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Existence of solutions to strongly damped quasilinear wave equations

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Abstract
In this paper, we study the strongly damped quasilinear wave equation. By using spatial sequence techniques and energy estimate methods, we obtain the existence theorem of the solution to abstract a strongly damped wave equation and to a class of strongly damped quasilinear wave equations.
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1 Introduction

This paper is concerned with the following initial-boundary problem of strongly damped quasilinear wave equations:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - k\frac{\partial \Delta u}{\partial t} = -\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,u,\dots,D^m u) + g(x,u,\dots,D^m u), \\ u|_{\partial\Omega} = \cdots = D^{m-1}u|_{\partial\Omega} = 0, \\ u(x,0) = \varphi, \qquad u_t(x,0) = \psi, \end{cases} \tag{1.1}$$

where $k > 0$, $m \geq 1$, Δ is the Laplacian operator, Ω denotes an open bounded set of R^n with smooth boundary $\partial\Omega$, and u denotes vertical displacement at (x,t).

Equation (1.1) is a quasilinear wave equation with strong damping, which has many applications. The existence and asymptotic behavior for the strongly damped wave equations have been extensively studied by many authors [1–15]. Local well-posedness for strongly damped wave equations with critical nonlinearities is studied in [2]. The existence and asymptotic behavior for a strongly damped nonlinear wave equation have been concerned in [1, 3–9, 12–15]. Fan [10] investigated the existence and the continuity of the inflated attractors for a class of nonautonomous strongly damped wave equations through differential inclusion. Li [11] obtained the existence of a global periodic attractor attracting any bounded set exponentially in the phase space by introducing a new norm, which is equivalent to the usual norm.

The quasilinear wave equation has been investigated by many authors in the last years [16–28]. In [16–20], it is considered the boundary value problem for the quasilinear wave equation. Under certain assumptions, the global smooth solvability is obtained. It has been shown by Alinhac [21, 22] that the null condition implies global existence of smooth solutions in two space dimensions. Zhang [23] studies the global existence, singularities,

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and life span of smooth solutions of the Cauchy problem for a class of quasilinear hyperbolic systems with higher order dissipative terms and gives their applications to nonlinear wave equations with higher order dissipative terms. Metcalfe and Sogge [24] give a simple proof of global existence for quadratic quasilinear Dirichlet-wave equations outside of a wide class of compact obstacles in the critical case where the spatial dimension is three. Yin [25] gives the lower bound of a lifespan of classical solutions and discusses the long time asymptotic behavior of solutions away from the blowup time. Weidemaier [26] establishes local (in time) existence of classical solutions to the initial-boundary value problem for a quasilinear wave equation. In [27], the existence and uniqueness of the classical solutions for the initial value problems and the first boundary problems of a quasilinear wave equation are proved by the Galerkin method. In [28], the numerical solution for a type of quasilinear wave equation is studied. The three-level difference scheme for quasilinear waver equation with strong dissipative term is constructed and the convergence is proved.

The strongly damped wave equations and the quasilinear wave equation have a lot of results. But up to now, we find several results on the strongly damped quasilinear wave equation. Chen [29] shows that the initial boundary value problem for the strongly damped quasilinear wave equation has a global solution and that there exists a compact global attractor with finite dimension. Comparing Eq. (1.1) and [29], we find that $A_\alpha(x,u,\ldots,D^m u) = \sigma(u_x)_x$, $g(x,u,\ldots,D^m u) = -f(u) + g(x)$, and $x \in \Omega = [0,1]$. In this article, our interest is to study that Eq. (1.1) has a solution under which condition of A and g. This article uses the spatial sequence techniques, each side of the equation to be treated in different spaces, which is an important way to get more extensive and wonderful results.

The outline of the paper is as follows. In Section 2, we provide essential preliminaries, which include definitions and lemmas from [30]. In Section 3, we give existence of solutions to abstract strongly damped wave equations. In Section 4, we present the main results and their proof. Existence of solutions to a class of strongly damped quasilinear wave equations is given.

2 Preliminaries

We introduce two spatial sequences:

$$\begin{cases} X \subset H_3 \subset X_2 \subset X_1 \subset H, \\ X_2 \subset H_2 \subset H_1 \subset H, \end{cases} \tag{2.1}$$

where H, H_1, H_2, H_3 are Hilbert spaces, X is a linear space, and X_1, X_2 are Banach spaces. All imbeddings of (2.1) are dense. Let

$$\begin{cases} L: X \to X_1 \quad \text{is one-one dense linear operator,} \\ \langle Lu, v\rangle_H = \langle u, v\rangle_{H_1}, \quad \forall u, v \in X. \end{cases} \tag{2.2}$$

Furthermore, L has eigenvectors $\{e_k\}$ satisfying

$$Le_k = \lambda_k e_k \quad (k = 1, 2, \ldots), \tag{2.3}$$

and $\{e_k\}$ constitutes a common orthogonal basis of H and H_3.

