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Variationally stable dynamic systems on time scales

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Abstract

In this paper we give a Lyapunov functional characterization of h-stability for nonlinear dynamic systems on time scales under the condition of u_∞-similarity between their variational systems. Furthermore, we give some examples related to the notions of u_∞-similarity and h-stability.

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1 Introduction

The theory on time scales has been developed as a generalization of both continuous and discrete time theory and applied to many different fields of mathematics [2, 3, 5, 6, 21, 23].

It is widely known that the various types of stability of nonlinear differential equations or difference equations can be characterized by using Lyapunov's second method, the method of variation of parameters, and inequalities, *etc.* [1, 21, 22, 24, 38, 40].

Pinto [32] introduced the notion of h-stability for differential equations with the intention of obtaining results about stability for weakly stable differential systems under some perturbations. Also, Medina and Pinto [29] applied the h-stability to obtain a uniform treatment for the various stability notions in difference systems and extended the study of exponential stability to a variety of reasonable systems called h-systems. Pinto and Medina obtained the important properties about h-stability for the various differential systems and difference systems [26–28, 30, 33–35].

Choi *et al.* [7] investigated h-stability for the nonlinear differential systems by means of the notions of Lyapunov functions and t_∞-similarity introduced by Conti [19]. Trench [39] introduced summable similarity as a discrete analog of Conti's definition of t_∞-similarity and investigated the various stabilities of linear difference systems by using summable similarity. Choi and Koo [8] studied the variational stability for nonlinear difference systems by means of n_∞-similarity. Also, Choi *et al.* studied the asymptotic property and the h-stability of difference systems via discrete similarities and comparison principle [9–11]. For detailed results about the various stabilities including the notions of h-stability and strong stability of dynamic systems on time scales, see [12–18, 20, 25, 31].

In this paper we introduce the notion of u_∞-similarity which extends the continuous t_∞-similarity [19] and the discrete n_∞-summable similarity [11]. Then we give a Lyapunov functional characterization of h-stability for nonlinear dynamic systems on time scales by

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assuming the condition of u_∞-similarity between its variational systems. Furthermore, we give some examples related to the notions of u_∞-similarity and h-stability on time scales.

2 Main results

We refer the reader to Ref. [5] for all the basic definitions and results on time scales necessary for this work (*e.g.*, delta differentiability, rd-continuity, exponential function and its properties).

It is assumed throughout that a time scale $\mathbb{T}$ will be unbounded above. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space. $C_{\mathrm{rd}}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ denotes the set of all rd-continuous functions from $\mathbb{T} \times \mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}_+ = [0, \infty)$.

We consider the dynamic system

$$x^\Delta = f(t, x), \quad x(t_0) = x_0, t_0 \in \mathbb{T}, \tag{2.1}$$

where $f \in C_{\mathrm{rd}}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ with $f(t, 0) = 0$, and x^Δ is the delta derivative of $x : \mathbb{T} \to \mathbb{R}^n$ with respect to $t \in \mathbb{T}$. We assume that $f_x = \frac{\partial f}{\partial x}$ exists and is rd-continuous on $\mathbb{T} \times \mathbb{R}^n$. Let $x(t) = x(t, t_0, x_0)$ be the unique solution of (2.1) satisfying the initial condition $x(t_0, t_0, x_0) = x_0$. For the existence and uniqueness of solutions of nonlinear dynamic system (2.1), see [23].

Also, we consider its associated variational systems

$$v^\Delta = f_x(t, 0)v \tag{2.2}$$

and

$$z^\Delta = f_x\big(t, x(t, t_0, x_0)\big)z, \tag{2.3}$$

where $I + \mu(t)f_x(t, x(t))$ is invertible for all $t \in \mathbb{T}$ and I denotes the $n \times n$ identity matrix.

To establish our main results we will use the following lemmas.

