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A coupled system of fractional differential equations with nonlocal integral boundary conditions

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Abstract

In this paper, we prove the existence and uniqueness of solutions for a system of fractional differential equations with Riemann-Liouville integral boundary conditions of different order. Our results are based on the nonlinear alternative of Leray-Schauder type and Banach's fixed-point theorem. An illustrative example is also presented.

MSC: 34A08; 34A12; 34B15

Keywords: Caputo fractional derivative; fractional differential systems; integral boundary conditions; fixed-point theorems

1 Introduction

In this paper, we investigate a boundary value problem of first-order fractional differential equations with Riemann-Liouville integral boundary conditions of different order given by

$$\begin{cases} {}^c\mathcal{D}^{\alpha}_{0+}u(t) = f(t,u(t),v(t)), & t \in [0,1], \\ {}^c\mathcal{D}^{\beta}_{0+}v(t) = g(t,u(t),v(t)), & t \in [0,1], \\ u(0) = \gamma I^p u(\eta) = \gamma \int_0^{\eta} \frac{(\eta-s)^{p-1}}{\Gamma(p)} u(s)\,ds, & 0 < \eta < 1, \\ v(0) = \delta I^q v(\zeta) = \delta \int_0^{\zeta} \frac{(\zeta-s)^{q-1}}{\Gamma(q)} v(s)\,ds, & 0 < \zeta < 1, \end{cases} \tag{1.1}$$

where ${}^c\mathcal{D}^{\alpha}_{0+}$, ${}^c\mathcal{D}^{\beta}_{0+}$ denote the Caputo fractional derivatives, $0 < \alpha, \beta \leq 1$, $f, g \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$, and $p, q, \gamma, \delta \in \mathbb{R}$.

Fractional differential equations have recently been addressed by several researchers for a variety of problems. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, *etc.* [1–5]. Fractional-order differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations. With this advantage, fractional-order models become more realistic and practical than the classical integer-order models, in which such effects are not taken into account. For some recent development on the topic, see [6–18], and the references therein. The study of a coupled system of fractional

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order is also very significant because this kind of system can often occur in applications. The reader is referred to the papers [19–22], and the references cited therein.

This paper is organized as follows: In Sect. 2, we present some basic materials needed to prove our main results. In Sect. 3, we prove the existence and uniqueness of solutions for the system (1.1) by applying some standard fixed-point principles.

2 Preliminaries

Let us introduce the space $X = \{u(t)|u(t) \in C^1([0,1])\}$ endowed with the norm $\|u\| = \max\{|u(t)|, t \in [0,1]\}$. Obviously, $(X, \|\cdot\|)$ is a Banach space. Also, let $Y = \{v(t)|v(t) \in C^1([0,1])\}$ endowed with the norm $\|v\| = \max\{|v(t)|, t \in [0,1]\}$. The product space $(X \times Y, \|(u,v)\|)$ is also a Banach space with norm $\|(u,v)\| = \|u\| + \|v\|$.

For the convenience of the readers, we now present some useful definitions and fundamental facts of fractional calculus [1, 4].

Definition 2.1 For at least n-times continuously differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\, ds, \quad q > 0,$$

provided the integral exists.

The following lemmas gives some properties of Riemann-Liouville fractional integrals and Caputo fractional derivative [1].

Lemma 2.3 *Let $p, q \geq 0$, $f \in L_1[a,b]$. Then $I^p I^q f(t) = I^{p+q} f(t)$ and $^cD^q I^q f(t) = f(t)$, for all $t \in [a,b]$.*

Lemma 2.4 *Let $\beta > \alpha > 0$, $f \in L_1[a,b]$. Then $^cD^\alpha I^\beta f(t) = I^{\beta-\alpha} f(t)$, for all $t \in [a,b]$.*

To define the solution of the boundary value problem (1.1), we need the following lemma, which deals with a linear variant of the problem (1.1).

