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Homomorphisms and derivations in C^*-ternary algebras *via* fixed point method

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Abstract

Park (J. Math. Phys. 47:103512, 2006) proved the Hyers-Ulam stability of homomorphisms in C^*-ternary algebras and of derivations on C^*-ternary algebras for the following generalized Cauchy-Jensen additive mapping:

$$2f\left(\frac{\sum_{j=1}^{p} x_j}{2} + \sum_{j=1}^{d} y_j\right) = \sum_{j=1}^{p} f(x_j) + 2\sum_{j=1}^{d} f(y_j).$$

In this paper, we improve and generalize some results concerning this functional equation *via* the fixed-point method.

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1 Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1 (Th.M. Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [6] following the same approach as in Rassias [4], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6], as well as by Rassias and Šemrl [7] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [6], as well as of Rassias and Šemrl [7] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [8], Jung [9], who among others studied the Hyers-Ulam stability of functional equations. The inequality (1.1) that was introduced for the first time by Rassias [4] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept (cf. the books of Czerwik [10], Hyers, Isac, and Rassias [11]).

Following the terminology of [12], a nonempty set G with a ternary operation $[\cdot,\cdot,\cdot] : G \times G \times G \to G$ is called a *ternary groupoid* and is denoted by $(G, [\cdot,\cdot,\cdot])$. The ternary groupoid $(G, [\cdot,\cdot,\cdot])$ is called *commutative* if $[x_1, x_2, x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ for all $x_1, x_2, x_3 \in G$ and all permutations σ of $\{1, 2, 3\}$.

If a binary operation $\circ$ is defined on G such that $[x, y, z] = (x \circ y) \circ z$ for all $x, y, z \in G$, then we say that $[\cdot,\cdot,\cdot]$ is derived from $\circ$. We say that $(G, [\cdot,\cdot,\cdot])$ is a *ternary semigroup* if the operation $[\cdot,\cdot,\cdot]$ is *associative*, *i.e.*, if $[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$ holds for all $x, y, z, u, v \in G$ (see [13]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which are $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [12, 14]). Every left Hilbert C^*-module is a C^*-ternary algebra *via* the ternary product $[x, y, z] := \langle x, y\rangle z$.

If a C^*-ternary algebra $(A, [\cdot,\cdot,\cdot])$ has an identity, *i.e.*, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary algebra homomorphism* if

$$H\bigl([x, y, z]\bigr) = \bigl[H(x), H(y), H(z)\bigr]$$

for all $x, y, z \in A$. If, in addition, the mapping H is bijective, then the mapping $H : A \to B$ is called a *C^*-ternary algebra isomorphism*. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta\bigl([x, y, z]\bigr) = \bigl[\delta(x), y, z\bigr] + \bigl[x, \delta(y), z\bigr] + \bigl[x, y, \delta(z)\bigr]$$

for all $x, y, z \in A$ (see [12, 15]).

There are some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang-Baxter equation (cf. [16–18]).

Throughout this paper, assume that p, d are nonnegative integers with $p + d \geq 3$, and that A and B are C^*-ternary algebras.

The aim of the present paper is to establish the stability problem of homomorphisms and derivations in C^*-ternary algebras by using the fixed-point method.

Let E be a set. A function $d : E \times E \to [0,1]$ is called a generalized metric on E if d satisfies

(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in E$;
(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in E$.

Theorem 1.2 *Let (E,d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with constant $L < 1$. Then for each given element $x \in E$, either*

$$d\big(J^n x, J^{n+1} x\big) = \infty$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
(2) the sequence $J^n x$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = y \in E : d(J^{n_0}, y) < \infty$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

2 Stability of homomorphisms in C^*-ternary algebras

Throughout this section, assume that A is a unital C^*-ternary algebra with norm $\|\cdot\|$ and unit e, and that B is a unital C^*-ternary algebra with norm $\|\cdot\|$ and unit e'.

The stability of homomorphisms in C^*-ternary algebras has been investigated in [19] *via* direct method. In this note, we improve some results in [19] *via* the fixed-point method. For a given mapping $f : A \to B$, we define

$$C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) := 2f\left(\frac{\sum_{j=1}^{p} \mu x_j}{2} + \sum_{j=1}^{d} \mu y_j\right) - \sum_{j=1}^{p} \mu f(x_j) - 2\sum_{j=1}^{d} \mu f(y_j)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x_1, \dots, x_p, y_1, \dots, y_d \in A$.

