \tag{3.12}$$

Therefore, (3.8) and (3.12), together with Lemma 2.3, guarantee that (1.3) has at least one positive solution in $(B_R \setminus \overline{B}_r) \cap P$. □

Theorem 3.2 *Suppose that (H3)-(H4) are satisfied. Then (1.3) has at least one positive solution.*

Proof If (H3) is satisfied, we will prove that there exists a sufficiently large number $R > 0$ such that $u \ngeq Au$, $\forall u \in P \cap \partial B_R$. Suppose there exists $u \in P \cap \partial B_R$ such that $u \geq Au$, and then

$$u(t) \ge \int_0^t \varphi_q\left(\int_s^1 f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{k=1}^m H(t,t_k)I_k\big(u(t_k)\big).$$

In view of $0 < \frac{p^*}{p-1} \le 1$, from (3.3), we know $(a_3u^{p-1})^{\frac{p^*}{p-1}} \le (f+c)^{\frac{p^*}{p-1}} \le f^{\frac{p^*}{p-1}} + c^{\frac{p^*}{p-1}}$. Accordingly, $f^{\frac{p^*}{p-1}} \ge (a_3u^{p-1})^{\frac{p^*}{p-1}} - c^{\frac{p^*}{p-1}}$. Similarly, $I_k^{p^*} \ge (a_4u)^{p^*} - c^{p^*}$. These and (3.6) imply that

$$\begin{aligned}\int_0^1 u^{p^*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t &\ge \frac{4\kappa_1}{\pi^2}\int_0^1 \left[\big(a_3u^{p-1}(t)\big)^{\frac{p^*}{p-1}} - c^{\frac{p^*}{p-1}}\right]\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\ &\quad + \frac{2\kappa_2}{\pi}\sum_{k=1}^m \left[\big(a_4u(t_k)\big)^{p^*} - c^{p^*}\right]\cos\left(\frac{\pi t_k}{2}\right) \\ &\ge \frac{4a_3^{\frac{p^*}{p-1}}\kappa_1}{\pi^2}\int_0^1 u^{p^*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\ &\quad + \frac{2a_4^{p^*}\kappa_2}{\pi}\sum_{k=1}^m u^{p^*}(t_k)\cos\left(\frac{\pi t_k}{2}\right) - c_1,\end{aligned} \tag{3.13}$$

where $c_1 = \frac{8c^{\frac{p^*}{p-1}}\kappa_1}{\pi^3} + \frac{2\kappa_2 c^{p^*}}{\pi}\sum_{k=1}^m \cos(\frac{\pi t_k}{2})$.

Now, we consider two cases.

Case 1. If $1 \ge \frac{4a_3^{\frac{p^*}{p-1}}\kappa_1}{\pi^2}$, by (2.2) we obtain

$$\begin{aligned}\frac{2(\pi^2 - 4a_3^{\frac{p^*}{p-1}}\kappa_1)\|u\|^{p^*}}{\pi} &\ge \left(\pi^2 - 4a_3^{\frac{p^*}{p-1}}\kappa_1\right)\int_0^1 u^{p^*}(t)\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t \\ &\ge 2\pi a_4^{p^*} t_1^{p^*}\kappa_2\|u\|^{p^*}\sum_{k=1}^m \cos\left(\frac{\pi t_k}{2}\right) - \pi^2 c_1,\end{aligned}$$

i.e.,

$$\|u\|^{p_*} \le \frac{\frac{\pi^3 c_1}{2}}{4a_3^{\frac{p_*}{p-1}}\kappa_1 + \pi^2 a_4^{p_*} t_1^{p_*}\kappa_2 \sum_{k=1}^m \cos(\frac{\pi t_k}{2}) - \pi^2} := K_2 > 0. \tag{3.14}$$

Case 2. If $1 < \frac{4a_3^{\frac{p_*}{p-1}}\kappa_1}{\pi^2}$, by (2.2) we have

$$c_1\pi^2 \ge \left(4a_3^{\frac{p_*}{p-1}}\kappa_1 - \pi^2\right)\|u\|^{p_*} \int_0^1 t^{p_*} \sin\left(\frac{\pi t}{2}\right) \mathrm{d}t + 2\pi t_1^{p_*} \|u\|^{p^*} a_4^{p_*} \kappa_2 \sum_{k=1}^m \cos\left(\frac{\pi t_k}{2}\right).$$

In view of $\frac{4}{\pi^2} \le \int_0^1 t^{p_*} \sin(\frac{\pi t}{2})\,\mathrm{d}t \le \frac{2}{\pi}$, we obtain

$$\|u\|^{p_*} \le \frac{\frac{\pi^4 c_1}{4}}{4a_3^{\frac{p_*}{p-1}}\kappa_1 + \frac{\pi^3}{2} a_4^{p_*} t_1^{p_*}\kappa_2 \sum_{k=1}^m \cos(\frac{\pi t_k}{2}) - \pi^2} := K_3 > 0. \tag{3.15}$$