Lemma 2.5 *Let $\gamma \neq \frac{\Gamma(p+1)}{\eta^p}$. Then for a given $g \in C([0,1],\mathbb{R})$, the solution of the fractional differential equation*

$$^cD^\alpha x(t) = g(t), \quad 0 < \alpha \leq 1 \tag{2.1}$$

subject to the boundary condition

$$x(0) = \gamma I^p x(\eta) = \gamma \int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} x(s)\, ds, \quad 0 < \eta < 1 \tag{2.2}$$

is given by

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,ds + \frac{\gamma\Gamma(p+1)}{\Gamma(p+1)-\gamma\eta^p} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} g(s)\,ds, \quad t \in [0,1]. \tag{2.3}$$

Proof For some constant $c_0 \in \mathbb{R}$, we have [1]

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,ds - c_0. \tag{2.4}$$

Using the Riemann-Liouville integral of order p for (2.4), we have

$$\begin{aligned} I^p x(t) &= \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \left[\int_0^s \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)} g(r)\,dr - c_0 \right] ds \\ &= I^p I^\alpha g(t) - c_0 \frac{t^p}{\Gamma(p+1)} = I^{p+\alpha} g(t) - c_0 \frac{t^p}{\Gamma(p+1)}, \end{aligned}$$

where we have used Lemma 2.3. Using the condition (2.2) in the above expression, we get

$$c_0 = -\frac{\gamma\Gamma(p+1)}{\Gamma(p+1)-\gamma\eta^p} I^{p+\alpha} g(\eta).$$

Substituting the value of c_0 in (2.4), we obtain (2.3). □

3 Main results

For the sake of convenience, we set

$$M_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{|\gamma|\eta^{p+\alpha}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1)-\gamma\eta^p|}, \tag{3.1}$$

$$M_2 = \frac{1}{\Gamma(\beta+1)} + \frac{|\delta|\zeta^{q+\beta}\Gamma(q+1)}{\Gamma(q+\beta+1)|\Gamma(q+1)-\delta\zeta^q|} \tag{3.2}$$

and

$$M_0 = \min\bigl\{1 - (M_1 k_1 + M_2 \lambda_1), 1 - (M_1 k_2 + M_2 \lambda_2)\bigr\}. \tag{3.3}$$

Define the operator $T : X \times Y \to X \times Y$ by

$$\begin{aligned} &T(u,v)(t) \\ &\quad = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix} \\ &\quad = \begin{pmatrix} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s),v(s))\,ds + \frac{\gamma\Gamma(p+1)}{\Gamma(p+1)-\gamma\eta^\alpha} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f(s,u(s),v(s))\,ds \\ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s,u(s),v(s))\,ds + \frac{\delta\Gamma(q+1)}{\Gamma(q+1)-\delta\zeta^\beta} \int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} g(s,u(s),v(s))\,ds \end{pmatrix}. \end{aligned}$$

The first result is based on Leray-Schauder alternative.

Lemma 3.1 (Leray-Schauder alternative, [23] p.4) *Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2 *Suppose that $\gamma \neq \frac{\Gamma(p+1)}{\eta^p}$ and $\delta \neq \frac{\Gamma(q+1)}{\zeta^q}$. Assume that there exist real constants k_i, $\lambda_i \geq 0$ $(i = 1, 2)$ and $k_0 > 0$, $\lambda_0 > 0$ such that $\forall x_i \in \mathbb{R}$ $(i = 1, 2)$, we have*

$$\begin{aligned} |f(t, x_1, x_2)| &\leq k_0 + k_1|x_1| + k_2|x_2|, \\ |g(t, x_1, x_2)| &\leq \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|. \end{aligned}$$

In addition, it is assumed that

$$M_1k_1 + M_2\lambda_1 < 1 \quad \text{and} \quad M_1k_2 + M_2\lambda_2 < 1,$$

where M_1 and M_2 are given by (3.1) and (3.2), respectively. Then the boundary value problem (1.1) has at least one solution.

Proof First, we show that the operator $T : X \times Y \to X \times Y$ is completely continuous. By continuity of functions f and g, the operator T is continuous.