One can easily show that a mapping $f : A \to B$ satisfies

$$C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \dots, x_p, y_1, \dots, y_d \in A$ if and only if

$$f(\mu x + \lambda y) = \mu f(x) + \lambda f(y)$$

for all $\mu, \lambda \in \mathbb{T}^1$ and all $x, y \in A$.

We will use the following lemma in this paper.

Lemma 2.1 ([20]) *Let $f : A \to B$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.2 *Let $\{x_n\}_n$, $\{y_n\}_n$ and $\{z_n\}_n$ be convergent sequences in A. Then the sequence $\{[x_n, y_n, z_n]\}$ is convergent in A.*

Proof Let $x, y, z \in A$ such that

$$\lim_{n\to\infty} x_n = x, \qquad \lim_{n\to\infty} v_n = y, \qquad \lim_{n\to\infty} z_n = z.$$

Since

$$\begin{aligned}[x_n, y_n, z_n] - [x, y, z] = [x_n - x, y_n - y, z_n, z] + [x_n, y_n, z] \\ + [x, y_n - y, z_n] + [x_n, y, z_n - z]\end{aligned}$$

for all n, we get

$$\begin{aligned}\big\|[x_n, y_n, z_n] - [x, y, z]\big\| = \|x_n - x\|\|y_n - y\|\|z_n - z\| + \|x_n - x\|\|y_n\|\|z\| \\ + \|x\|\|y_n - y\|\|z_n\| + \|x_n\|\|y\|\|z_n - z\|\end{aligned}$$

for all n. So

$$\lim_{n\to\infty} [x_n, y_n, z_n] = [x, y, z].$$

This completes the proof. □

Theorem 2.3 *Let $f : A \to B$ be a mapping for which there exist functions $\varphi : A^{p+d} \to [0,\infty)$ and $\psi : A^3 \to [0,\infty)$ such that*

$$\lim_{n\to\infty} \gamma^{-n}\varphi\big(\gamma^n x_1, \ldots, \gamma^n x_p, \gamma^n y_1, \ldots, \gamma^n y_d\big) = 0,$$

$$\lim_{n\to\infty} \gamma^{-3n}\psi\big(\gamma^n x, \gamma^n y, \gamma^n z\big) = 0,$$

$$\big\|C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d)\big\| \le \varphi(x_1, \ldots, x_p, y_1, \ldots, y_d), \tag{2.1}$$

$$\big\|f[x, y, z] - \big[f(x), f(y), f(z)\big]\big\| \le \psi(x, y, z) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \ldots, x_p, y_1, \ldots, y_d \in A$, where $\gamma = \frac{p+2d}{2}$. If there exists constant $L < 1$ such that

$$\varphi(\gamma x, \ldots, \gamma x) \le \gamma L \varphi(x, \ldots, x)$$

for all $x \in A$, then there exists a unique C^*-ternary algebras homomorphism $H : A \to B$ satisfying

$$\big\|f(x) - H(x)\big\| \le \frac{1}{(1-L)2\gamma}\varphi(x, \ldots, x) \tag{2.3}$$

for all $x \in A$.

Proof Let us assume $\mu = 1$ and $x_1 = \cdots = x_p = y_1 = \cdots = y_d = x$ in (2.1). Then we get

$$\big\|f(\gamma x) - \gamma f(x)\big\| \le \frac{1}{2}\varphi(x, \ldots, x) \tag{2.4}$$

for all $x \in A$. Let $E := \{g : A \to B\}$. We introduce a generalized metric on E as follows:

$$d(g,h) := \inf\{C \in [0,\infty] : \|g(x) - h(x)\| \le C\varphi(x,\dots,x) \text{ for all } x \in A\}.$$

It is easy to show that (E,d) is a generalized complete metric space.