Choosing $R > \max\{\sqrt[p_*]{K_2}, \sqrt[p_*]{K_3}, r\}$ (r is determined by (H4)), we get

$$u \ngeq Au, \quad \forall u \in P \cap \partial B_R. \tag{3.16}$$

On the other hand, if (H4) is satisfied, then $u \nleq Au$, $\forall u \in P \cap \partial B_r$. If not, there exists $u \in P \cap \partial B_r$ such that $u \le Au$. It follows from (3.10) and (H4) that

$$\begin{aligned}\int_0^1 u^{p^*}(t) \sin\left(\frac{\pi t}{2}\right) \mathrm{d}t &\le \frac{4\kappa_3}{\pi^2} \int_0^1 \left(b_3 u^{p-1}(t)\right)^{\frac{p^*}{p-1}} \sin\left(\frac{\pi t}{2}\right) \mathrm{d}t \\ &\quad + \frac{2\kappa_4}{\pi} \sum_{k=1}^m \left(b_4 u(t_k)\right)^{p^*} \cos\left(\frac{\pi t_k}{2}\right) \\ &\le \frac{4b_3^{\frac{p^*}{p-1}}\kappa_3}{\pi^2} \int_0^1 u^{p^*}(t) \sin\left(\frac{\pi t}{2}\right) \mathrm{d}t \\ &\quad + \frac{2b_4^{p^*}\kappa_4}{\pi} \sum_{k=1}^m u^{p^*}(t_k) \cos\left(\frac{\pi t_k}{2}\right).\end{aligned} \tag{3.17}$$

Therefore,

$$\begin{aligned}\|u\|^{p^*}\left(\pi^2 - 4b_3^{\frac{p^*}{p-1}}\kappa_3\right) \int_0^1 t^{p^*} \sin\left(\frac{\pi t}{2}\right) \mathrm{d}t &\le \left(\pi^2 - 4b_3^{\frac{p^*}{p-1}}\kappa_3\right) \int_0^1 u^{p^*} \sin\left(\frac{\pi t}{2}\right) \mathrm{d}t \\ &\le 2\pi b_4^{p^*}\kappa_4 \|u\|^{p^*} \sum_{k=1}^m \cos\left(\frac{\pi t_k}{2}\right),\end{aligned}$$

i.e.,

$$4b_3^{\frac{p^*}{p-1}}\kappa_3 + \frac{2\pi b_4^{p^*}\kappa_4 \sum_{k=1}^m \cos(\frac{\pi t_k}{2})}{\int_0^1 t^{p^*} \sin(\frac{\pi t}{2})\,\mathrm{d}t} \ge \pi^2,$$

which contradicts (H4). Thus

$$u \nleq Au, \quad \forall u \in \partial B_r \cap P. \tag{3.18}$$

By Lemma 2.3, (3.16) and (3.18) imply that (1.3) has at least one positive solution in $(B_R \setminus \overline{B}_r) \cap P$. □

Theorem 3.3 *Suppose that (H1), (H3) and (H5) are satisfied. Then (1.3) has at least two positive solutions.*

Proof If $u \in P \cap \partial B_\rho$, it follows from (H5) that

$$\begin{aligned}
\|Au\| &\le \int_0^1 \varphi_q\left(\int_s^1 f\bigl(\tau, u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{k=1}^m I_k\bigl(u(t_k)\bigr) \\
&\le \int_0^1 \varphi_q\left(\int_s^1 \eta^{p-1}\rho^{p-1}\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{k=1}^m \rho\eta_k \\
&\le \rho\left(\eta\frac{p-1}{p} + \sum_{k=1}^m \eta_k\right) < \rho\left(\eta + \sum_{k=1}^m \eta_k\right) \le \|u\|,
\end{aligned}$$

from which we obtain

$$u \nleq Au, \quad \forall u \in \partial B\rho \cap P. \tag{3.19}$$

On the other hand, by (H1) and (H3), we may take $0 < r < \rho$ and $R > \rho$ such that $u \ngeq Au$, $\forall u \in P \cap \partial B_r$ and $u \ngeq Au$, $\forall u \in P \cap \partial B_R$ (see the proofs of Theorems 3.1 and 3.2). Now Lemma 2.3 guarantees that the operator A has at least two fixed points, one in $(B_R \setminus \overline{B}_\rho) \cap P$ and the other in $(B_\rho \setminus \overline{B}_r) \cap P$. The proof is completed. □

Theorem 3.4 *Suppose that (H2), (H4) and (H6) are satisfied. Then (1.3) has at least two positive solutions.*

Proof For any $u \in P \cap \partial B_\rho$, $u(t) \ge t_1\|u\| = t_1\rho$ for all $t \in [t_1, 1]$. It follows from (H6) that

$$\begin{aligned}
\|Au\| &= \int_0^1 \varphi_q\left(\int_s^1 f\bigl(\tau, u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{k=1}^m I_k\bigl(u(t_k)\bigr) \\
&\ge \int_{t_1}^1 \varphi_q\left(\int_s^1 f\bigl(\tau, u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{k=1}^m I_k\bigl(u(t_k)\bigr) \\
&\ge \int_{t_1}^1 \varphi_q\left(\int_s^1 \eta^{p-1}\rho^{p-1}\,\mathrm{d}\tau\right)\mathrm{d}s + \sum_{k=1}^m \eta_k\rho \\
&\ge \rho\left(\eta p^{-1}(p-1)(1-t_1)^{p/(p-1)} + \sum_{k=1}^m \eta_k\right) > \rho = \|u\|.
\end{aligned} \tag{3.20}$$