Let $\Omega \subset X \times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t, u(t), v(t))| \leq L_1, \qquad |g(t, u(t), v(t))| \leq L_2, \quad \forall (u, v) \in \Omega.$$

Then for any $(u, v) \in \Omega$, we have

$$\begin{aligned} |T_1(u, v)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s), v(s))| \, ds \\ &\quad + \frac{|\gamma|\Gamma(p+1)}{|\Gamma(p+1) - \gamma\eta^p|} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} |f(s, u(s), v(s))| \, ds \\ &\leq L_1 \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|\gamma|\eta^{p+\alpha}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1) - \gamma\eta^p|} \right\} = L_1M_1. \end{aligned}$$

Similarly, we get

$$\|T_2(u, v)\| \leq L_2 \left\{ \frac{1}{\Gamma(\beta+1)} + \frac{|\delta|\zeta^{q+\beta}\Gamma(q+1)}{\Gamma(q+\beta+1)|\Gamma(q+1) - \delta\zeta^q|} \right\} = L_2M_2,$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $0 \leq t_1 \leq t_2 \leq 1$. Then we have

$$\begin{aligned} &|T_1(u(t_2), v(t_2)) - T_1(u(t_1), v(t_1))| \\ &\quad = \left| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), v(s)) \, ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), v(s)) \, ds \right| \end{aligned}$$

$$\begin{aligned}&\leq \frac{L_1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\right|\\&\leq \frac{L_1}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}\right).\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}&\left|T_2\left(u(t_2),v(t_2)\right)-T_2\left(u(t_1),v(t_1)\right)\right|\\&\quad\leq \frac{L_2}{\Gamma(\beta)}\left|\int_0^{t_1}\left[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right]ds+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\,ds\right|\\&\quad\leq \frac{L_2}{\Gamma(\beta+1)}\left(t_2^{\beta}-t_1^{\beta}\right).\end{aligned}$$

Therefore, the operator $T(u,v)$ is equicontinuous, and thus the operator $T(u,v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E}=\{(u,v)\in X\times Y|(u,v)=\lambda T(u,v), 0\leq\lambda\leq 1\}$ is bounded. Let $(u,v)\in\mathcal{E}$, then $(u,v)=\lambda T(u,v)$. For any $t\in[0,1]$, we have

$$u(t)=\lambda T_1(u,v)(t),\qquad v(t)=\lambda T_2(u,v)(t).$$

Then

$$\left|u(t)\right|\leq\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{|\gamma|\eta^{p+\alpha}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1)-\gamma\eta^{p}|}\right\}\left(k_0+k_1\left|u(t)\right|+k_2\left|v(t)\right|\right)$$

and

$$\left|v(t)\right|\leq\left\{\frac{1}{\Gamma(\beta+1)}+\frac{|\delta|\zeta^{q+\beta}\Gamma(q+1)}{\Gamma(q+\beta+1)|\Gamma(q+1)-\delta\zeta^{q}|}\right\}\left(\lambda_0+\lambda_1\left|u(t)\right|+\lambda_2\left|v(t)\right|\right).$$

Hence, we have

$$\|u\|\leq M_1\left(k_0+k_1\|u\|+k_2\|v\|\right)$$

and

$$\|v\|\leq M_2\left(\lambda_0+\lambda_1\|u\|+\lambda_2\|v\|\right),$$

which imply that

$$\|u\|+\|v\|=(M_1k_0+M_2\lambda_0)+(M_1k_1+M_2\lambda_1)\|u\|+(M_1k_2+M_2\lambda_2)\|v\|.$$

Consequently,

$$\left\|(u,v)\right\|\leq\frac{M_1k_0+M_2\lambda_0}{M_0},$$

for any $t\in[0,1]$, where M_0 is defined by (3.3), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator T has at least one fixed point. Hence, the boundary value problem (1.1) has at least one solution. The proof is complete. □

In the second result, we prove existence and uniqueness of solutions of the boundary value problem (1.1) via Banach's contraction principle.