Now, we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = \frac{1}{\gamma} g(\gamma x), \quad \text{for all } g \in E \text{ and } x \in A.$$

Let $g, h \in E$ and let $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le C$. From the definition of d, we have

$$\|g(x) - h(x)\| \le C\varphi(x,\dots,x)$$

for all $x \in A$. By the assumption and the last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = \frac{1}{\gamma}\|g(\gamma x) - h(\gamma x)\| \le \frac{C}{\gamma}\varphi(\gamma x,\dots,\gamma x) \le CL\varphi(x,\dots,x)$$

for all $x \in A$. So $d(\Lambda g, \Lambda h) \le Ld(g,h)$ for any $g, h \in E$. It follows from (2.4) that $d(\Lambda f, f) \le \frac{1}{2\gamma}$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point H of Λ, *i.e.*,

$$H : A \to B, \quad H(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} \frac{1}{\gamma^n} f(\gamma^n x) \tag{2.5}$$

and $H(\gamma x) = \gamma H(x)$ for all $x \in A$. Also H is the unique fixed point of Λ in the set $E = \{g \in E : d(f,g) < \infty\}$ and

$$d(H,f) \le \frac{1}{1-L} d(\Lambda f, f) \le \frac{1}{(1-L)2\gamma}$$

i.e., the inequality (2.3) holds true for all $x \in A$. It follows from the definition of H that

$$\begin{aligned}
&\left\| 2H\left(\frac{\sum_{j=1}^p \mu x_j}{2} + \sum_{j=1}^d \mu y_j\right) - \sum_{j=1}^p \mu H(x_j) - 2\sum_{j=1}^d \mu H(y_j)\right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{\gamma^n}\left\| 2f\left(\gamma^n \frac{\sum_{j=1}^p \mu x_j}{2} + \gamma^n \sum_{j=1}^d \mu y_j\right) - \sum_{j=1}^p \mu f(\gamma^n x_j) - 2\sum_{j=1}^d \mu f(\gamma^n y_j)\right\| \\
&\quad \le \lim_{n\to\infty} \gamma^{-n}\varphi(\gamma^n x_1,\dots,\gamma^n x_p, \gamma^n y_1,\dots,\gamma^n y_d) = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1,\dots,x_p, y_1,\dots,y_d \in A$. Hence

$$2H\left(\frac{\sum_{j=1}^p \mu x_j}{2} + \sum_{j=1}^d \mu y_j\right) = \sum_{j=1}^p \mu H(x_j) + 2\sum_{j=1}^d \mu H(y_j)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1,\dots,x_p, y_1,\dots,y_d \in A$. So $H(\lambda x + \mu y) = \lambda H(x) + \mu H(y)$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y \in A$.

Therefore, by Lemma 2.1, the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) and (2.5) that

$$\begin{aligned}
&\big\|H\big([x,y,z]\big) - \big[H(x),H(y),H(z)\big]\big\| \\
&\quad = \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \big\|f\big(\big[\gamma^n x,\gamma^n y,\gamma^n z\big]\big) - \big[f\big(\gamma^n x\big),f\big(\gamma^n y\big),f\big(\gamma^n z\big)\big]\big\| \\
&\quad \le \lim_{n\to\infty} \gamma^{-3n}\psi\big(\gamma^n x,\gamma^n y,\gamma^n z\big) = 0
\end{aligned}$$

for all $x,y,z \in A$. Thus

$$H\big([x,y,z]\big) = \big[H(x),H(y),H(z)\big]$$

for all $x,y,z \in A$. Therefore, the mapping H is a C^*-ternary algebras homomorphism.

Now, let $T : A \to B$ be another C^*-ternary algebras homomorphism satisfying (2.3). Since $d(f,T) \le \frac{1}{(1-L)2\gamma}$ and T is $\mathbb{C}$-linear, we get $T \in E'$ and $(\Lambda T)(x) = \frac{1}{\gamma}(T\gamma x) = T(x)$ for all $x \in A$, *i.e.*, T is a fixed point of Λ. Since H is the unique fixed point of $\Lambda \in E'$, we get $H = T$. □

Theorem 2.4 *Let $f : A \to B$ be a mapping for which there exist functions $\varphi : A^{p+d} \to [0,\infty)$ and $\psi : A^3 \to [0,\infty)$ satisfying (2.1), (2.2),*

$$\begin{aligned}
&\lim_{n\to\infty} \gamma^n \varphi\left(\frac{x_1}{\gamma^n},\dots,\frac{x_p}{\gamma^n},\frac{y_1}{\gamma^n},\dots,\frac{y_d}{\gamma^n}\right) = 0, \\
&\lim_{n\to\infty} \gamma^{3n}\psi\left(\frac{x}{\gamma^n},\frac{y}{\gamma^n},\frac{z}{\gamma^n}\right) = 0,
\end{aligned}$$