Lemma 2.1 [22, Theorem 2.6.4] *Assume that $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ are the solutions of system (2.1) through (t_0, x_0) and (t_0, y_0) respectively, which exist for each $t \in \mathbb{T}_0$ and are such that x_0 and y_0 belong to a convex subset D of $\mathbb{R}^n$. Then*

$$x(t, t_0, y_0) - x(t, t_0, x_0) = \int_0^1 \Phi\big(t, t_0, x_0 + \tau(y_0 - x_0)\big) d\tau \cdot (y_0 - x_0), \quad t \in \mathbb{T}_0,$$

where Φ is a fundamental matrix of (2.3) and $\mathbb{T}_0 = \mathbb{T} \cap [t_0, \infty)$.

This lemma can be proved in the same manner as that of Theorem 2.6.4 in [22], so we omit the detail.

Lemma 2.2 [23, Lemma 2.7.1] *Let $f \in C_{\mathrm{rd}}(\mathbb{T}^\kappa \times D, \mathbb{R}^n)$, where D is an open convex set in $\mathbb{R}^n$. Suppose that f_x exists and is rd-continuous. Then*

$$f(t, x_2) - f(t, x_1) = \int_0^1 f_x\big(t, sx_2 + (1 - s)x_1\big) ds(x_2 - x_1), \quad t \in \mathbb{T}.$$

In order to prove the variation of parameters formula on time scales, we need the following result on differentiability of solutions with respect to initial values.

Lemma 2.3 *Assume that $f : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$ possesses partial derivatives on $\mathbb{T}_0 \times \mathbb{R}^n$ and $f_x(t, x(t, t_0, x_0))$ is rd-continuous on $\mathbb{T}$. Let $x(t) = x(t, t_0, x_0)$ be the solution of (2.1), which exists for $t \geq t_0$ and let*

$$H(t, t_0, x_0) = \frac{\partial f(t, x(t, t_0, x_0))}{\partial x}. \tag{2.4}$$

Then

$$\Phi(t, t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial x_0} \tag{2.5}$$

exists and is the solution of

$$\Phi^\Delta(t, t_0, x_0) = H(t, t_0, x_0)\Phi(t, t_0, x_0), \quad t \in \mathbb{T}_0, \tag{2.6}$$

$$\Phi(t_0, t_0, x_0) = I. \tag{2.7}$$

The proof of Lemma 2.3 follows simply by differentiating the solution identity

$$x^\Delta(t, t_0, x_0) = f\big(t, x(t, t_0, x_0)\big), \quad t \geq t_0,$$

with respect to x_0. It is a special case of [36, Satz 1.2.22].

Remark 2.1 [23, Theorem 2.7.1] $H(t, t_0, x_0)$ in Lemma 2.3 is given by

$$H(t, t_0, x_0) = \lim_{\xi \to 0} \int_0^1 f_x\big(t, sx(t, t_0, x_0) + (1 - s)x(t, t_0, x_0 + \xi)\big)\, ds.$$

It follows from Lemma 2.3 that the fundamental matrix solution $\Phi(t, t_0, 0)$ of (2.2) is given by

$$\Phi(t, t_0, 0) = \frac{\partial x(t, t_0, 0)}{\partial x_0}$$

and the fundamental matrix solution $\Phi(t, t_0, x_0)$ of (2.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial x_0}$$

or equivalently

$$x(t, t_0, x_0) = \left[\int_0^1 \Phi(t, t_0, sx_0)\, ds\right] x_0. \tag{2.8}$$

Let $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices over $\mathbb{R}$. The class of all rd-continuous functions $A : \mathbb{T} \to M_n(\mathbb{R})$ is denoted by

$$C_{\mathrm{rd}}\big(\mathbb{T}, M_n(\mathbb{R})\big).$$

Consider the quasilinear dynamic system

$$y^\Delta = A(t)y + g(t, y), \quad y(t_0) = y_0, t \in \mathbb{T}, \tag{2.9}$$

where $A \in C_{\mathrm{rd}}(\mathbb{T}, M_n(\mathbb{R}))$ and $g : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous in the first argument with $g(t, 0) = 0$.

We need the following result which is a slight modification of the variation of constants formula in [24, Theorem 4.6.1].