We consider the following abstract equation:

$$\begin{cases} \frac{d^2u}{dt^2} + k\frac{d}{dt}\mathcal{L}u = G(u), \quad k > 0, \\ u(0) = \varphi, \qquad u_t(0) = \psi, \end{cases} \tag{2.4}$$

where $G : X_2 \times R^+ \to X_1^*$ is a map, $R^+ = [0, \infty)$ and $\mathcal{L} : X_2 \to X_1$ is a bounded linear operator satisfying

$$\langle \mathcal{L}u, Lv\rangle_H = \langle u, v\rangle_{H_2}, \quad \forall u, v \in X_2. \tag{2.5}$$

Definition 2.1 We say $u \in W^{1,\infty}((0,T), H_1) \cap L^\infty((0,T), X_2)$ is a global weak solution of Eq. (2.4) provided that $(\varphi, \psi) \in X_2 \times H_1$

$$\langle u_t, v\rangle_H + k\langle \mathcal{L}u, v\rangle_H = \int_0^t \langle G(u), v\rangle\, dt + \langle \psi, v\rangle_H + k\langle \mathcal{L}\varphi, v\rangle_H, \tag{2.6}$$

for all $v \in X_1$ and $0 \le t \le T < \infty$.

Definition 2.2 Let $u_n, u_0 \in L^p((0,T), X_2)$. We say $u_n \rightharpoonup u_0$ in $L^p((0,T), X_2)$ is uniformly weakly convergent if $\{u_n\} \subset L^p((0,T), H)$ is bounded, and

$$\begin{cases} u_n \rightharpoonup u_0 \quad \text{in } L^p((0,T), X_2), \\ \lim_{n\to\infty} \int_0^T |\langle u_n - u_0, v\rangle_H|^2\, dt = 0, \quad \forall v \in H. \end{cases} \tag{2.7}$$

Definition 2.3 We say that a map $G : X_2 \times (0,\infty) \to X_1^*$ is T-coercive weakly continuous if for all $\{u_n\} \subset L^p_{\text{loc}}((0,\infty), X_2) \cap L^\infty_{\text{loc}}((0,\infty), H)$, $u_n \rightharpoonup u_0$ in $L^p((0,T), X_2)$ is uniformly weakly convergent, and

$$\lim_{n\to\infty} \int_0^t \left|\langle Gu_n - Gu_0, Lu_n - Lu_0\rangle\right| dt = 0, \quad 0 < T < \infty,$$

then

$$\lim_{n\to\infty} \int_0^t \left|\langle Gu_n, v\rangle\right| dt = \lim_{n\to\infty} \int_0^T \left|\langle Gu_0, v\rangle\right| dt, \quad \forall v \in X_1, 0 < t < \infty.$$

Lemma 2.4 ([30]) *Let $\{u_n\} \in L^p((0,T), W^{m,p}(\Omega))$ ($m \ge 1$) be bounded sequences, and $\{u_n\}$ uniformly weakly convergent to $\{u_0\} \in L^p((0,T), W^{m,p}(\Omega))$. Then, for all $|\alpha| \le m - 1$, it follows that*

$$D^\alpha u_n \to D^\alpha u_0 \quad \text{in } L^2\big((0,T) \times \Omega\big). \tag{2.8}$$

Lemma 2.5 ([30]) *Let $\Omega \subset R^n$ be a open set, and $f : \Omega \times R^N \to R^1$ satisfy the Caratheodory condition and*

$$\left|f(x,\xi)\right| \le C \sum_{i=1}^N |\xi_i|^{\frac{p_i}{p}} + b(x). \tag{2.9}$$

If $\{u_{i_k}\} \subset L^{p_i}(\Omega)$ $(1 \le i \le N)$ is bounded and u_{i_k} convergent to u_i in Ω_0 for all bounded $\Omega_0 \subset \Omega$, then for all $v \in L^{p'}$, the following equality holds:

$$\lim_{k\to\infty} \int_\Omega f(x, u_{1_k}, \ldots, u_{N_k}) v \, dx = \int_\Omega f(x, u_1, \ldots, u_N) v \, dx.$$

3 Existence of solutions to abstract equations

Let $G = A + B : X_2 \times R^+ \to X_1^*$. Assume:

(A1) There is a C^1 functional $F : X_2 \to R^1$ such that

$$\langle Au, Lv\rangle = \langle -DF(u), v\rangle, \quad \forall u, v \in X. \tag{3.1}$$

(A2) Functional $F : X_2 \to R^1$ is coercive, *i.e.*,

$$F(u) \to \infty \quad \Leftrightarrow \quad \|u\|_{X_2} \to \infty. \tag{3.2}$$

(A3) B satisfies

$$\big|\langle Bu, Lv\rangle\big| \le C_1 F(u) + \frac{k}{2}\|v\|_{H_1}^2 + g(t), \quad \forall u, v \in X, \tag{3.3}$$

for $g \in L^1_{\text{loc}}(0, \infty)$.