Theorem 3.3 *Assume that $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions and there exist constants m_i, n_i, $i=1,2$ such that for all $t\in[0,1]$ and $u_i,v_i\in\mathbb{R}$, $i=1,2$,*

$$\bigl|f(t,u_1,u_2)-f(t,v_1,v_2)\bigr|\le m_1|u_1-v_1|+m_2|u_2-v_2|$$

and

$$\bigl|g(t,u_1,u_2)-g(t,v_1,v_2)\bigr|\le n_1|u_1-v_1|+n_2|u_2-v_2|.$$

In addition, assume that

$$M_1(m_1+m_2)+M_2(n_1+n_2)<1,$$

where M_1 and M_2 are given by (3.1) and (3.2), respectively. Then the boundary value problem (1.1) has a unique solution.

Proof Define $\sup_{t\in[0,1]}f(t,0,0)=N_1<\infty$ and $\sup_{t\in[0,1]}g(t,0,0)=N_2<\infty$ such that

$$r\ge\frac{N_1M_1+N_2M_2}{1-M_1(m_1+m_2)-M_2(n_1+n_2)}.$$

We show that $TB_r\subset B_r$, where $B_r=\{(u,v)\in X\times Y:\|(u,v)\|\le r\}$.

For $(u,v)\in B_r$, we have

$$\begin{aligned}
&\bigl|T_1(u,v)(t)\bigr|\\
&\quad\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bigl|f\bigl(s,u(s),v(s)\bigr)\bigr|\,ds\\
&\qquad+\frac{|\gamma|\Gamma(p+1)}{|\Gamma(p+1)-\gamma\eta^p|}\int_0^\eta\frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)}\bigl|f\bigl(s,u(s),v(s)\bigr)\bigr|\,ds\\
&\quad\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bigl(\bigl|f\bigl(s,u(s),v(s)\bigr)-f(t,0,0)\bigr|+\bigl|f(t,0,0)\bigr|\bigr)\,ds\\
&\qquad+\frac{|\gamma|\Gamma(p+1)}{|\Gamma(p+1)-\gamma\eta^p|}\int_0^\eta\frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)}\bigl(\bigl|f\bigl(s,u(s),v(s)\bigr)-f(t,0,0)\bigr|+\bigl|f(t,0,0)\bigr|\bigr)\,ds\\
&\quad\le\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{|\gamma|\eta^{p+\alpha}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1)-\gamma\eta^p|}\right\}\bigl(m_1\|u\|+m_2\|v\|+N_1\bigr)\\
&\quad\le M_1\bigl[(m_1+m_2)r+N_1\bigr].
\end{aligned}$$

Hence,

$$\bigl\|T_1(u,v)(t)\bigr\|\le M_1\bigl[(m_1+m_2)r+N_1\bigr].$$

In the same way, we can obtain that

$$\bigl\|T_2(u,v)(t)\bigr\|\le M_2\bigl[(n_1+n_2)r+N_2\bigr].$$

Consequently, $\|T(u,v)(t)\|\le r$.

Now for $(u_2, v_2), (u_1, v_1) \in X \times Y$, and for any $t \in [0,1]$, we get

$$\begin{aligned}
&\bigl|T_1(u_2,v_2)(t) - T_1(u_1,v_1)(t)\bigr| \\
&\quad\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl|f\bigl(s,u_2(s),v_2(s)\bigr) - f\bigl(s,u_1(s),v_1(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{|\gamma|\Gamma(p+1)}{|\Gamma(p+1)-\gamma\eta^p|} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} \bigl|f\bigl(s,u_2(s),v_2(s)\bigr) - f\bigl(s,u_1(s),v_1(s)\bigr)\bigr|\,ds \\
&\quad\le \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|\gamma|\eta^{p+\alpha}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1)-\gamma\eta^p|} \right\} \bigl(m_1|u_2-u_1| + m_2|v_2-v_1|\bigr) \\
&\quad\le M_1\bigl(m_1\|u_2-u_1\| + m_2\|v_2-v_1\|\bigr) \\
&\quad\le M_1(m_1+m_2)\bigl(\|u_2-u_1\| + \|v_2-v_1\|\bigr),
\end{aligned}$$

and consequently we obtain

$$\bigl\|T_1(u_2,v_2)(t) - T_1(u_1,v_1)\bigr\| \le M_1(m_1+m_2)\bigl(\|u_2-u_1\| + \|v_2-v_1\|\bigr). \tag{3.4}$$