Lemma 2.4 [36] *The solution $y(t, t_0, y_0)$ of (2.9) satisfies the equation*

$$y(t) = \Phi(t, t_0)y_0 + \int_{t_0}^t \Phi(t, \sigma(s))g(s, y(s)) \Delta s, \quad t \geq t_0, \tag{2.10}$$

where Φ is a transition matrix of the linear system

$$y^\Delta = A(t)y, \quad y(t_0) = y_0, \tag{2.11}$$

where $A \in C_{\mathrm{rd}}(\mathbb{T}^\kappa, M_n(\mathbb{R}))$.

For the Lyapunov-like function $V \in C_{\mathrm{rd}}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+)$, we recall the following definition.

Definition 2.2 [23, Definition 3.1.1] We define the *generalized derivative* $D^+ V_{(2.1)}^\Delta(t, x(t))$ of $V(t, x)$ relative to (2.1) as follows: given $\varepsilon > 0$, there exists a neighborhood $N(\varepsilon)$ of $t \in \mathbb{T}$ such that

$$\frac{1}{\sigma(t) - s}\big[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t)) - (\sigma(t) - s)f(t, x(t)))\big]$$
$$< D^+ V_{(2.1)}^\Delta(t, x(t)) + \varepsilon, \quad s \in N(\varepsilon), s > t,$$

where $x(t)$ is any solution of (2.1) and the *upper right Dini derivative* $V_*^\Delta(t)$ of V_* is given by

$$V_*^\Delta(t) = \begin{cases} \overline{\lim}_{\eta \to 0^+, \eta + t \in \mathbb{T}} \frac{V_*(t+\eta) - V_*(t)}{\eta}, & \text{if } t = \sigma(t), \\ \frac{V_*(\sigma(t)) - V_*(t)}{\mu(t)}, & \text{if } t < \sigma(t), \end{cases} \tag{2.12}$$

where $V_*(t) = V(t, x(t))$.

Then it is well known that

$$D^+ V_{(2.1)}^\Delta(t, x(t)) = V_*^\Delta(t)$$

if $V(t, x)$ is Lipschitzian in x for each $t \in \mathbb{T}$ [22, 40].

In case $t \in \mathbb{T}$ is right dense, we have

$$D^+ V_{(2.1)}^\Delta(t, x(t)) = DV_{(2.1)}(t, x(t)) = \overline{\lim}_{\eta \to 0^+} \frac{1}{\eta}\big[V(t + \eta, x(t) + \eta f(t, x(t))) - V(t, x(t))\big]$$
$$= \overline{\lim}_{\eta \to 0^+, \eta + t \in \mathbb{T}} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} = D^+ V_*(t).$$

In case $t \in \mathbb{T}$ is right scattered and $V(t, x(t))$ is continuous at t, we have

$$D^+ V_{(2.1)}^\Delta \big(t, x(t)\big) = \frac{1}{\mu(t)} \big[V\big(\sigma(t), x(\sigma(t))\big) - V\big(t, x(t)\big)\big].$$

In fact, if $x(t)$ is a solution of (2.1), we have

$$V^\Delta\big(t, x(t)\big) = V^{\Delta_t}\big(t, x(t)\big) + \left[\int_0^1 D_2 V\big(\sigma(t), x(t) + \eta\mu(t)x^\Delta(t)\big)\, d\eta\right] x^\Delta(t)$$

$$= V^{\Delta_t}\big(t, x(\sigma(t))\big) + \left[\int_0^1 D_2 V\big(t, x(t) + \eta\mu(t)x^\Delta(t)\big)\, d\eta\right] x^\Delta(t),$$

by the chain rule of a differentiable function $V(t, x(t))$ [37, Theorem 1].

We note that the total difference of the function V along the solutions x of (2.1) is given by

$$\Delta V_{(2.1)}(n, x) = V\big(n+1, x(n+1, n, x)\big) - V\big(n, x(n, n, x)\big).$$

Choi *et al.* [7] investigated h-stability for nonlinear differential systems using the notions of t_∞-similarity and Lyapunov functions. Also, Choi *et al.* [11] introduced the notion of n_∞-summable similarity which is the corresponding t_∞-similarity for the discrete case and then characterized h-stability in variation and asymptotic equilibrium in variation for nonlinear difference system via n_∞-summable similarity and comparison principle.

Now, we define u_∞-similarity on time scales in order to unify (continuous) t_∞-similarity and (discrete) n_∞-similarity for matrix-valued functions.