Theorem 3.1 *If $G : X_2 \times R^+ \to X_1^*$ is T-coercively weakly continuous, and*

$$\lim_{n\to\infty} \int_0^t \big|\langle Gu_n - Gu_0, Lu_n - Lu_0\rangle\big| \, dt + \lim_{n\to\infty} \|u_n - u_0\|_{H_2}^2 = 0,$$

then for all $(\varphi, \psi) \in X_2 \times H_1$, then the following assertions hold:

(1) If $G = A$ satisfies (A1) and (A2), then Eq. (2.4) has a global weak solution

$$u \in W^{1,\infty}\big((0,\infty), H_1\big) \cap W^{1,2}\big((0,\infty), H_2\big) \cap L^\infty\big((0,\infty), X_2\big). \tag{3.4}$$

(2) If $G = A + B$ satisfies (A1)-(A3), then Eq. (2.4) has a global weak solution

$$u \in W^{1,\infty}_{\text{loc}}\big((0,\infty), H_1\big) \cap W^{1,2}_{\text{loc}}\big((0,\infty), H_2\big) \cap L^\infty_{\text{loc}}\big((0,\infty), X_2\big). \tag{3.5}$$

(3) Furthermore, if $\mathcal{L} : X_2 \to X_1$ is a symmetric sectorial operator, i.e., $\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}v\rangle$, and $G = A + B$ satisfies

$$\big|\langle Gu, v\rangle\big| \le CF(u) + \frac{1}{2}\|v\|_H^2 + g(t), \tag{3.6}$$

for $g \in L^1(0, T)$, then $u \in W^{2,2}_{\text{loc}}((0,\infty); H)$.

Proof Let $\{e_k\} \subset X$ be a common orthogonal basis of H and H_3, satisfying (2.3). Set

$$\begin{cases} X_n = \{\sum_{i=1}^n \alpha_i e_i | \alpha_i \in R^1\}, \\ \tilde{X}_n = \{\sum_{j=1}^n \beta_j(t) e_j | \beta_j(t) \in C^2[0,\infty)\}. \end{cases} \tag{3.7}$$

Clearly, $LX_n = X_n$, $L\tilde{X}_n = \tilde{X}_n$.

By using the Galerkin method, there exists $u_n \in C^2([0,\infty), X_n)$ satisfying

$$\begin{cases} \langle \frac{du_n}{dt}, v\rangle_H + k\langle \mathcal{L}u_n, v\rangle_H = \int_0^t \langle G(u_n), v\rangle\, dt + \langle \psi_n, v\rangle_H + k\langle \mathcal{L}\varphi_n, v\rangle_H, \\ u_n(0) = \varphi_n, \qquad u_n'(0) = \psi_n, \end{cases} \tag{3.8}$$

for $\forall v \in X_n$, and

$$\int_0^t \left[\left\langle \frac{d^2u_n}{d^2t}, v\right\rangle_H + k\left\langle \mathcal{L}\frac{du_n}{dt}, v\right\rangle_H \right] dt = \int_0^t \langle Gu_n, v\rangle\, dt \tag{3.9}$$

for $\forall v \in \tilde{X}_n$.

Firstly, we consider $G = A$. Let $v = \frac{d}{dt}Lu_n$ in (3.9). Taking into account (2.2) and (3.1), it follows that

$$\begin{aligned} 0 &= \int_0^t \left[\frac{1}{2}\frac{d}{dt}\left\langle \frac{du_n}{dt}, \frac{du_n}{dt}\right\rangle_{H_1} + k\left\langle \frac{du_n}{dt}, \frac{du_n}{dt}\right\rangle_{H_2} + \left\langle DF(u_n), \frac{du_n}{dt}\right\rangle \right] dt \\ &= \frac{1}{2}\left\| \frac{du_n}{dt}\right\|_{H_1}^2 - \frac{1}{2}\|\psi_n\|_{H_1}^2 + k\int_0^t \left\| \frac{du_n}{dt}\right\|_{H_2}^2 dt + F(u_n) - F(\varphi_n). \end{aligned}$$