for all $x,y,z,x_1,\dots,x_p,y_1,\dots,y_d \in A$, where $\gamma = \frac{p+2d}{2}$. If there exists constant $L < 1$ such that

$$\varphi\left(\frac{1}{\gamma}x,\dots,\frac{1}{\gamma}x\right) \le \frac{1}{\gamma}L\varphi(x,\dots,x)$$

for all $x \in A$, then there exists a unique C^*-ternary algebras homomorphism $H : A \to B$ satisfying

$$\big\|f(x) - H(x)\big\| \le \frac{1}{(1-L)2\gamma}\varphi(x,\dots,x)$$

for all $x \in A$.

Proof If we replace x in (2.4) by $\frac{x}{\gamma}$, then we get

$$\left\|f(x) - \gamma f\left(\frac{x}{\gamma}\right)\right\| \le \frac{1}{2}\varphi\left(\frac{x}{\gamma},\dots,\frac{x}{\gamma}\right) \le \frac{L}{2\gamma}\varphi(x,\dots,x) \tag{2.6}$$

for all $x \in A$. Let $E := \{g : A \to A\}$. We introduce a generalized metric on E as follows:

$$d(g,h) := \inf\big\{C \in [0,\infty] : \big\|g(x) - h(x)\big\| \le C\varphi(x,\dots,x) \text{ for all } x \in A\big\}.$$

It is easy to show that (E,d) is a generalized complete metric space.

Now, we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = \gamma g\left(\frac{x}{\gamma}\right), \quad \text{for all } g \in E \text{ and } x \in A.$$

Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C$. From the definition of d, we have

$$\|g(x) - h(x)\| \le C\varphi(x, \dots, x)$$

for all $x \in A$. By the assumption and the last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = \left\|\gamma g\left(\frac{x}{\gamma}\right) - \gamma h\left(\frac{x}{\gamma}\right)\right\| \le \gamma C\varphi\left(\frac{x}{\gamma}, \dots, \frac{x}{\gamma}\right) \le CL\varphi(x, \dots, x)$$

for all $x \in A$, and so $d(\Lambda g, \Lambda h) \le Ld(g, h)$ for any $g, h \in E$. It follows from (2.6) that $d(\Lambda f, f) \le \frac{1}{2\gamma}$. Thus, according to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point H of Λ, *i.e.*,

$$H : A \to B, \quad H(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} \gamma^n f\left(\frac{x}{\gamma^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.3, and we omit it. □

Corollary 2.5 ([19]) *Let r and θ be nonnegative real numbers such that $r \notin [1, 3]$, and let $f : A \to B$ be a mapping such that*

$$\|C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d)\| \le \theta\left(\sum_{j=1}^{p} \|x_j\|^r + \sum_{j=1}^{d} \|y_j\|^r\right) \tag{2.7}$$

and

$$\left\|f([x, y, z]) - [f(x), f(y), f(z)]\right\| \le \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right) \tag{2.8}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \dots, x_p, y_1, \dots, y_d \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{2^r(p + d)\theta}{|2(p + 2d)^r - (p + 2d)2^r|}\|x\|^r \tag{2.9}$$

for all $x \in A$.

Proof The proof follows from Theorems 2.3 and 2.4 by taking

$$\varphi(x_1, \dots, x_p, y_1, \dots, y_d) := \theta\left(\sum_{j=1}^{p} \|x_j\|^r + \sum_{j=1}^{d} \|y_j\|^r\right),$$

$$\psi(x, y, z) := \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \dots, x_p, y_1, \dots, y_d \in A$. Then we can choose $L = 2^{1-r}(p+2d)^{r-1}$, when $0 < r < 1$ and $L = 2 - 2^{1-r}(p+2d)^{r-1}$, when $r > 3$ and we get the desired results. $\square$

3 Superstability of homomorphisms in C^*-ternary algebras

Throughout this section, assume that A is a unital C^*-ternary algebra with norm $\|\cdot\|$ and unit e, and that B is a unital C^*-ternary algebra with norm $\|\cdot\|$ and unit e'.

We investigate homomorphisms in C^*-ternary algebras associated with the functional equation $C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) = 0$.

