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Intuitionistic random almost additive-quadratic mappings

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Abstract

In this paper, we investigate the Hyers-Ulam stability of the additive-quadratic functional equation $\sum_{i=1}^{n} f(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j) = \sum_{i=1}^{n} f(x_i) - nf(\frac{1}{n}\sum_{i=1}^{n} x_i)$ $(n \geq 2)$ in intuitionistic random normed spaces.

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1 Introduction

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940 and affirmatively solved by Hyers [2]. The result of Hyers was generalized by Aoki [3] for approximate additive mappings and by Rassias [4] for approximate linear mappings by allowing the difference Cauchy equation $\|f(x+y) - f(x) - f(y)\|$ to be controlled by $\varepsilon(\|x\|^p + \|y\|^p)$. In 1994, a generalization of the Th.M. Rassias' theorem was obtained by Găvruta [5], who replaced $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. For more details about the results concerning such problems, the reader is referred to [6–16].

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive mapping [17, 18]. It is natural that this equation is called a *quadratic functional equation*. In particular, every solution of the quadratic equation (1.1) is said to be a *quadratic mapping*. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive mapping B_1 such that $f(x) = B_1(x,x)$ for all x. The bi-additive mapping B_1 is given by $B_1(x,y) = \frac{1}{4}(f(x+y) - f(x-y))$. The Hyers-Ulam stability problem for the quadratic functional equation was solved by Skof [19]. In [20], Czerwik proved the Hyers-Ulam stability of the function equation (1.1).

Eshaghi Gordji and Khodaei [21] have established the general solution and investigated the Hyers-Ulam stability for a mixed type of cubic, quadratic and additive functional equation

$$f(x+ky) + f(x-ky) = k^2 f(x+y) + k^2 f(x-y) + 2(1-k^2)f(x) \tag{1.2}$$

in quasi-Banach spaces, where k is a nonzero integer with $k \neq \pm 1$. Obviously, the function $f(x) = ax + bx^2 + cx^3$ is a solution of the functional equation (1.2). Interesting new results concerning mixed functional equations have recently been obtained by Najati *et al.* [22–24], Jun and Kim [25, 26] as well as for the fuzzy stability of a mixed-type functional equation by Park *et al.* [27–29].

The stability of the mixed functional equation

$$\sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \sum_{i=1}^{n} f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \tag{1.3}$$

was investigated by Najati and Rassias [23].

The theory of random normed spaces (RN-spaces) is important as a generalization of the deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us with the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The Hyers-Ulam stability of different functional equations in random normed spaces and RN-spaces has been recently studied in Alsina [30], Eshaghi Gordji *et al.* [31, 32], Miheţ and Radu [33–35], Miheţ, Saadati and Vaezpour [36, 37], and Saadati *et al.* [38]. Recently, Zhang *et al.* [39] investigated the intuitionistic random stability problems for the cubic functional equation.

In this paper, we prove the Hyers-Ulam stability of the additive and quadratic functional equation (1.3) in intuitionistic random spaces.

2 Preliminaries

We start our work with the following notion of intuitionistic random normed spaces. In the sequel, we adopt the usual terminology, notations and conventions of the theory of intuitionistic Menger probabilistic normed spaces as in [33] and [40–44].

A *measure distribution function* is a function $\mu : \mathbb{R} \to [0,1]$, which is left continuous on $\mathbb{R}$, non-decreasing, $\inf_{t\in\mathbb{R}} \mu(t) = 0$ and $\sup_{t\in\mathbb{R}} \mu(t) = 1$.

We denote by D the family of all measure distribution functions, and by H a special element of D defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, then $\mu : X \to D$ is called a probabilistic measure on X and $\mu(x)$ is denoted by μ_x.

A *non-measure distribution function* is a function $\nu : \mathbb{R} \to [0,1]$, which is right continuous on $\mathbb{R}$, non-increasing, $\inf_{t\in\mathbb{R}} \nu(t) = 0$ and $\sup_{t\in\mathbb{R}} \nu(t) = 1$.

We denote by B the family of all non-measure distribution functions, and by G a special element of B defined by

$$G(t) = \begin{cases} 1, & \text{if } t \leq 0, \\ 0, & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, then $\nu : X \to B$ is called a probabilistic non-measure on X and $\nu(x)$ is denoted by ν_x.

Lemma 2.1 [45, 46] *Consider the set* L^* *and the operation* $\leq_{L^*}$ *defined by*

$$L^* = \{(x_1, x_2) \in [0,1] \times [0,1] : x_1 + x_2 \leq 1\},$$

$$\forall (x_1, x_2), (y_1, y_2) \in L^*, \quad (x_1, x_2) \leq_{L^*} (y_1, y_2) \quad \Longleftrightarrow \quad x_1 \leq y_1, x_2 \geq y_2.$$

Then $(L^*, \leq_{L^*})$ *is a complete lattice.*

We denote the units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, for all $x \in [0,1]$, a triangular norm $* = T$ on [0,1] is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \longrightarrow [0,1]$ satisfying $T(1,x) = 1 * x = x$, and a triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0,x) = 0 \diamond x = x$.

By use of the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 2.2 [46] A *triangular norm* (*t-norm*) *on* L^* is a mapping $\Upsilon : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

(i) $\forall x \in L^*$, $\Upsilon(x, 1_{L^*}) = x$ (boundary condition);
(ii) $\forall (x,y) \in (L^*)^2$, $\Upsilon(x,y) = \Upsilon(y,x)$ (commutativity);
(iii) $\forall (x,y,z) \in (L^*)^3$, $\Upsilon(x, \Upsilon(y,z)) = \Upsilon(\Upsilon(x,y), z)$ (associativity);
(iv) $\forall (x,x',y,y') \in (L^*)^4$, $x \leq_{L^*} x'$, $y \leq_{L^*} y' \Longrightarrow \Upsilon(x,y) \leq_{L^*} \Upsilon(x',y')$ (monotonicity).

If $(L^*, \leq_{L^*}, \Upsilon)$ is an Abelian topological monoid with unit 1_{L^*}, then Υ is said to be a continuous t-norm.

Definition 2.3 [46] A *continuous t-norm* Υ *on* L^* is said to be continuous t-representable if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on [0,1] such that

$$\forall x = (x_1, x_2), \quad y = (y_1, y_2) \in L^*, \qquad \Upsilon(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

Typical examples of continuous t-representable are $\Upsilon(a,b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ and $M(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$.

Now, we define a sequence Υ^n recursively by $\Upsilon^1 = \Upsilon$ as

$$\Upsilon^n\left(x^{(1)}, x^{(2)}, \ldots, x^{(n+1)}\right) = \Upsilon\left(\Upsilon^{n-1}\left(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right), x^{(n+1)}\right)$$

for all $x^{(1)}, \ldots, x^{(n+1)} \in L^*$ and $n \geq 2$.