We get

$$\frac{1}{2}\left\| \frac{du_n}{dt}\right\|_{H_1}^2 + k\int_0^t \left\| \frac{du_n}{dt}\right\|_{H_2}^2 dt + F(u_n) = F(\varphi_n) + \frac{1}{2}\|\psi_n\|_{H_1}^2. \tag{3.10}$$

Let $\varphi \in H_3$. From (2.1) and (2.2), it is known that $\{e_n\}$ is an orthogonal basis of H_1. We find that $\varphi_n \to \varphi$ in H_3, and $\psi_n \to \psi$ in H_1. From that $H_3 \subset X_2$ is an imbedding, it follows that

$$\begin{cases} \varphi_n \to \varphi & \text{in } X_2, \\ \psi_n \to \psi & \text{in } H_1. \end{cases} \tag{3.11}$$

From (3.2), (3.10), and (3.11), we obtain

$$\{u_n\} \subset W^{1,\infty}\big((0,\infty), H_1\big) \cap W^{1,2}\big((0,\infty), H_2\big) \cap L^\infty\big((0,\infty), X_2\big) \quad \text{is bounded.}$$

Let

$$\begin{cases} u_n \rightharpoonup {}^*u_0 & \text{in } W^{1,\infty}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2), \\ u_n \rightharpoonup u_0 & \text{in } W^{1,2}((0,\infty), H_2), \end{cases} \tag{3.12}$$

which implies that $u_n \to u_0$ in $W^{1,2}((0,\infty), H)$ is uniformly weakly convergent from that $H_2 \subset H$ is a compact imbedding.

If we have the following equality,

$$\lim_{n\to\infty} \left[-\int_0^t \big|\langle Gu_n - Gu_0, Lu_n - Lu_0\rangle\big|\, dt + \frac{k}{2}\|u_n - u_0\|_{H_2} \right] = 0, \tag{3.13}$$

then u_0 is a weak solution of Eq. (2.4) in view of (3.8), (3.12), and T-coercively weakly continuous of G.

We will show (3.13) as follows. It follows that from (2.5)

$$\begin{aligned}\int_0^t \left\langle \frac{d}{dt}\mathcal{L}u_n - \frac{d}{dt}\mathcal{L}u_0, Lu_n - Lu_0 \right\rangle_H dt &= \frac{1}{2}\int_0^t \frac{d}{dt}\langle u_n - u_0, u_n - u_0\rangle_{H_2}\, dt \\ &= \frac{1}{2}\left\| u_n(t) - u_0(t)\right\|_{H_2}^2 - \frac{1}{2}\|\varphi_n - \varphi\|_{H_2}^2.\end{aligned}$$

Taking into account (2.2), (2.5) and (3.9), we get that

$$\begin{aligned}&-\int_0^t \langle Gu_n - Gu_0, Lu_n - Lu_0\rangle\, dt + \frac{k}{2}\|u_n - u_0\|_{H_2} \\ &\quad= \int_0^t \bigg[\langle Gu_0 - Gu_n, Lu_n - Lu_0\rangle \\ &\qquad + k\left\langle \frac{d}{dt}\mathcal{L}u_n - \frac{d}{dt}\mathcal{L}u_0, Lu_n - Lu_0\right\rangle_H \bigg] dt + \frac{k}{2}\|\varphi_n - \varphi\|_{H_2}^2 \\ &\quad= \int_0^t \bigg[\langle Gu_0, Lu_n - Lu_0\rangle + \langle Gu_n, Lu_0\rangle - k\left\langle \frac{du_n}{dt}, u_0\right\rangle_{H_2} - k\left\langle \frac{d}{dt}u_0, u_n - u_0\right\rangle_{H_2} \\ &\qquad + \left\langle \frac{du_n}{dt}, \frac{du_n}{dt}\right\rangle_{H_1}\bigg] dt - \left\langle \frac{du_n}{dt}, u_n\right\rangle_{H_1} + \langle \psi_n, \varphi_n\rangle_{H_1} + \frac{k}{2}\|\varphi_n - \varphi\|_{H_2}^2.\end{aligned}$$

From (2.1) and (3.12), we have

$$\begin{aligned}&\lim_{n\to\infty} \|\varphi_n - \varphi\|_{H_2} = 0, \\ &\lim_{n\to\infty} \int_0^t \langle Gu_0, Lu_n - Lu_0\rangle\, dt = 0, \\ &\lim_{n\to\infty} \int_0^t \left\langle \frac{d}{dt}u_0, u_n - u_0\right\rangle_{H_2} dt = 0.\end{aligned}$$