Theorem 3.1 ([19]) *Let $r > 1$ (*resp.*, $r < 1$) and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying (2.1) and*

$$f([x,y,z]) = [f(x), f(y), f(z)]$$

for all $x, y, z \in A$. If $\lim_{n\to\infty} \frac{(p+2d)^n}{2^n} f(\frac{2^n e}{(p+2d)^n}) = e'$ (resp., $\lim_{n\to\infty} \frac{2^n}{(p+2d)^n} f(\frac{(p+2d)^n}{2^n} e) = e'$), then the mapping $f : A \to B$ is a C^*-ternary algebra isomorphism.

In the following theorems we have alternative results of Theorem 3.1.

Theorem 3.2 *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (2.7) and (2.8). If there exist a real number $\lambda > 1$ (*resp.*, $0 < \lambda < 1$) and an element $x_0 \in A$ such that $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$ (resp., $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$), then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism.*

Proof By using the proof of Corollary 2.5, there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ satisfying (2.9). It follows from (2.9) that

$$H(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x), \qquad \left(H(x) = \lim_{n\to\infty} \lambda^n f\left(\frac{x}{\lambda^n}\right)\right)$$

for all $x \in A$ and all real numbers $\lambda > 1$ $(0 < \lambda < 1)$. Therefore, by the assumption, we get that $H(x_0) = e'$.

Let $\lambda > 1$ and $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$. It follows from (2.8) that

$$\begin{aligned}
&\left\| [H(x), H(y), H(z)] - [H(x), H(y), f(z)] \right\| \\
&\quad = \left\| H[x,y,z] - [H(x), H(y), f(z)] \right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{\lambda^{3n}} \left\| f([\lambda^n x, \lambda^n y, \lambda^n z]) - [f(\lambda^n x), f(\lambda^n y), f(\lambda^n z)] \right\| \\
&\quad \le \lim_{n\to\infty} \frac{\lambda^{rn}}{\lambda^{3n}} \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right) = 0
\end{aligned}$$

for all $x \in A$. So $[H(x), H(y), H(z)] = [H(x), H(y), f(z)]$ for all $x, y, z \in A$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in A$. Similarly, one can show that $H(x) = f(x)$ for all $x \in A$ when $0 < \lambda < 1$ and $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$.

Similarly, one can show the theorem for the case $\lambda > 1$.

Therefore, the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism. $\square$

Theorem 3.3 *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (2.7) and (2.8). If there exist a real number $0 < \lambda < 1$ (resp., $\lambda > 1$) and an element $x_0 \in A$ such that $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$ (resp., $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$), then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism.*

Proof The proof is similar to the proof of Theorem 3.2 and we omit it. □

4 Stability of derivations on C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|$.

Park [19] proved the Hyers-Ulam stability of derivations on C^*-ternary algebras for the functional equation $C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) = 0$.

For a given mapping $f : A \to A$, let

$$\mathbf{D}f(x,y,z) = f\big([x,y,z]\big) - \big[f(x),y,z\big] - \big[x,f(y),z\big] - \big[x,y,f(z)\big]$$

for all $x, y, z \in A$.

Theorem 4.1 ([19]) *Let r and θ be nonnegative real numbers such that $r \notin [1,3]$, and let $f : A \to A$ be a mapping satisfying (2.7) and*

$$\big\|\mathbf{D}f(x,y,z)\big\| \le \theta\big(\|x\|^r + \|y\|^r + \|z\|^r\big)$$

for all $x, y, z \in A$. Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that

$$\big\|f(x) - \delta(x)\big\| \le \frac{2^r(p+d)}{|2(p+2d)^r - (p+2d)2^r|}\theta\|x\|^r$$

for all $x \in A$.

In the following theorem, we generalize and improve the result in Theorems 4.1.