Thus,

$$Au \nleq u, \quad \forall u \in P \cap \partial B_\rho. \tag{3.21}$$

On the other hand, by (H2) and (H4), we may take $0 < r < \rho$ and $R > \rho$ such that $u \nleq Au$, $\forall u \in P \cap \partial B_r$ and $u \nleq Au$, $\forall u \in P \cap \partial B_R$ (see the proofs of Theorems 3.1 and 3.2). Thus Lemma 2.3 indicates that the operator A has at least two fixed points, one in $(B_R \setminus \overline{B}_\rho) \cap P$ and the other in $(B_\rho \setminus \overline{B}_r) \cap P$. The proof is completed. □

4 Examples

Example 4.1 Consider the impulsive boundary value problem

$$\begin{cases} (\varphi_p(u'))' = -f(t,u), \quad t \neq \frac{1}{2}, 0 < t < 1, \\ \Delta u|_{t=\frac{1}{2}} = I_1(u(\frac{1}{2})), \\ \Delta u'|_{t=\frac{1}{2}} = 0, \\ u(0) = u'(1) = 0, \end{cases} \tag{4.1}$$

where $p = \frac{3}{2}$.

Case 1. Let $f(t,u) = u^\alpha$, $0 < \alpha < \frac{1}{2}$, $I_1(u) = u^{\alpha_1}$, $0 < \alpha_1 < 1$. Then

$$\lim_{u\to 0^+} \frac{f(t,u)}{u^{p-1}} = \lim_{u\to 0^+} \frac{u^\alpha}{u^{p-1}} = \infty, \qquad \lim_{u\to 0^+} \frac{I_1(u)}{u} = \lim_{u\to 0^+} u^{\alpha_1 - 1} = \infty. \tag{4.2}$$

From (4.2) we see that (H1) is satisfied. In fact, since $p = 3/2$ and $t_1 = 1/2$, $p_* = 1/2$, $\kappa_1 = \kappa_2 = 1/\sqrt{2}$. Taking $a_1 = 3$, $a_2 = 1$, we get

$$a_1^{\frac{p_*}{p-1}} \kappa_1 + \frac{\pi^2}{4} a_2^{p_*} t_1^{p_*} \kappa_2 \sum_{k=1}^{m} \cos\left(\frac{\pi t_k}{2}\right) > \frac{\pi^2}{4}.$$

Moreover,

$$\lim_{u\to \infty} \frac{f(t,u)}{u^{p-1}} = \lim_{u\to \infty} \frac{u^\alpha}{u^{p-1}} = 0, \qquad \lim_{u\to \infty} \frac{I_1(u)}{u} = \lim_{u\to \infty} u^{\alpha_1 - 1} = 0. \tag{4.3}$$

From (4.3) we see that (H2) is satisfied, for example taking $b_1 = 1$, $b_2 = \frac{1}{10}$. All the assumptions in Theorem 3.1 are satisfied, and the problem (4.1) has at least one positive solution by Theorem 3.1.

Case 2. Take $f(t,u) = u^\beta$, $\beta > \frac{1}{2}$, $I_1(u) = u^{\beta_1}$, $\beta_1 > 1$. One can easily verify conditions (H3) and (H4). Thus the problem (4.1) has at least one positive solution by Theorem 3.2.

Case 3. Let $f(t,u) = \frac{u^\alpha + u^\beta}{5}$, $0 < \alpha < \frac{1}{2} < \beta$, $I_1(u) = \frac{u}{5}$. Thus, we get

$$\lim_{u\to 0^+} \frac{f(t,u)}{u^{p-1}} = \lim_{u\to 0^+} \frac{u^\alpha + u^\beta}{5u^{p-1}} = \infty. \tag{4.4}$$

From (4.4) we see that (H1) is satisfied. Note

$$\lim_{u\to \infty} \frac{f(t,u)}{u^{p-1}} = \lim_{u\to \infty} \frac{u^\alpha + u^\beta}{5u^{p-1}} = \infty. \tag{4.5}$$

From (4.5) we see that (H3) is satisfied. Take $\rho = 1$, $\eta = \frac{4}{5}$, $\eta_1 = \frac{1}{5}$ in (H5), and note for $0 \le u \le \rho$ and $t \in [0,1]$ that $f(t,u) \le \frac{\rho^\alpha + \rho^\beta}{5} = \frac{2}{5} < \frac{2\sqrt{5}}{5} = \eta^{\frac{1}{2}}$, $I_1(u) = \frac{u}{5} \le \frac{1}{5} = \eta_1$. As a result, (H5) holds. From Theorem 3.3, the problem (4.1) has at least two positive solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first author conceived of the study and carried out the proof. Authors read and approved the final manuscript.

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