Then we get

$$\begin{aligned}&\lim_{n\to\infty} -\int_0^t \langle Gu_n - Gu_0, Lu_n - Lu_0\rangle\, dt + \frac{k}{2}\lim_{n\to\infty}\|u_n - u_0\|_{H_2}^2 \\ &\quad= \lim_{n\to\infty}\int_0^t \left[\langle Gu_n, Lu_0\rangle - k\left\langle \frac{du_n}{dt}, u_0\right\rangle_{H_2} + \left\| \frac{du_n}{dt}\right\|_{H_1}^2 \right] dt \\ &\qquad - \lim_{n\to\infty}\left\langle \frac{du_n}{dt}, u_n\right\rangle_{H_1} + \langle \psi, \varphi\rangle_{H_1}.\end{aligned} \tag{3.14}$$

In view of (3.9), (3.12), we obtain for all $v \in \bigcup_{n=1}^\infty \tilde{X}_n$

$$\begin{aligned}\lim_{n\to\infty}\int_0^t \langle Gu_n, Lv\rangle\, dt &= \int_0^t \left[k\left\langle \frac{du_0}{dt}, v\right\rangle_{H_2} - \left\langle \frac{du_0}{dt}, \frac{dv}{dt}\right\rangle_{H_1}\right] dt \\ &\quad + \left\langle \frac{du_0}{dt}, v\right\rangle_{H_1} - \langle \psi, v(0)\rangle_{H_1}.\end{aligned} \tag{3.15}$$

Since $\bigcup_{n=1}^{\infty}\tilde{X}_n$ is dense in $W^{1,2}((0,T),H_2)\cap L^p((0,T),X_2)$, $\forall p<\infty$, (3.15) holds for all $v\in W^{1,2}((0,T),H_2)\cap L^p((0,T),X_2)$. Thus, we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^t\langle Gu_n,Lu_0\rangle\,dt=\int_0^t\left[k\left\langle\frac{du_0}{dt},u_0\right\rangle_{H_2}-\left\|\frac{du_0}{dt}\right\|_{H_1}^2\right]dt\\+\left\langle\frac{du_0}{dt},u_0\right\rangle_{H_1}-\langle\psi,\varphi\rangle_{H_1}.\end{aligned}\tag{3.16}$$

From (3.12) and $H_2\subset H_1$ is compact imbedding, it follows that

$$\lim_{n\to\infty}\int_0^t\left\|\frac{du_n}{dt}\right\|_{H_1}^2dt=\int_0^t\left\|\frac{du_0}{dt}\right\|_{H_1}^2dt,$$

$$\lim_{n\to\infty}\left\langle\frac{du_n}{dt},u_n\right\rangle_{H_1}=\left\langle\frac{du_0}{dt},u_0\right\rangle_{H_1},\quad\text{a.e. }t\ge0.$$

Clearly,

$$\lim_{n\to\infty}\int_0^t\left\langle\frac{du_n}{dt},u_n\right\rangle_{H_1}dt=\int_0^t\left\langle\frac{du_0}{dt},u_0\right\rangle_{H_1}dt.$$

Then (3.13) follows from (3.14)-(3.16), which imply assertion (1).

Secondly, we consider $G=A+B$. Let $v=\frac{d}{dt}Lu_n$ in (3.9). In view of (2.2) and (3.1), it follows that

$$\frac{1}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2+k\int_0^t\left\|\frac{du_n}{dt}\right\|_{H_2}^2dt+F(u_n)=\int_0^t\left\langle B(u_n),\frac{d}{dt}Lu_n\right\rangle dt+F(\varphi_n)+\frac{1}{2}\|\psi_n\|_{H_1}^2.$$

From (3.3), we have

$$\frac{1}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2+F(u_n)+k\int_0^t\left\|\frac{du_n}{dt}\right\|_{H_2}^2dt\le C\int_0^t\left[F(u_n)+\frac{1}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2\right]dt+f(t),\tag{3.17}$$

where $f(t)=\int_0^t g(\tau)\,d\tau+\frac{1}{2}\|\psi\|_{H_1}^2+\sup_n F(\varphi_n)$.