Recall that if Υ is a t-norm and $\{x^{(n)}\}$ is a given sequence of numbers in L^*, $\Upsilon_{i=1}^n x^{(i)}$ is defined recurrently by

$$\Upsilon_{i=1}^n x^{(i)} = \begin{cases} x^{(1)}, & \text{if } n = 1, \\ \Upsilon(\Upsilon_{i=1}^{n-1} x^{(i)}, x^{(n)}), & \text{if } n \geq 2, \end{cases}$$

for all $x^{(i)} \in L^*$. $\Upsilon_{i=n}^{\infty} x^{(i)}$ is defined as $\Upsilon_{i=1}^{\infty} x^{(n+i)}$.

A *negator on* L^* is any decreasing mapping $\aleph : L^* \longrightarrow L^*$ satisfying $\aleph(0_{L^*}) = (1_{L^*})$ and $\aleph(1_{L^*}) = (0_{L^*})$. If $\aleph\aleph(x) = x$ for all $x \in L^*$, then $\aleph$ is called an involutive negator. A negator on [0,1] is a decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on [0,1] defined by $N_s(x) = 1 - x$ for all $x \in [0,1]$.

Definition 2.4 [39]. Let μ and ν be measure and non-measure distribution functions from $X \times (0,+\infty)$ to $[0,1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and all $t > 0$. The triple $(X, \Lambda_{\mu,\nu}, \Upsilon)$ is said to be an *intuitionistic random normed space (briefly IRN-space)* if X is a vector space, Υ is a continuous t-representable, and $\Lambda_{\mu,\nu} : X \times (0,+\infty) \to L^*$ is a mapping such that the following conditions hold for all $x, y \in X$ and all $t, s \geq 0$:

(IRN_1) $\Lambda_{\mu,\nu}(x,0) = 0_{L^*}$;
(IRN_2) $\Lambda_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x = 0$;
(IRN_3) $\Lambda_{\mu,\nu}(\alpha x,t) = \Lambda_{\mu,\nu}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(IRN_4) $\Lambda_{\mu,\nu}(x+y,t+s) \geq_{L^*} \Upsilon(\Lambda_{\mu,\nu}(x,t), \Lambda_{\mu,\nu}(y,s))$.

In this case, $\Lambda_{\mu,\nu}$ is called an intuitionistic random norm. Here, $\Lambda_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t))$.

Every normed space $(X, \|\cdot\|)$ defines an IRN-space $(X, \Lambda_{\mu,\nu}, \Upsilon)$, where $\Lambda_{\mu,\nu}(x,t) = (\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|})$ for all $t > 0$ and $\Upsilon(a,b) = (a_1b_1, \min\{a_2+b_2, 1\})$ for all $a = (a_1,a_2)$, $b = (b_1,b_2) \in L^*$. This space is called the induced IRN-space.

Definition 2.5 Let $(X, \Lambda_{\mu,\nu}, \Upsilon)$ be an IRN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, $\Lambda_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.
(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\epsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\Lambda_{\mu,\nu}(x_n - x_m, t) \geq_{L^*} (N_s(\varepsilon), \varepsilon)$ for every $m, n \geq n_0$, where N_s is a standard negator.
(3) An IRN-space $(X, \Lambda_{\mu,\nu}, \Upsilon)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

From now on, let X be a linear space and $(Y, \Lambda_{\mu,\nu}, \Upsilon)$ be a complete IRN-space.

For convenience, we use the following abbreviation for a given mapping $f : X \to Y$:

$$\Delta f(x_1,\ldots,x_n) = \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

for all $x_1,\ldots,x_n \in X$, where $n \geq 2$ is a fixed integer.

3 Results in intuitionistic random spaces

In the following theorem, we prove the Hyers-Ulam stability of the functional equation (1.3) in IRN-spaces for quadratic mappings.

Theorem 3.1 *Let $\xi, \zeta : X^n \to D^+$ ($\xi(x_1,\ldots,x_n)$ is denoted by $\xi_{x_1,\ldots,x_n}$, $\zeta(x_1,\ldots,x_n)$ is denoted by $\zeta_{x_1,\ldots,x_n}$ and $(\xi_{x_1,\ldots,x_n}(t), \zeta_{x_1,\ldots,x_n}(t))$ is denoted by $\Phi_{\xi,\zeta}(x_1,\ldots,x_n,t)$) be mappings such that*

$$\lim_{m\to\infty} \Phi_{\xi,\zeta}\left(2^m x_1,\ldots,2^m x_n, 2^{2m} t\right) = 1_{L^*} \tag{3.1}$$

for all $x_1,\ldots,x_n \in X$ and all $t > 0$, and

$$\lim_{m\to\infty} \Upsilon_{i=1}^{\infty}\left(M^e_{\mu,\nu}\left(2^{m+i-1}x, 2^{2m+i}t\right)\right) = 1_{L^*} \tag{3.2}$$

for all $x \in X$ and all $t > 0$. Suppose that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\Lambda_{\mu,\nu}\big(\Delta f(x_1,\ldots,x_n),t\big) \geq_{L^*} \Phi_{\xi,\zeta}(x_1,\ldots,x_n,t) \tag{3.3}$$

for all $x_1,\ldots,x_n \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\Lambda_{\mu,\nu}\big(f(x) - Q(x),t\big) \geq_{L^*} \Upsilon_{i=1}^{\infty}\big(M^{e}_{\mu,\nu}\big(2^{i-1}x,2^{i}t\big)\big) \tag{3.4}$$

for all $x \in X$ and all $t > 0$, where

$$\begin{aligned} M^{e}_{\mu,\nu}(x,t) = \Upsilon\bigg(&\Phi_{\xi,\zeta}\big(nx,nx,0,\ldots,0,(n-1)t\big),\\ &\Upsilon\bigg(\Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx,0,\ldots,0,\frac{n-1}{8}t\bigg),\Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(0,nx,\ldots,nx,\frac{n-1}{16}t\bigg),\\ &\Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx,0,\ldots,0,\frac{n-1}{16}t\bigg),\Phi_{\xi,\zeta}\bigg(nx,0,\ldots,0,\frac{n-1}{8n}t\bigg)\bigg)\bigg)\bigg),\\ &\Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(x,(n-1)x,0,\ldots,0,\frac{n-1}{8}t\bigg),\Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(0,nx,\ldots,nx,\frac{n-1}{16}t\bigg),\\ &\Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx,0,\ldots,0,\frac{(n-1)}{16}t\bigg),\Phi_{\xi,\zeta}\bigg(nx,0,\ldots,0,\frac{n-1}{8n}t\bigg)\bigg)\bigg)\bigg)\bigg)\bigg). \end{aligned} \tag{3.5}$$