Similarly,

$$\bigl\|T_2(u_2,v_2)(t) - T_2(u_1,v_1)\bigr\| \le M_2(n_1+n_2)\bigl(\|u_2-u_1\| + \|v_2-v_1\|\bigr). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\bigl\|T(u_2,v_2)(t) - T(u_1,v_1)(t)\bigr\| \le \bigl[M_1(m_1+m_2) + M_2(n_1+n_2)\bigr]\bigl(\|u_2-u_1\| + \|v_2-v_1\|\bigr).$$

Since $M_1(m_1+m_2) + M_2(n_1+n_2) < 1$, therefore, T is a contraction operator. So, by Banach's fixed-point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1.1). This completes the proof. □

Example 3.4 Consider the following system of fractional boundary value problem:

$$\begin{cases}
{}^cD^{1/2}x(t) = \frac{1}{4(t+2)^2}\frac{|u(t)|}{1+|u(t)|} + 1 + \frac{1}{16}\sin^2 v(t), & t \in [0,1], \\
{}^cD^{1/2}x(t) = \frac{1}{32\pi}\sin(2\pi u(t)) + \frac{|v(t)|}{16(1+|v(t)|)} + \frac{1}{2}, & t \in [0,1], \\
u(0) = \sqrt{3}I^{3/2}u(\frac{1}{3}), \\
v(0) = \sqrt{2}I^{1/2}v(\frac{1}{2}).
\end{cases} \tag{3.6}$$

Here, $\alpha = 1/2$, $\gamma = \sqrt{3}$, $p = 3/2$, $\eta = 1/3$, $\beta = 1/2$, $\delta = \sqrt{2}$, $q = 1/2$, $\zeta = 1/2$, and $f(t,u,v) = \frac{1}{4(t+2)^2}\frac{|u|}{1+|u|} + 1 + \frac{1}{8}\sin^2 v$ and $g(t,u,v) = \frac{1}{32\pi}\sin(2\pi u) + \frac{|v|}{16(1+|v|)} + \frac{1}{2}$. Note that $\gamma = \sqrt{3} \neq \Gamma(p+1)/\eta^p = \Gamma(5/2)/(1/3)^{3/2}$ and $\delta = \sqrt{2} \neq \Gamma(q+1)/\zeta^q = \Gamma(3/2)/(1/2)^{1/2}$. Furthermore, $|f(t,u_1,u_2) - f(t,v_1,v_2)| \le \frac{1}{16}|u_1-u_2| + \frac{1}{16}|v_1-v_2|$, $|g(t,u_1,u_2) - g(t,v_1,v_2)| \le \frac{1}{16}|u_1-u_2| + \frac{1}{16}|v_1-v_2|$, and

$$\begin{aligned}
M_1(m_1+m_2) + M_2(n_1+n_2) &= \frac{1}{8}\left\{\frac{2}{\sqrt{\pi}} + \frac{\sqrt{3\pi}}{2(9\sqrt{\pi}-4)}\right\} + \frac{1}{16}\left\{\frac{2}{\sqrt{\pi}} + \frac{\sqrt{2\pi}}{2(2-\sqrt{\pi})}\right\} \\
&\approx 0.712679 < 1.
\end{aligned}$$

Thus, all the conditions of Theorem 3.3 are satisfied and consequently, its conclusion applies to the problem (3.6).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors SKN and MO contributed to each part of this study equaly and read and approved the final version of the manuscript.

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