By using the Gronwall inequality, it follows that

$$\frac{1}{2}\left\|\frac{du_n}{dt}\right\|_{H_1}^2+F(u_n)\le f(0)e^{Ct}+\int_0^t f(\tau)e^{C(t-\tau)}\,d\tau,\tag{3.18}$$

which implies that for all $0<T<\infty$,

$$\{u_n\}\subset W^{1,\infty}\big((0,T),H_1\big)\cap L^\infty\big((0,T),X_2\big)\quad\text{is bounded.}$$

From (3.17) and (3.18), it follows that

$$\{u_n\}\subset W^{1,2}\big((0,T),H_2\big)\quad\text{is bounded.}$$

Let

$$\begin{cases}u_n\rightharpoonup{}^*u_0&\text{in }W^{1,\infty}((0,T),H_1)\cap L^\infty((0,T),X_2),\\u_n\rightharpoonup u_0&\text{in }W^{1,2}((0,T),H_2),\end{cases}\tag{3.19}$$

which implies that $u_n \to u_0$ in $W^{1,2}((0,T),H)$ is uniformly weakly convergent from that $H_2 \subset H$ is an compact imbedding.

The left proof is same as assertion (1).

Lastly, assume (3.6) holds. Let $v = \frac{d^2u_n}{d^2t}$ in (3.9). It follows that

$$\begin{aligned}
&\int_0^t \left\langle \frac{d^2u_n}{dt^2}, \frac{d^2u_n}{dt^2} \right\rangle_H dt + \frac{k}{2} \left\| \frac{du_n}{dt} \right\|_{H_1}^2 \\
&\quad \le \frac{k}{2} \|\psi_n\|_H^2 + \int_0^t \left[\frac{1}{2} \left\| \frac{d^2u_n}{dt^2} \right\|_H^2 + CF(u_n) + g(\tau) \right] d\tau.
\end{aligned}$$

From (3.18), the above inequality implies

$$\int_0^t \left\| \frac{d^2u_n}{dt^2} \right\|_H^2 d\tau \le C \quad (C > 0 \text{ is constant}). \tag{3.20}$$

We see that for all $0 < T < \infty$, $\{u_n\} \subset W^{2,2}((0,T),H)$ is bounded. Thus, $u \in W^{2,2}((0,T),H)$. □

4 Main result

We consider the strongly damped quasilinear wave equations (1.1). We give the following assumption for (1.1). There exists an C^1 function $F(x,\zeta)$, where $\zeta = \{\zeta_\alpha | |\alpha| \le m\}$, ζ_α corresponds to $D^\alpha u$ such that

$$A_\alpha(x,\zeta) = \frac{\partial}{\partial \zeta_\alpha} F(x,\zeta), \tag{4.1}$$

$$F(x,\zeta) \ge C_1 \sum_{|\beta|=m} |\zeta_\beta|^p - C_2, \quad p \ge 2, \tag{4.2}$$

$$\sum_{|\beta|=m} \left[A_\beta(x,\xi,\eta_1) - A_\beta(x,\xi,\eta_2)\right](\eta_{1\beta} - \eta_{2\beta}) \ge \lambda |\eta_1 - \eta_2|^2, \tag{4.3}$$

where $\lambda > 0$, $\eta = \{\eta_\beta | |\beta| = m\}$, $\xi = \{\xi_\alpha | |\alpha| \le m-1\}$,

$$\left|A_\alpha(x,\zeta)\right| \le C\left(\sum_{|\alpha|\le m} |\zeta_\alpha|^{p-1} + 1\right), \tag{4.4}$$

$$\left|g(x,\zeta)\right| \le C\left(\sum_{|\beta|\le m} |\zeta_\beta|^{\frac{p}{2}} + 1\right). \tag{4.5}$$

Definition 4.1 We say $u \in W^{1,2}_{\text{loc}}((0,\infty), L^2(\Omega)) \cap L^\infty_{\text{loc}}((0,\infty), W_0^{m,p}(\Omega))$ is the weak solution of (1.1), if $u(0) = \varphi$, and for $\forall v \in C_0^\infty(\Omega)$, the following equality holds:

$$\begin{aligned}
&\int_\Omega \frac{\partial u}{\partial t} v\,dx + k \int_\Omega \nabla u \nabla v\,dx \\
&\quad = -\int_0^t \int_\Omega \sum_{|\alpha|\le m} D^\alpha A_\alpha\big(x,u,\ldots,D^m u\big) D^\alpha v\,dx\,d\tau \\
&\qquad + \int_0^t \int_\Omega g\big(x,u,\ldots,D^m u\big) v\,dx\,dt + \int_\Omega \psi v\,dx + k \int_\Omega \nabla\varphi \nabla v\,dx.
\end{aligned} \tag{4.6}$$

Theorem 4.2 *Under conditions (4.1)-(4.5), for* $(\varphi,\psi) \in W_0^{m,p}(\Omega) \times L^p(\Omega)$, *there exists a global weak solution for (1.1)*

$$u \in L^\infty_{\mathrm{loc}}\big((0,\infty), W_0^{m,p}(\Omega)\big),$$
$$u_t \in L^\infty_{\mathrm{loc}}\big((0,\infty), L^2(\Omega)\big) \cap L^2_{\mathrm{loc}}\big((0,\infty), H_0^1(\Omega)\big).$$