Proof Letting $x_1 = nx_1$ and $x_i = nx_2$ $(i = 2,\ldots,n)$ in (3.3) and using the evenness of f, we get

$$\begin{aligned} &\Lambda_{\mu,\nu}\big(nf\big(x_1 + (n-1)x_2\big) + f\big((n-1)(x_1 - x_2)\big)\\ &\qquad + (n-1)f(x_1 - x_2) - f(nx_1) - (n-1)f(nx_2),t\big)\\ &\quad \geq_{L^*} \Phi_{\xi,\zeta}(nx_1,nx_2,\ldots,nx_2,t) \end{aligned} \tag{3.6}$$

for all $x_1, x_2 \in X$ and all $t > 0$. Interchanging x_1 with x_2 in (3.6) and using the evenness of f, we get

$$\begin{aligned} &\Lambda_{\mu,\nu}\big(nf\big((n-1)x_1 + x_2\big) + f\big((n-1)(x_1 - x_2)\big)\\ &\qquad + (n-1)f(x_1 - x_2) - (n-1)f(nx_1) - f(nx_2),t\big)\\ &\quad \geq_{L^*} \Phi_{\xi,\zeta}(nx_2,nx_1,\ldots,nx_1,t) \end{aligned} \tag{3.7}$$

for all $x_1, x_2 \in X$ and all $t > 0$. It follows from (3.6) and (3.7) that

$$\begin{aligned} &\Lambda_{\mu,\nu}\big(nf\big((n-1)x_1 + x_2\big) + nf\big(x_1 + (n-1)x_2\big) + 2f\big((n-1)(x_1 - x_2)\big) + 2(n-1)f(x_1 - x_2)\\ &\qquad - nf(nx_1) - nf(nx_2),t\big)\\ &\quad \geq_{L^*} \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx_1,nx_2,\ldots,nx_2,\frac{t}{2}\bigg),\Phi_{\xi,\zeta}\bigg(nx_2,nx_1,\ldots,nx_1,\frac{t}{2}\bigg)\bigg) \end{aligned} \tag{3.8}$$

for all $x_1, x_2 \in X$ and all $t > 0$. Setting $x_1 = nx_1$, $x_2 = -nx_2$ and $x_i = 0$ $(i = 3, \dots, n)$ in (3.3) and using the evenness of f, we get

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f\big((n-1)x_1+x_2\big)+f\big(x_1+(n-1)x_2\big)+2(n-1)f(x_1-x_2)-f(nx_1)-f(nx_2),t\big)\\&\quad\geq_{L^*}\Phi_{\xi,\zeta}(nx_1,-nx_2,0,\dots,0,t)\end{aligned}\tag{3.9}$$

for all $x_1, x_2 \in X$ and all $t > 0$. So it follows from (3.8) and (3.9) that

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f\big((n-1)(x_1-x_2)\big)-(n-1)^2f(x_1-x_2),t\big)\\&\quad\geq_{L^*}\Upsilon\left(\Phi_{\xi,\zeta}\left(nx_1,-nx_2,0,\dots,0,\frac{t}{n}\right),\Upsilon\left(\Phi_{\xi,\zeta}\left(nx_1,nx_2,\dots,nx_2,\frac{t}{2}\right),\right.\right.\\&\qquad\left.\left.\Phi_{\xi,\zeta}\left(nx_2,nx_1,\dots,nx_1,\frac{t}{2}\right)\right)\right)\end{aligned}\tag{3.10}$$

for all $x_1, x_2 \in X$ and all $t > 0$. So

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f\big((n-1)x\big)-(n-1)^2f(x),t\big)\\&\quad\geq_{L^*}\Upsilon\left(\Phi_{\xi,\zeta}\left(0,nx,\dots,nx,\frac{t}{2}\right),\Upsilon\left(\Phi_{\xi,\zeta}\left(nx,0,\dots,0,\frac{t}{2}\right),\right.\right.\\&\qquad\left.\left.\Phi_{\xi,\zeta}\left(nx,0,\dots,0,\frac{t}{n}\right)\right)\right)\end{aligned}\tag{3.11}$$

for all $x \in X$ and all $t > 0$. Putting $x_1 = nx$ and $x_i = 0$ $(i = 2, \dots, n)$ in (3.3), we obtain

$$\Lambda_{\mu,\nu}\big(f(nx)-f\big((n-1)x\big)-(2n-1)f(x),t\big)\geq_{L^*}\Phi_{\xi,\zeta}(nx,0,\dots,0,t)\tag{3.12}$$

for all $x \in X$ and all $t > 0$. It follows from (3.11) and (3.12) that

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f(nx)-n^2f(x),t\big)\\&\quad\geq_{L^*}\Upsilon\left(\Phi_{\xi,\zeta}\left(nx,0,\dots,0,\frac{t}{2}\right),\Upsilon\left(\Phi_{\xi,\zeta}\left(0,nx,\dots,nx,\frac{t}{4}\right),\right.\right.\\&\qquad\left.\left.\Upsilon\left(\Phi_{\xi,\zeta}\left(nx,0,\dots,0,\frac{t}{4}\right),\Phi_{\xi,\zeta}\left(nx,0,\dots,0,\frac{t}{2n}\right)\right)\right)\right)\end{aligned}\tag{3.13}$$

for all $x \in X$ and all $t > 0$. Letting $x_2 = -(n-1)x_1$ in (3.9) and replacing x_1 by $\frac{x}{n}$ in the obtained inequality, we get

$$\Lambda_{\mu,\nu}\big(f\big((n-1)x\big)-f\big((n-2)x\big)-(2n-3)f(x),t\big)\geq_{L^*}\Phi_{\xi,\zeta}\big(x,(n-1)x,0,\dots,0,t\big)\tag{3.14}$$

for all $x \in X$ and all $t > 0$. It follows from (3.11) and (3.14) that

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f\big((n-2)x\big)-(n-2)^2f(x),t\big)\\&\quad\geq_{L^*}\Upsilon\left(\Phi_{\xi,\zeta}\left(x,(n-1)x,0,\dots,0,\frac{t}{2}\right),\Upsilon\left(\Phi_{\xi,\zeta}\left(0,nx,\dots,nx,\frac{t}{4}\right),\right.\right.\\&\qquad\left.\left.\Upsilon\left(\Phi_{\xi,\zeta}\left(nx,0,\dots,0,\frac{t}{4}\right),\Phi_{\xi,\zeta}\left(nx,0,\dots,0,\frac{t}{2n}\right)\right)\right)\right)\end{aligned}\tag{3.15}$$