Theorem 4.2 *Let $\varphi : A^{p+d} \to [0,\infty)$ and $\psi : A^3 \to [0,\infty)$ be functions such that*

$$\lim_{n\to\infty} \gamma^{-n}\varphi\big(\gamma^n x_1, \dots, \gamma^n x_p, \gamma^n y_1, \dots, \gamma^n y_d\big) = 0, \tag{4.1}$$

$$\lim_{n\to\infty} \gamma^{-3n}\psi\big(\gamma^n x, \gamma^n y, \gamma^n z\big) = 0, \qquad \lim_{n\to\infty} \gamma^{-2n}\psi\big(\gamma^n x, \gamma^n y, z\big) = 0 \tag{4.2}$$

for all $x, y, z, x_1, \dots, x_p, y_1, \dots, y_d \in A$, where $\gamma = \frac{p+2d}{2}$. Suppose that $f : A \to A$ is a mapping satisfying

$$\big\|C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d)\big\| \le \varphi(x_1, \dots, x_p, y_1, \dots, y_d), \tag{4.3}$$

$$\big\|\mathbf{D}f(x,y,z)\big\|_A \le \psi(x,y,z) \tag{4.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \dots, x_p, y_1, \dots, y_d \in A$. If there exists a constant $L < 1$ such that

$$\varphi(\gamma x, \dots, \gamma x) \le \gamma\varphi(x, \dots, x),$$

then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof Let us assume $\mu = 1$ and $x_1 = \cdots = x_p = y_1 = \cdots = y_d = x$ in (4.3). Then we get

$$\|f(\gamma x) - \gamma f(x)\| \le \frac{1}{2}\varphi(x,\ldots,x) \tag{4.5}$$

for all $x \in A$. Let $E := \{g : A \to A\}$. We introduce a generalized metric on E as follows:

$$d(g,h) := \inf\bigl\{C \in [0,\infty] : \|g(x) - h(x)\| \le C\varphi(x,\ldots,x) \text{ for all } x \in A\bigr\}.$$

It is easy to show that (E,d) is a generalized complete metric space.

Now, we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = \frac{1}{\gamma}g(\gamma x), \quad \text{for all } g \in E \text{ and } x \in A.$$

Let $g,h \in E$ and let $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le C$. From the definition of d, we have

$$\|g(x) - h(x)\| \le C\varphi(x,\ldots,x)$$

for all $x \in A$. By the assumption and the last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = \frac{1}{\gamma}\|g(\gamma x) - h(\gamma x)\| \le \frac{C}{\gamma}\varphi(\gamma x,\ldots,\gamma x) \le CL\varphi(x,\ldots,x)$$

for all $x \in A$. Then $d(\Lambda g,\Lambda h) \le Ld(g,h)$ for any $g,h \in E$. It follows from (2.4) that $d(\Lambda f,f) \le \frac{1}{2\gamma}$. Thus according to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point δ of Λ, *i.e.*,

$$\delta : A \to A, \quad \delta(x) = \lim_{n\to\infty}\bigl(\Lambda^n f\bigr)(x) = \lim_{n\to\infty}\frac{1}{\gamma^n}f\bigl(\gamma^n x\bigr) \tag{4.6}$$

and $\delta(\gamma x) = \gamma\delta(x)$ for all $x \in A$. Also δ is the unique fixed point of Λ in the set $E = \{g \in E : d(f,g) < \infty\}$ and

$$d(\delta,f) \le \frac{1}{1-L}d(\Lambda f,f) \le \frac{1}{(1-L)2\gamma}$$

i.e., the inequality (2.3) holds true for all $x \in A$. It follows from the definition of δ, (4.1), (4.3), and (4.6) that

$$\begin{aligned}
&\|C_\mu\delta(x_1,\ldots,x_p,y_1,\ldots,y_d)\| \\
&\quad = \lim_{n\to\infty}\frac{1}{\gamma^n}\|C_\mu f\bigl(\gamma^n x_1,\ldots,\gamma^n x_p,\gamma^n y_1,\ldots,\gamma^n y_d\bigr)\| \\
&\quad \le \lim_{n\to\infty}\frac{1}{\gamma^n}\varphi\bigl(\gamma^n x_1,\ldots,\gamma^n x_p,\gamma^n y_1,\ldots,\gamma^n y_d\bigr) = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z,x_1,\ldots,x_p,y_1,\ldots,y_d \in A$. Hence,

$$2\delta\left(\frac{\sum_{j=1}^p \mu x_j}{2} + \sum_{j=1}^d \mu y_j\right) = \sum_{j=1}^p \mu\delta(x_j) + 2\sum_{j=1}^d \mu\delta(y_j)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \dots, x_p, y_1, \dots, y_d \in A$. So $\delta(\lambda x + \mu y) = \lambda\delta(x) + \mu\delta(y)$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y \in A$.

Therefore, by Lemma 2.1 the mapping $\delta : A \to A$ is $\mathbb{C}$-linear.