Proof We introduce spatial sequences

$$X = C_0^\infty(\Omega), \qquad X_1 = X_2 = W_0^{m,p}(\Omega),$$
$$H = H_1 = L^2(\Omega), \qquad H_2 = H_0^1(\Omega),$$
$$L = id : X \to X_1, \qquad \mathcal{L} = -\Delta u.$$

Define map $G = A + B : X_2 \to X_1^*$ by

$$\langle Au, v\rangle = -\int_\Omega \sum_{|\alpha|\le m} A_\alpha\big(x,u,\dots,D^m u\big) D^\alpha v\,dx,$$
$$\langle Bu, v\rangle = \int_\Omega g\big(x,u,\dots,D^m u\big) v\,dx.$$

We show that $G = A + B : X_2 \to X_1^*$ is T-coercively weakly continuous. Let $\{u_n\} \subset L^\infty(0,T,W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))$ satisfying (2.7) and

$$\lim_{n\to\infty}\int_0^T\int_\Omega\bigg[\bigg(\sum_{|\alpha|\le m} A_\alpha\big(x,u_n,\dots,D^m u_n\big) - \sum_{|\alpha|\le m} A_\alpha\big(x,u_0,\dots,D^m u_0\big)\bigg)\big(D^\alpha u_n - D^\alpha u_0\big)$$
$$+ \big(g\big(x,u_n,\dots,D^m u_n\big) - g\big(x,u_0,\dots,D^m u_0\big)\big)(u_n - u_0)\bigg]\,dx\,dt = 0. \tag{4.7}$$

We need prove that

$$\lim_{n\to\infty}\int_0^T\int_\Omega\bigg[\sum_{|\alpha|\le m} A_\alpha\big(x,u_n,\dots,D^m u_n\big) + g\big(x,u_n,\dots,D^m u_n\big)\bigg] v\,dx\,dt$$
$$= \int_0^T\int_\Omega\bigg[\sum_{|\alpha|\le m} A_\alpha\big(x,u_0,\dots,D^m u_0\big) + g\big(x,u_0,\dots,D^m u_0\big)\bigg] v\,dx\,dt. \tag{4.8}$$

From (2.7) and Lemma 2.4, we obtain

$$u_n \to u_0, \qquad Du_n \to Du_0,\dots, \qquad D^{m-1}u_n \to D^{m-1}u_0, \quad \text{in } L^2\big((0,T)\times\Omega\big). \tag{4.9}$$

We have the deformation

$$\int_0^T\int_\Omega\bigg[\bigg(\sum_{|\alpha|\le m} A_\alpha\big(x,u_n,\dots,D^m u_n\big) - \sum_{|\alpha|\le m} A_\alpha\big(x,u_0,\dots,D^m u_0\big)\bigg)\big(D^\alpha u_n - D^\alpha u_0\big)\bigg]\,dx\,dt$$
$$+ \int_0^T\int_\Omega\big[g\big(x,u_n,\dots,D^m u_n\big) - g\big(x,u_0,\dots,D^m u_0\big)\big](u_n - u_0)\,dx\,dt$$

$$
\begin{aligned}
&= \int_0^T \int_\Omega \biggl[\biggl(\sum_{|\alpha|\le m} A_\alpha \bigl(x, u_n, \ldots, D^{m-1}u_n, D^m u_0\bigr) \\
&\quad - \sum_{|\alpha|\le m} A_\alpha \bigl(x, u_0, \ldots, D^{m-1}u_0, D^m u_0\bigr) \biggr) \bigl(D^\alpha u_n - D^\alpha u_0\bigr) \biggr] \, dx\, dt \\
&\quad + \int_0^T \int_\Omega \biggl[\biggl(\sum_{|\alpha|\le m} A_\alpha \bigl(x, u_n, \ldots, D^{m-1}u_n, D^m u_n\bigr) \\
&\quad - \sum_{|\alpha|\le m} A_\alpha \bigl(x, u_n, \ldots, D^{m-1}u_n, D^m u_0\bigr) \biggr) \bigl(D^\alpha u_n - D^\alpha u_0\bigr) \biggr] \, dx\, dt \\
&\quad + \int_0^T \int_\Omega \bigl[g\bigl(x, u_n, \ldots, D^m u_n\bigr) - g\bigl(x, u_0, \ldots, D^m u_0\bigr) \bigr] (u_n - u_0) \, dx\, dt. \qquad (4.10)
\end{aligned}
$$