for all $x \in X$ and all $t > 0$. Applying (3.13) and (3.15), we get

$$\begin{aligned}
&\Lambda_{\mu,\nu}\big(f(nx) - f\big((n-2)x\big) - 4(n-1)f(x), t\big) \\
&\quad \geq_{L^*} \Upsilon\bigg(\Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{t}{4}\bigg), \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(0, nx, \ldots, nx, \frac{t}{8}\bigg), \\
&\qquad \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{t}{8}\bigg), \Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{t}{4n}\bigg)\bigg)\bigg)\bigg), \\
&\qquad \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(x, (n-1)x, 0, \ldots, 0, \frac{t}{4}\bigg), \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(0, nx, \ldots, nx, \frac{t}{8}\bigg), \\
&\qquad \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{t}{8}\bigg), \Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{t}{4n}\bigg)\bigg)\bigg)\bigg)\bigg) \qquad (3.16)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Setting $x_1 = x_2 = nx$ and $x_i = 0$ $(i = 3, \ldots, n)$ in (3.3), we obtain

$$\Lambda_{\mu,\nu}\big(f\big((n-2)x\big) + (n-1)f(2x) - f(nx), t\big) \geq_{L^*} \Phi_{\xi,\zeta}(nx, nx, 0, \ldots, 0, 2t) \qquad (3.17)$$

for all $x \in X$ and all $t > 0$. It follows from (3.16) and (3.17) that

$$\begin{aligned}
&\Lambda_{\mu,\nu}\big(f(2x) - 4f(x), t\big) \\
&\quad \geq_{L^*} \Upsilon\bigg(\Phi_{\xi,\zeta}\big(nx, nx, 0, \ldots, 0, (n-1)t\big), \\
&\qquad \Upsilon\bigg(\Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{n-1}{8}t\bigg), \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(0, nx, \ldots, nx, \frac{n-1}{16}t\bigg), \\
&\qquad \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{n-1}{16}t\bigg), \Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{n-1}{8n}t\bigg)\bigg)\bigg)\bigg), \\
&\qquad \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(x, (n-1)x, 0, \ldots, 0, \frac{n-1}{8}t\bigg), \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(0, nx, \ldots, nx, \frac{n-1}{16}t\bigg), \\
&\qquad \Upsilon\bigg(\Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{n-1}{16}t\bigg), \Phi_{\xi,\zeta}\bigg(nx, 0, \ldots, 0, \frac{n-1}{8n}t\bigg)\bigg)\bigg)\bigg)\bigg)\bigg)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. It follows from (3.5) that

$$\Lambda_{\mu,\nu}\bigg(\frac{f(2x)}{2^2} - f(x), t\bigg) \geq_{L^*} M^e_{\mu,\nu}\big(x, 2^2 t\big) \geq_{L^*} M^e_{\mu,\nu}(x, 2t) \qquad (3.18)$$

for all $x \in X$ and all $t > 0$, which implies that

$$\Lambda_{\mu,\nu}\bigg(\frac{f(2^{k+1}x)}{2^{2(k+1)}} - \frac{f(2^k x)}{2^{2k}}, t\bigg) \geq_{L^*} M^e_{\mu,\nu}\big(2^k x, 2^{2(k+1)}t\big) \qquad (3.19)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. It follows from (3.19) and (IRN_4) that

$$\begin{aligned}
\Lambda_{\mu,\nu}\bigg(\frac{f(2^2x)}{2^4} - f(x), t\bigg) &\geq_{L^*} \Upsilon\bigg(\Lambda_{\mu,\nu}\bigg(\frac{f(2^2x)}{2^4} - \frac{f(2x)}{2^2}, \frac{t}{2}\bigg), \Lambda_{\mu,\nu}\bigg(\frac{f(2x)}{2^2} - f(x), \frac{t}{2}\bigg)\bigg) \\
&\geq_{L^*} \Upsilon\big(M^e_{\mu,\nu}\big(2x, 2^3 t\big), M^e_{\mu,\nu}(x, 2t)\big) \\
&\geq_{L^*} \Upsilon\big(M^e_{\mu,\nu}\big(2x, 2^2 t\big), M^e_{\mu,\nu}(x, 2t)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\Lambda_{\mu,\nu}\left(\frac{f(2^3x)}{2^6}-f(x),t\right)\\
&\quad\geq_{L^*}\Upsilon\left(\Lambda_{\mu,\nu}\left(\frac{f(2^3x)}{2^6}-\frac{f(2x)}{2^2},\frac{t}{2}\right),\Lambda_{\mu,\nu}\left(\frac{f(2x)}{2^2}-f(x),\frac{t}{2}\right)\right)\\
&\quad\geq_{L^*}\Upsilon\left(\Upsilon\left(\Lambda_{\mu,\nu}\left(\frac{f(2^3x)}{2^6}-\frac{f(2^2x)}{2^4},\frac{t}{4}\right),\Lambda_{\mu,\nu}\left(\frac{f(2^2x)}{2^4}-\frac{f(2x)}{2^2},\frac{t}{4}\right)\right),\right.\\
&\qquad\left.\Lambda_{\mu,\nu}\left(\frac{f(2x)}{2^2}-f(x),\frac{t}{2}\right)\right)\\
&\quad\geq_{L^*}\Upsilon\left(\Upsilon\left(M^e_{\mu,\nu}\left(2^2x,2^4t\right),M^e_{\mu,\nu}\left(2x,2^2t\right)\right),M^e_{\mu,\nu}(x,2t)\right)\\
&\quad\geq_{L^*}\Upsilon\left(\Upsilon\left(M^e_{\mu,\nu}\left(2^2x,2^3t\right),M^e_{\mu,\nu}\left(2x,2^2t\right)\right),M^e_{\mu,\nu}(x,2t)\right)\\
&\quad=\Upsilon\left(M^e_{\mu,\nu}(x,2t),\Upsilon\left(M^e_{\mu,\nu}\left(2x,2^2t\right),M^e_{\mu,\nu}\left(2^2x,2^3t\right)\right)\right)\\
&\quad=\Upsilon\left(\Upsilon\left(M^e_{\mu,\nu}(x,2t),M^e_{\mu,\nu}\left(2x,2^2t\right)\right),M^e_{\mu,\nu}\left(2^2x,2^3t\right)\right)
\end{aligned}$$

for all $x\in X$ and all $t>0$. Thus

$$\Lambda_{\mu,\nu}\left(\frac{f(2^mx)}{2^{2m}}-f(x),t\right)\geq_{L^*}\Upsilon_{i=1}^m\left(M^e_{\mu,\nu}\left(2^{i-1}x,2^it\right)\right)\tag{3.20}$$

for all $x\in X$ and all $t>0$. In order to prove the convergence of the sequence $\{\frac{f(2^mx)}{2^{2m}}\}$, we replace x with $2^{m'}x$ in (3.20) to find that

$$\Lambda_{\mu,\nu}\left(\frac{f(2^{m+m'}x)}{2^{2(m+m')}}-\frac{f(2^{m'}x)}{2^{2m'}},t\right)\geq_{L^*}\Upsilon_{i=1}^m\left(M^e_{\mu,\nu}\left(2^{m'+i-1}x,2^{2m'+i}t\right)\right)\tag{3.21}$$