It follows from (4.2) and (4.4) that

$$\left\|\mathbf{D}\delta(x,y,z)\right\| = \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \left\|\mathbf{D}f\left(\gamma^n x, \gamma^n y, \gamma^n z\right)\right\| \le \lim_{n\to\infty} \frac{1}{\gamma^{3n}} \psi\left(\gamma^n x, \gamma^n y, \gamma^n z\right) = 0$$

for all $x, y, z \in A$. Hence

$$\delta\left([x,y,z]\right) = \left[\delta(x), y, z\right] + \left[x, \delta(y), z\right] + \left[x, y, \delta(z)\right] \tag{4.7}$$

for all $x, y, z \in A$. So the mapping $\delta : A \to A$ is a C^*-ternary derivation.

It follows from (4.2) and (4.4)

$$\begin{aligned}
&\left\|\delta[x,y,z] - \left[\delta(x), y, z\right] - \left[x, \delta(y), z\right] - \left[x, y, f(z)\right]\right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{\gamma^{2n}} \left\|f\left[\gamma^n x, \gamma^n y, z\right] - \left[f\left(\gamma^n x\right), \gamma^n y, z\right]\right. \\
&\qquad \left. - \left[\gamma^n x, f\left(\gamma^n y\right), z\right] - \left[\gamma^n x, \gamma^n y, f(z)\right]\right\| \\
&\quad \le \lim_{n\to\infty} \frac{1}{\gamma^{2n}} \psi\left(\gamma^n x, \gamma^n y, z\right) = 0
\end{aligned}$$

for all $x, y, z \in A$. Thus

$$\delta[x,y,z] = \left[\delta(x), y, z\right] + \left[x, \delta(y), z\right] + \left[x, y, f(z)\right] \tag{4.8}$$

for all $x, y, z \in A$. Hence, we get from (4.7) and (4.8) that

$$\left[x, y, \delta(z)\right] = \left[x, y, f(z)\right] \tag{4.9}$$

for all $x, y, z \in A$. Letting $x = y = f(z) - \delta(z)$ in (4.9), we get

$$\left\|f(z) - \delta(z)\right\|^3 = \left\|\left[f(z) - \delta(z), f(z) - \delta(z), f(z) - \delta(z)\right]\right\| = 0$$

for all $z \in A$. Hence, $f(z) = \delta(z)$ for all $z \in A$. So the mapping $f : A \to A$ is a C^*-ternary derivation, as desired. □

Corollary 4.3 *Let $r < 1$, $s < 2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (2.7) and*

$$\left\|\mathbf{D}f(x,y,z)\right\|_A \le \theta\left(\|x\|_A^s + \|y\|_A^s + \|z\|_A^s\right)$$

for all $x, y, z \in A$. Then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof Defining

$$\varphi(x_1, \dots, x_p, y_1, \dots, y_d) = \theta\left(\sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{d} \|y_j\|_A^r\right)$$

and

$$\psi(x,y,z) = \theta\left(\|x\|_A^s + \|y\|_A^s + \|z\|_A^s\right)$$

for all $x,y,z,x_1,\dots,x_p,y_1,\dots,y_d \in A$, and applying Theorem 4.2, we get the desired result. □

Theorem 4.4 *Let $\varphi : A^{p+d} \to [0,\infty)$ and $\psi : A^3 \to [0,\infty)$ be functions such that*

$$\lim_{n\to\infty} \gamma^n \varphi\left(\frac{x_1}{\gamma^n},\dots,\frac{x_p}{\gamma^n},\frac{y_1}{\gamma^n},\dots,\frac{y_d}{\gamma^n}\right) = 0,$$
$$\lim_{n\to\infty} \gamma^{3n} \psi\left(\frac{x}{\gamma^n},\frac{y}{\gamma^n},\frac{z}{\gamma^n}\right) = 0, \qquad \lim_{n\to\infty} \gamma^{2n} \psi\left(\frac{x}{\gamma^n},\frac{y}{\gamma^n},z\right) = 0$$

for all $x,y,z,x_1,\dots,x_p,y_1,\dots,y_d \in A$ where $\gamma = \frac{p+2d}{2}$. Suppose that $f : A \to A$ is a mapping satisfying (4.3) and (4.4). If there exists a constant $L < 1$ such that