From (4.9), (4.4), (4.5), and Lemma 2.5, we have

$$
\lim_{n\to\infty} \int_0^T \int_\Omega \bigl[g\bigl(x, u_n, \ldots, D^m u_n\bigr) - g\bigl(x, u_0, \ldots, D^m u_0\bigr) \bigr] (u_n - u_0) \, dx\, dt = 0, \qquad (4.11)
$$

$$
\begin{aligned}
&\int_0^T \int_\Omega \biggl[\sum_{|\alpha|\le m} A_\alpha \bigl(x, u_n, \ldots, D^{m-1}u_n, D^m u_0\bigr) \\
&\quad - \sum_{|\alpha|\le m} A_\alpha \bigl(x, u_0, \ldots, D^{m-1}u_0, D^m u_0\bigr) \biggr] \bigl(D^\alpha u_n - D^\alpha u_0\bigr) \, dx\, dt = 0. \qquad (4.12)
\end{aligned}
$$

From (4.7), (4.3), (4.10)-(4.12), it follows that

$$
\begin{aligned}
0 &= \int_0^T \int_\Omega \biggl[\biggl(\sum_{|\alpha|\le m} A_\alpha \bigl(x, u_n, \ldots, D^{m-1}u_n, D^m u_n\bigr) \\
&\quad - \sum_{|\alpha|\le m} A_\alpha \bigl(x, u_n, \ldots, D^{m-1}u_n, D^m u_0\bigr) \biggr) \bigl(D^\alpha u_n - D^\alpha u_0\bigr) \biggr] \, dx\, dt \\
&\ge \lambda \int_0^T \int_\Omega \bigl| D^m u_n - D^m u_0 \bigr|^2 \, dx\, dt.
\end{aligned}
$$

Since $\lambda > 0$, we have

$$
\lim_{n\to\infty} \int_0^T \int_\Omega \bigl| D^m u_n - D^m u_0 \bigr|^2 \, dx\, dt = 0. \qquad (4.13)
$$

From (4.9), (4.13), (4.4), (4.5), and Lemma 2.5, we get (4.8). Hence, $G = A + B : X_2 \to X_1^*$ is T-coercively weakly continuous.

From (4.1) and (4.2), we get

$$
\begin{aligned}
&\langle Au, Lu\rangle = -\bigl\langle DF(x,\zeta), v\bigr\rangle, \\
&F(x,u) \to \infty \quad \Leftrightarrow \quad \|u\|_{X_2} \to \infty,
\end{aligned}
$$

which imply conditions (A1), (A2) of Theorem 3.1.

We will show (3.3) as follows. It follows that

$$\begin{aligned}\bigl|\langle Bu, Lv\rangle\bigr| &= \int_\Omega \bigl|g(x,u,\ldots,D^m u)\bigr||v|\,dx \\ &\le \frac{k}{2}\int_\Omega |v|^2\,dx + \frac{2}{k}\int_\Omega \bigl|g(x,u,\ldots,D^m u)\bigr|^2\,dx \\ &\le \frac{k}{2}\|v\|_{H_2}^2 + C\int_\Omega \biggl[\sum_{|\alpha|\le m} |\zeta|^{\frac{p}{2}} + 1\biggr]^2\,dx \\ &\le \frac{k}{2}\|v\|_{H_2}^2 + CF(u) + C,\end{aligned}$$

which implies condition (A3) of Lemma 2.4. From Lemma 2.4, Eq. (1.1) has a solution

$$\begin{aligned}&u \in L^\infty_{\mathrm{loc}}\bigl((0,\infty), W_0^{m,p}(\Omega)\bigr), \\ &u_t \in L^\infty_{\mathrm{loc}}\bigl((0,\infty), L^2(\Omega)\bigr) \cap L^2_{\mathrm{loc}}\bigl((0,\infty), H_0^1(\Omega)\bigr),\end{aligned}$$

satisfying

$$\begin{aligned}&\int_\Omega \frac{\partial u}{\partial t} v\,dx + k\int_\Omega \nabla u \nabla v\,dx \\ &\quad = -\int_0^t \int_\Omega \sum_{|\alpha|\le m} D^\alpha A_\alpha\bigl(x,u,\ldots,D^m u\bigr) D^\alpha v\,dx\,d\tau \\ &\qquad + \int_0^t \int_\Omega g\bigl(x,u,\ldots,D^m u\bigr) v\,dx\,dt + \int_\Omega \psi v\,dx + k\int_\Omega \nabla\varphi\nabla v\,dx.\end{aligned}$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read and approved the final manuscript.

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