for all $x\in X$ and all $t>0$. Since the right-hand side of the inequality (3.21) tends to 1_{L^*} as m' and m tend to infinity, the sequence $\{\frac{f(2^mx)}{2^{2m}}\}$ is a Cauchy sequence. Therefore, one can define the mapping $Q:X\to Y$ by $Q(x):=\lim_{m\to\infty}\frac{1}{2^{2m}}f(2^mx)$ for all $x\in X$. Now, if we replace $x_1,\dots,x_n$ with $2^mx_1,\dots,2^mx_n$ in (3.3) respectively, then

$$\Lambda_{\mu,\nu}\left(\frac{\Delta f(2^mx_1,\dots,2^mx_n)}{2^{2m}},t\right)\geq_{L^*}\Phi_{\xi,\zeta}\left(2^mx_1,\dots,2^mx_n,2^{2m}t\right)\tag{3.22}$$

for all $x_1,\dots,x_n\in X$ and all $t>0$. By letting $m\to\infty$ in (3.22), we find that $\Lambda_{\mu,\nu}(\Delta Q(x_1,\dots,x_n),t)=1_{L^*}$ for all $t>0$, which implies $\Delta Q(x_1,\dots,x_n)=0$. Thus Q satisfies (1.3). Hence the mapping $Q:X\to Y$ is quadratic.

To prove (3.4), take the limit as $m\to\infty$ in (3.20).

Finally, to prove the uniqueness of the quadratic mapping Q subject to (3.4), let us assume that there exists a quadratic mapping Q' which satisfies (3.4). Since $Q(2^mx)=2^{2m}Q(x)$ and $Q'(2^mx)=2^{2m}Q'(x)$ for all $x\in X$ and $m\in\mathbb{N}$, from (3.4) and (3.2) it follows that

$$\begin{aligned}
&\Lambda_{\mu,\nu}\left(Q(x)-Q'(x),t\right)\\
&\quad=\Lambda_{\mu,\nu}\left(Q\left(2^mx\right)-Q'\left(2^mx\right),2^{2m}t\right)
\end{aligned}$$

$$\begin{aligned}&\geq_{L^*} \Upsilon\big(\Lambda_{\mu,\nu}\big(Q\big(2^m x\big) - f\big(2^m x\big), 2^{2m-1}t\big), \Lambda_{\mu,\nu}\big(f\big(2^m x\big) - Q'\big(2^m x\big), 2^{2m-1}t\big)\big)\\ &\geq_{L^*} \Upsilon\big(\Upsilon_{i=1}^{\infty}\big(M^e_{\mu,\nu}\big(2^{m+i-1}x, 2^{2m+i}t\big)\big), \Upsilon_{i=1}^{\infty}\big(M^e_{\mu,\nu}\big(2^{m+i-1}x, 2^{2m+i}t\big)\big)\big)\end{aligned} \tag{3.23}$$

for all $x \in X$ and all $t > 0$. By letting $m \to \infty$ in (3.23), we find that $Q = Q'$. □

Corollary 3.2 *Let $(X, \Lambda_{\mu',\nu'}, \Upsilon)$ be an IRN-space and let $(Y, \Lambda_{\mu,\nu}, \Upsilon)$ be a complete IRN-space. If $f : X \to Y$ is a mapping such that*

$$\Lambda_{\mu,\nu}\big(\Delta f(x_1, \ldots, x_n), t\big) \geq_{L^*} \Lambda_{\mu',\nu'}(x_1 + \cdots + x_n, t)$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$, and

$$\lim_{m\to\infty} \Upsilon_{i=1}^{\infty}\big(M^e_{\mu',\nu'}\big(2^{m+i-1}x, 2^{2m+i}t\big)\big) = 1_{L^*}$$

for all $x \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\Lambda_{\mu,\nu}\big(f(x) - Q(x), t\big) \geq_{L^*} \Upsilon_{i=1}^{\infty}\big(M^e_{\mu',\nu'}\big(2^{i-1}x, 2^i t\big)\big)$$

for all $x \in X$ and all $t > 0$, where

$$\begin{aligned}M^e_{\mu',\nu'}(x,t) = \Upsilon\Bigg(&\Lambda_{\mu',\nu'}\bigg(x, \frac{n-1}{2n}t\bigg), \Upsilon\Bigg(\Upsilon\Bigg(\Lambda_{\mu',\nu'}\bigg(x, \frac{n-1}{8n}t\bigg), \Upsilon\Bigg(\Lambda_{\mu',\nu'}\bigg(x, \frac{1}{16n}t\bigg),\\ &\Upsilon\Bigg(\Lambda_{\mu',\nu'}\bigg(x, \frac{n-1}{16n}t\bigg), \Lambda_{\mu',\nu'}\bigg(x, \frac{n-1}{8n^2}t\bigg)\Bigg)\Bigg)\Bigg), \Upsilon\Bigg(\Lambda_{\mu',\nu'}\bigg(x, \frac{n-1}{8n}t\bigg),\\ &\Upsilon\Bigg(\Lambda_{\mu',\nu'}\bigg(x, \frac{1}{16n}t\bigg), \Upsilon\Bigg(\Lambda_{\mu',\nu'}\bigg(x, \frac{(n-1)}{16n}t\bigg), \Lambda_{\mu',\nu'}\bigg(x, \frac{n-1}{8n^2}t\bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg).\end{aligned}$$

Proof Let $\Phi_{\xi,\zeta}(x_1, \ldots, x_n, t) = \Lambda_{\mu',\nu'}(x_1 + \cdots + x_n, t)$. Then the corollary follows immediately from Theorem 3.1. □

Now, we prove the Hyers-Ulam stability of the functional equation (1.3) in IRN-spaces for additive mappings.

Theorem 3.3 *Let $\xi, \zeta : X^n \to D^+$ be mappings such that*

$$\lim_{m\to\infty} \Phi_{\xi,\zeta}\big(2^m x_1, \ldots, 2^m x_n, 2^m t\big) = 1_{L^*} \tag{3.24}$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$, and

$$\lim_{m\to\infty} \Upsilon_{i=1}^{\infty}\big(M^o_{\mu,\nu}\big(2^{m+i-1}x, 2^{m-1}t\big)\big) = 1_{L^*} \tag{3.25}$$

for all $x \in X$ and all $t > 0$. Suppose that an odd mapping $f : X \to Y$ satisfies (3.3) for all $x_1, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\Lambda_{\mu,\nu}\big(f(x) - A(x), t\big) \geq_{L^*} \Upsilon_{i=1}^{\infty}\big(M^o_{\mu,\nu}\big(2^{i-1}x, t\big)\big) \tag{3.26}$$

for all $x \in X$ and all $t > 0$, where

$$M^{o}_{\mu,\nu}(x,t) = \Upsilon\left(\Phi_{\xi,\zeta}\left(2x,0,\dots,0,\frac{t}{2}\right),\Upsilon\left(\Phi_{\xi,\zeta}\left(x,x,0,\dots,0,\frac{t}{2n}\right),\right.\right.$$
$$\left.\left.\Upsilon\left(\Phi_{\xi,\zeta}\left(x,-x,\dots,-x,\frac{t}{4}\right),\Phi_{\xi,\zeta}\left(-x,x,\dots,x,\frac{t}{4}\right)\right)\right)\right).$$