$$\varphi\left(\frac{x}{\gamma},\dots,\frac{x}{\gamma}\right) \le \frac{L}{\gamma}\varphi(x,\dots,x),$$

then the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof If we replace x in (4.5) by $\frac{x}{\gamma}$, then we get

$$\left\| f(x) - \gamma f\left(\frac{x}{\gamma}\right)\right\|_A \le \frac{1}{2}\varphi\left(\frac{x}{\gamma},\dots,\frac{x}{\gamma}\right)$$

for all $x \in A$. Let $E := \{g : A \to A\}$. We introduce a generalized metric on E as follows:

$$d(g,h) := \inf\left\{C \in [0,\infty] : \left\|g(x) - h(x)\right\| \le C\varphi(x,\dots,x) \text{ for all } x \in A\right\}$$

It is easy to show that (E,d) is a generalized complete metric space.

Now, we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = \gamma g\left(\frac{x}{\gamma}\right), \quad \text{for all } g \in E \text{ and } x \in A.$$

Let $g,h \in E$ and let $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le C$. From the definition of d, we have

$$\left\|g(x) - h(x)\right\| \le C\varphi(x,\dots,x)$$

for all $x \in A$. By the assumption and last inequality, we have

$$\left\|(\Lambda g)(x) - (\Lambda h)(x)\right\| = \left\|\gamma g\left(\frac{x}{\gamma}\right) - \gamma h\left(\frac{x}{\gamma}\right)\right\| \le \gamma C\varphi\left(\frac{x}{\gamma},\dots,\frac{x}{\gamma}\right) \le CL\varphi(x,\dots,x)$$

for all $x \in A$. Then $d(\Lambda g,\Lambda h) \le Ld(g,h)$ for any $g,h \in E$. It follows from (4.5) that $d(\Lambda f,f) \le \frac{1}{2\gamma}$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point δ of Λ, *i.e.*,

$$\delta : A \to A, \quad \delta(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} \gamma^n f\left(\frac{x}{\gamma^n}\right)$$

and $\delta(\gamma x) = \gamma\delta(x)$ for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.2, and we omit it. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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References
1. Ulam, SM: A Collection of the Mathematical Problems. Interscience, New York (1960)
2. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
3. Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64-66 (1950)
4. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
5. Rassias, TM: Problem 16; 2. Report of the 27th international symp. on functional equations. Aequ. Math. **39**, 292-293 (1990)
6. Gajda, Z: On stability of additive mappings. Int. J. Math. Math. Sci. **14**, 431-434 (1991)
7. Rassias, TM, Šemrl, P: On the behaviour of mappings which do not satisfy Hyers-Ulam stability. Proc. Am. Math. Soc. **114**, 989-993 (1992)
8. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431-436 (1994)
9. Jung, S: On the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **204**, 221-226 (1996)
10. Czerwik, P: Functional Equations and Inequalities in Several Variables. World Scientific, Singapore (2002)
11. Hyers, DH, Isac, G, Rassias, TM: Stability of Functional Equations in Several Variables. Birkhäuser, Basel (1998)
12. Amyari, M, Moslehian, MS: Approximately ternary semigroup homomorphisms. Lett. Math. Phys. **77**, 1-9 (2006)
13. Bazunova, N, Borowiec, A, Kerner, R: Universal differential calculus on ternary algebras. Lett. Math. Phys. **67**, 195-206 (2004)
14. Zettl, H: A characterization of ternary rings of operators. Adv. Math. **48**, 117-143 (1983)
15. Moslehian, MS: Almost derivations on C^*-ternary rings. Bull. Belg. Math. Soc. Simon Stevin **14**, 135-142 (2007)
16. Kerner, R: Ternary algebraic structures and their applications in physics. Preprint
17. Vainerman, L, Kerner, R: On special classes of *n*-algebras. J. Math. Phys. **37**, 2553-2565 (1996)
18. Abramov, V, Kerner, R, Le Roy, B: Hypersymmetry: a $\mathbb{Z}_3$-graded generalization of supersymmetry. J. Math. Phys. **38**, 1650-1669 (1997)
19. Park, C: Isomorphisms between C^*-ternary algebras. J. Math. Phys. **47**(10), 103512 (2006)
20. Park, C: Homomorphisms between Poisson JC^*-algebras. Bull. Braz. Math. Soc. **36**, 79-97 (2005)

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