Proof Letting $x_1 = nx_1$ and $x_i = nx_1'$ $(i = 2,\dots,n)$ in (3.3) and using the oddness of f, we get

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(nf\big(x_1+(n-1)x_1'\big)+f\big((n-1)\big(x_1-x_1'\big)\big)\\&\quad-(n-1)f\big(x_1-x_1'\big)-f(nx_1)-(n-1)f\big(nx_1'\big),t\big)\\&\geq_{L^*}\Phi_{\xi,\zeta}\big(nx_1,nx_1',\dots,nx_1',t\big)\end{aligned}\tag{3.27}$$

for all $x_1, x_1' \in X$ and all $t > 0$. Interchanging x_1 with x_1' in (3.27) and using the oddness of f, we get

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(nf\big((n-1)x_1+x_1'\big)-f\big((n-1)\big(x_1-x_1'\big)\big)\\&\quad+(n-1)f\big(x_1-x_1'\big)-(n-1)f(nx_1)-f\big(nx_1'\big),t\big)\\&\geq_{L^*}\Phi_{\xi,\zeta}\big(nx_1',nx_1,\dots,nx_1,t\big)\end{aligned}\tag{3.28}$$

for all $x_1, x_1' \in X$ and all $t > 0$. It follows from (3.27) and (3.28) that

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(nf\big(x_1+(n-1)x_1'\big)-nf\big((n-1)x_1+x_1'\big)\\&\quad+2f\big((n-1)\big(x_1-x_1'\big)\big)-2(n-1)f\big(x_1-x_1'\big)\\&\quad+(n-2)f(nx_1)-(n-2)f\big(nx_1'\big),t\big)\\&\geq_{L^*}\Upsilon\left(\Phi_{\xi,\zeta}\left(nx_1,nx_1',\dots,nx_1',\frac{t}{2}\right),\Phi_{\xi,\zeta}\left(nx_1',nx_1,\dots,nx_1,\frac{t}{2}\right)\right)\end{aligned}\tag{3.29}$$

for all $x_1, x_1' \in X$ and all $t > 0$. Setting $x_1 = nx_1$, $x_2 = -nx_1'$ and $x_i = 0$ $(i = 3,\dots,n)$ in (3.3) and using the oddness of f, we get

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f\big((n-1)x_1+x_1'\big)-f\big(x_1+(n-1)x_1'\big)+2f\big(x_1-x_1'\big)-f(nx_1)+f\big(nx_1'\big),t\big)\\&\geq_{L^*}\Phi_{\xi,\zeta}\big(nx_1,-nx_1',0,\dots,0,t\big)\end{aligned}\tag{3.30}$$

for all $x_1, x_1' \in X$ and all $t > 0$. It follows from (3.29) and (3.30) that

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f\big((n-1)\big(x_1-x_1'\big)\big)+f\big(x_1-x_1'\big)-f(nx_1)+f\big(nx_1'\big),t\big)\\&\geq_{L^*}\Upsilon\left(\Phi_{\xi,\zeta}\left(nx_1,-nx_1',0,\dots,0,\frac{t}{n}\right),\Upsilon\left(\Phi_{\xi,\zeta}\left(nx_1,nx_1',\dots,nx_1',\frac{t}{2}\right),\right.\right.\\&\qquad\left.\left.\Phi_{\xi,\zeta}\left(nx_1',nx_1,\dots,nx_1,\frac{t}{2}\right)\right)\right)\end{aligned}\tag{3.31}$$

for all $x_1, x_1' \in X$ and all $t > 0$. Putting $x_1 = n(x_1 - x_1')$ and $x_i = 0$ $(i = 2, \dots, n)$ in (3.3), we obtain

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f\big(n\big(x_1 - x_1'\big)\big) - f\big((n-1)\big(x_1 - x_1'\big)\big) - f\big(\big(x_1 - x_1'\big)\big), t\big)\\&\quad \geq_{L^*} \Phi_{\xi,\zeta}\big(nx_1, -nx_1', 0, \dots, 0, t\big)\end{aligned} \tag{3.32}$$

for all $x_1, x_1' \in X$ and all $t > 0$. It follows from (3.31) and (3.32) that

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f\big(n\big(x_1 - x_1'\big)\big) - f(nx_1) + f\big(nx_1'\big), t\big)\\&\quad \geq_{L^*} \Upsilon\left(\Phi_{\xi,\zeta}\left(nx_1, -nx_1', 0, \dots, 0, \frac{t}{2}\right), \Upsilon\left(\Phi_{\xi,\zeta}\left(nx_1, -nx_1', 0, \dots, 0, \frac{t}{2n}\right),\right.\right.\\&\qquad \left.\left.\Upsilon\left(\Phi_{\xi,\zeta}\left(nx_1, nx_1', \dots, nx_1', \frac{t}{4}\right), \Phi_{\xi,\zeta}\left(nx_1', nx_1, \dots, nx_1, \frac{t}{4}\right)\right)\right)\right)\end{aligned} \tag{3.33}$$

for all $x_1, x_1' \in X$ and all $t > 0$. Replacing x_1 and x_1' by $\frac{x}{n}$ and $\frac{-x}{n}$ in (3.33), respectively, we obtain

$$\begin{aligned}&\Lambda_{\mu,\nu}\big(f(2x) - 2f(x), t\big)\\&\quad \geq_{L^*} \Upsilon\left(\Phi_{\xi,\zeta}\left(2x, 0, \dots, 0, \frac{t}{2}\right), \Upsilon\left(\Phi_{\xi,\zeta}\left(x, x, 0, \dots, 0, \frac{t}{2n}\right),\right.\right.\\&\qquad \left.\left.\Upsilon\left(\Phi_{\xi,\zeta}\left(x, -x, \dots, -x, \frac{t}{4}\right), \Phi_{\xi,\zeta}\left(-x, x, \dots, x, \frac{t}{4}\right)\right)\right)\right)\end{aligned}$$

for all $x \in X$ and all $t > 0$. Therefore,

$$\Lambda_{\mu,\nu}\left(\frac{f(2x)}{2} - f(x), t\right) \geq_{L^*} M^o_{\mu,\nu}(x, 2t) \geq_{L^*} M^o_{\mu,\nu}(x, t) \tag{3.34}$$

for all $x \in X$ and all $t > 0$, which implies that

$$\Lambda_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}, t\right) \geq_{L^*} M^o_{\mu,\nu}\big(2^k x, 2^{k+1} t\big) \tag{3.35}$$

for all $x \in X$, $t > 0$ and $k \in \mathbb{N}$. It follows from (3.35) and (IRN_4) that

$$\begin{aligned}&\Lambda_{\mu,\nu}\left(\frac{f(2^2x)}{2^2} - f(x), t\right)\\&\quad \geq_{L^*} \Upsilon\left(\Lambda_{\mu,\nu}\left(\frac{f(2^2x)}{2^2} - \frac{f(2x)}{2}, \frac{t}{2}\right), \Lambda_{\mu,\nu}\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right)\right)\\&\quad \geq_{L^*} \Upsilon\big(M^o_{\mu,\nu}(2x, 2t), M^0_{\mu,\nu}(x, t)\big) \geq_{L^*} \Upsilon\big(M^o_{\mu,\nu}(2x, t), M^o_{\mu,\nu}(x, t)\big)\end{aligned}$$

and

$$\begin{aligned}&\Lambda_{\mu,\nu}\left(\frac{f(2^3x)}{2^3} - f(x), t\right)\\&\quad \geq_{L^*} \Upsilon\left(\Lambda_{\mu,\nu}\left(\frac{f(2^3x)}{2^3} - \frac{f(2x)}{2}, \frac{t}{2}\right), \Lambda_{\mu,\nu}\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right)\right)\end{aligned}$$

$$\geq_{L^*} \Upsilon\left(\Upsilon\left(\Lambda_{\mu,\nu}\left(\frac{f(2^3x)}{2^3} - \frac{f(2^2x)}{2^2}, \frac{t}{4}\right), \Lambda_{\mu,\nu}\left(\frac{f(2^2x)}{2^2} - \frac{f(2x)}{2}, \frac{t}{4}\right)\right),\right.$$
$$\left.\Lambda_{\mu,\nu}\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right)\right)$$
$$\geq_{L^*} \Upsilon\left(\Upsilon\left(M^o_{\mu,\nu}\left(2^2x, 2t\right), M^o_{\mu,\nu}(2x,t)\right), M^o_{\mu,\nu}(x,t)\right)$$
$$\geq_{L^*} \Upsilon\left(\Upsilon\left(M^o_{\mu,\nu}\left(2^2x, t\right), M^o_{\mu,\nu}(2x,t)\right), M^o_{\mu,\nu}(x,t)\right)$$
$$= \Upsilon\left(M^o_{\mu,\nu}(x,t), \Upsilon\left(M^o_{\mu,\nu}(2x,t), M^o_{\mu,\nu}\left(2^2x,t\right)\right)\right)$$
$$= \Upsilon\left(\Upsilon\left(M^o_{\mu,\nu}(x,t), M^o_{\mu,\nu}(2x,t)\right), M^o_{\mu,\nu}\left(2^2x,t\right)\right)$$

for all $x \in X$ and all $t > 0$. Thus

$$\Lambda_{\mu,\nu}\left(\frac{f(2^m x)}{2^m} - f(x), t\right) \geq_{L^*} \Upsilon_{i=1}^m\left(M^o_{\mu,\nu}\left(2^{i-1}x, t\right)\right) \tag{3.36}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^m x)}{2^m}\}$, we replace x with $2^{m'}x$ in (3.36) to find that

$$\Lambda_{\mu,\nu}\left(\frac{f(2^{m+m'}x)}{2^{m+m'}} - \frac{f(2^{m'}x)}{2^{m'}}, t\right) \geq_{L^*} \Upsilon_{i=1}^m\left(M^o_{\mu,\nu}\left(2^{m'+i-1}x, 2^{m'}t\right)\right) \tag{3.37}$$

for all $x \in X$ and all $t > 0$. Since the right-hand side of the inequality (3.37) tends to 1_{L^*} as m' and m tend to infinity, the sequence $\{\frac{f(2^m x)}{2^m}\}$ is a Cauchy sequence. Therefore, one can define the mapping $A : X \to Y$ by $A(x) := \lim_{m\to\infty} \frac{1}{2^m} f(2^m x)$ for all $x \in X$. Now, if we replace $x_1, \ldots, x_n$ with $2^m x_1, \ldots, 2^m x_n$ in (3.3) respectively, then

$$\Lambda_{\mu,\nu}\left(\frac{\Delta f(2^m x_1, \ldots, 2^m x_n)}{2^m}, t\right) \geq_{L^*} \Phi_{\xi,\zeta}\left(2^m x_1, \ldots, 2^m x_n, 2^m t\right) \tag{3.38}$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$. By letting $m \to \infty$ in (3.38), we find that $\Lambda_{\mu,\nu}(\Delta A(x_1, \ldots, x_n), t) = 1_{L^*}$ for all $t > 0$, which implies $\Delta A(x_1, \ldots, x_n) = 0$. Thus A satisfies (1.3). Hence the mapping $A : X \to Y$ is additive. To prove (3.26), take the limit as $m \to \infty$ in (3.36).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4 *Let $(X, \Lambda_{\mu',\nu'}, \Upsilon)$ be an IRN-space and let $(Y, \Lambda_{\mu,\nu}, \Upsilon)$ be a complete IRN-space. If $f : X \to Y$ is a mapping such that*

$$\Lambda_{\mu,\nu}\left(\Delta f(x_1, \ldots, x_n), t\right) \geq_{L^*} \Lambda_{\mu',\nu'}(x_1 + \cdots + x_n, t)$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$, and

$$\lim_{m\to\infty} \Upsilon_{i=1}^\infty\left(M^o_{\mu',\nu'}\left(2^{m+i-1}x, 2^{m-1}t\right)\right) = 1_{L^*}$$

for all $x \in X$ and all $t > 0$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\Lambda_{\mu,\nu}\left(f(x) - A(x), t\right) \geq_{L^*} \Upsilon_{i=1}^\infty\left(M^o_{\mu',\nu'}\left(2^{i-1}x, t\right)\right)$$

for all $x \in X$ *and all* $t > 0$, *where*

$$M^{o}_{\mu',\nu'}(x,t) = \Upsilon\left(\Lambda_{\mu',\nu'}\left(x,\frac{t}{4}\right),\Upsilon\left(\Lambda_{\mu',\nu'}\left(x,\frac{t}{4n}\right),\right.\right.$$
$$\left.\left.\Upsilon\left(\Lambda_{\mu',\nu'}\left(x,\frac{t}{4(2-n)}\right),\Lambda_{\mu',\nu'}\left(x,\frac{t}{4(n-2)}\right)\right)\right)\right).$$

The main result of this paper is the following:

Theorem 3.5 *Let* $\xi,\zeta : X^n \to D^+$ *be mappings satisfying (3.24) and (3.25) for all* $x_1,\ldots,x_n$, $x \in X$ *and all* $t > 0$. *Suppose that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies (3.3) for all* $x_1,\ldots,x_n \in X$ *and all* $t > 0$. *Then there exist an additive mapping* $A : X \to Y$ *and a quadratic mapping* $Q : X \to Y$ *satisfying (1.3) and*

$$\begin{aligned}&\Lambda_{\mu,\nu}\left(f(x)-A(x)-Q(x),t\right)\\ &\quad\geq_{L^*}\Upsilon\Big(\Upsilon\big(\Upsilon_{i=1}^{\infty}\big(M^{e}_{\mu,\nu}\big(2^{i-1}x,2^{i-1}t\big)\big),\Upsilon_{i=1}^{\infty}\big(M^{e}_{\mu,\nu}\big(-2^{i-1}x,2^{i-1}t\big)\big)\big),\\ &\qquad\Upsilon\Big(\Upsilon_{i=1}^{\infty}\Big(M^{o}_{\mu,\nu}\Big(2^{i-1}x,\frac{t}{2}\Big)\Big),\Upsilon_{i=1}^{\infty}\Big(M^{o}_{\mu,\nu}\Big(-2^{i-1}x,\frac{t}{2}\Big)\Big)\Big)\Big).\end{aligned}\tag{3.39}$$

Proof Let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$ and

$$\begin{aligned}\Lambda_{\mu,\nu}\left(\Delta f_e(x_1,\ldots,x_n),t\right) &= \Lambda_{\mu,\nu}\left(\frac{\Delta f(x_1,\ldots,x_n)+\Delta f(-x_1,\ldots,-x_n)}{2},t\right)\\ &\geq_{L^*}\Upsilon\left(\Lambda_{\mu,\nu}\left(\Delta f(x_1,\ldots,x_n),t\right),\Lambda_{\mu,\nu}\left(\Delta f(-x_1,\ldots,-x_n),t\right)\right)\\ &\geq_{L^*}\Upsilon\left(\Phi_{\xi,\zeta}(x_1,\ldots,x_n,t),\Phi_{\xi,\zeta}(-x_1,\ldots,-x_n,t)\right)\end{aligned}$$

for all $x_1,\ldots,x_n \in X$ and all $t > 0$. By Theorem 3.1, there exists a quadratic mapping $Q : X \to Y$ such that

$$\Lambda_{\mu,\nu}\left(f_e(x)-Q(x),t\right)\geq_{L^*}\Upsilon\left(\Upsilon_{i=1}^{\infty}\left(M^{e}_{\mu,\nu}\left(2^{i-1}x,2^{i}t\right)\right),\Upsilon_{i=1}^{\infty}\left(M^{e}_{\mu,\nu}\left(-2^{i-1}x,2^{i}t\right)\right)\right)\tag{3.40}$$

for all $x \in X$ and all $t > 0$.

On the other hand, let $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$. By Theorem 3.3, there exists an additive mapping $A : X \to Y$ such that

$$\Lambda_{\mu,\nu}\left(f_o(x)-A(x),t\right)\geq_{L^*}\Upsilon\left(\Upsilon_{i=1}^{\infty}\left(M^{o}_{\mu,\nu}\left(2^{i-1}x,t\right)\right),\Upsilon_{i=1}^{\infty}\left(M^{o}_{\mu,\nu}\left(-2^{i-1}x,t\right)\right)\right)\tag{3.41}$$

for all $x \in X$ and all $t > 0$. Hence (3.39) follows from (3.40) and (3.41). □

Corollary 3.6 *Let* $(X,\Lambda_{\mu',\nu'},\Upsilon)$ *be an IRN-space and let* $(Y,\Lambda_{\mu,\nu},\Upsilon)$ *be a complete IRN-space. If* $f : X \to Y$ *is a mapping such that*

$$\Lambda_{\mu,\nu}\left(\Delta f(x_1,\ldots,x_n),t\right)\geq_{L^*}\Lambda_{\mu',\nu'}(x_1+\cdots+x_n,t)$$

for all $x_1,\ldots,x_n \in X$ *and all* $t > 0$, *and*

$$\lim_{m\to\infty}\Upsilon_{i=1}^{\infty}\left(M^{o}_{\mu',\nu'}\left(2^{m+i-1}x,2^{m-1}t\right)\right)=1_{L^*}$$

for all $x \in X$ and all $t > 0$, then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}
&\Lambda_{\mu,\nu}\big(f(x) - A(x) - Q(x), t\big) \\
&\quad \geq_{L^*} \Upsilon\Big(\Upsilon\big(\Upsilon_{i=1}^{\infty}\big(M^e_{\mu',\nu'}\big(2^{i-1}x, 2^{i-1}t\big)\big), \Upsilon_{i=1}^{\infty}\big(M^e_{\mu',\nu'}\big(-2^{i-1}x, 2^{i-1}t\big)\big)\big), \\
&\qquad \Upsilon\Big(\Upsilon_{i=1}^{\infty}\Big(M^o_{\mu',\nu'}\Big(2^{i-1}x, \frac{t}{2}\Big)\Big), \Upsilon_{i=1}^{\infty}\Big(M^o_{\mu',\nu'}\Big(-2^{i-1}x, \frac{t}{2}\Big)\Big)\Big)\Big)
\end{aligned}$$

for all $x \in X$ and all $t > 0$.

Now, we give an example to validate the result of quadratic mappings as follows:

Example Let $(X, \|\cdot\|)$ be a Banach space, $(X, \Lambda_{\mu,\nu}, \mathbf{M})$ an IRN-space in which

$$\Lambda_{\mu,\nu}(x,t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right)$$

and let $(Y, \Lambda_{\mu,\nu}, \mathbf{M})$ be a complete IRN-space for all $x \in X$. Define a mapping $f : X \to Y$ by $f(x) = x^2 + x_0$, where x_0 is a unit vector in X. A straightforward computation shows that, for all $t > 0$,

$$\Lambda_{\mu,\nu}\big(\Delta f(x_1, \ldots, x_n), t\big) \geq_{L^*} \Lambda_{\mu,\nu}(x_1 + \cdots + x_n, t)$$

and

$$\begin{aligned}
\lim_{m\to\infty} \mathbf{M}_{i=1}^{\infty}\big(M^e_{\mu,\nu}\big(2^{m+i-1}x, 2^{2m+i}t\big)\big) &= \lim_{m\to\infty}\lim_{k\to\infty} \mathbf{M}_{i=1}^{k}\big(M^e_{\mu,\nu}\big(x, 2^{m+1}t\big)\big) \\
&= \lim_{m\to\infty}\big(M^e_{\mu,\nu}\big(x, 2^{m+1}t\big)\big) = 1_{L^*}.
\end{aligned}$$

Therefore, all the conditions of Theorem 3.1 hold, and so there exists a unique quadratic mapping $Q : X \to Y$ such that $\Lambda_{\mu,\nu}(f(x) - Q(x), t) \geq_{L^*} M^e_{\mu,\nu}(x, 2t)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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