Let $\mathfrak{M}_n(\mathbb{R})$ be the set of all $n \times n$ invertible matrices over $\mathbb{R}$, and $C_{\mathrm{rd}}^1(\mathbb{T}^\kappa, \mathfrak{M}_n(\mathbb{R}))$ be the set of all rd-continuous differentiable functions S from $\mathbb{T}^\kappa$ to $\mathfrak{M}_n(\mathbb{R})$ such that S and S^{-1} are bounded on $\mathbb{T}_0$.

Definition 2.3 A function $A : \mathbb{T}^\kappa \to M_n(\mathbb{R})$ is called *regressive* if for each $t \in \mathbb{T}^\kappa$ the $n \times n$ matrix $I + \mu(t)A(t)$ is invertible.

The class of all rd-continuous and regressive functions from $\mathbb{T}^\kappa$ to $M_n(\mathbb{R})$ is denoted by

$$C_{\mathrm{rd}}\mathcal{R}\big(\mathbb{T}^\kappa, M_n(\mathbb{R})\big).$$

Definition 2.4 [17] Let $A, B \in C_{\mathrm{rd}}\mathcal{R}(\mathbb{T}^\kappa, M_n(\mathbb{R}))$ and $t_0 \in \mathbb{T}$. A function A is u_∞-*similar* to a function B if there exists an absolutely integrable function $F \in C_{\mathrm{rd}}(\mathbb{T}, M_n(\mathbb{R}))$, *i.e.*, $\int_{t_0}^\infty |F(t)|\Delta t < \infty$, such that

$$S^\Delta(t) + S^\sigma(t)B(t) - A(t)S(t) = F(t), \quad t \in \mathbb{T}^\kappa, \tag{2.13}$$

for some $S \in C_{\mathrm{rd}}^1(\mathbb{T}^\kappa, \mathfrak{M}_n(\mathbb{R}))$.

Remark 2.5 If $\mathbb{T} = \mathbb{R}$, then u_∞-similarity becomes t_∞-similarity and if $\mathbb{T} = \mathbb{Z}$, then u_∞-similarity becomes n_∞-similarity. Also if A and B are u_∞-similar with $F = 0$ defined on $\mathbb{T}$, then they are *kinematically similar* [4].

Let $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \ldots, n_0 + k, \ldots\}$, where n_0 is a nonnegative integer and $\mathfrak{M}$ denote the set of all $s \times s$ invertible matrix-valued functions defined on $\mathbb{N}(n_0)$.

Remark 2.6 [11, Definition 2.5] A matrix function $A \in \mathfrak{M}$ is n_∞-*summably similar* to a matrix function $B \in \mathfrak{M}$ if there exists an absolutely summable $s \times s$ matrix $F(n)$ over $\mathbb{N}(n_0)$, that is,

$$\sum_{l=n_0}^{\infty} |F(l)| < \infty$$

such that

$$\Delta S(n) + S(n+1)B(n) - A(n)S(n) = F(n) \tag{2.14}$$

for some $S \in \mathfrak{S}$.

For the example of n_∞-summable similarity, see [11].

Remark 2.7 We can easily show that the n_∞-summable similarity is an equivalence relation in the similar manner of Trench in [39]. Also, if A and B are n_∞-summably similar with $F(n) = 0$, then we say that they are *kinematically similar*.

Pinto [32] introduced the notion of h-stability which is an extension of the notions of exponential stability and uniform stability of the solutions of differential equations. The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$. We recall the notions of h-stability for dynamic systems on time scales in [14].

Definition 2.8 System (2.1) is called an *h-system* if there exist a positive rd-continuous function $h : \mathbb{T} \to \mathbb{R}$, a constant $c \geq 1$ and $\delta > 0$ such that

$$|x(t, t_0, x_0)| \leq c|x_0|h(t)h(t_0)^{-1}, \quad t \geq t_0$$

for $|x_0| < \delta$ (here $h(t)^{-1} = \frac{1}{h(t)}$).

Moreover, system (2.1) is said to be

(hS) *h-stable* if h is a bounded function in the definition of h-system,
(GhS) *globally h-stable* if system (2.1) is hS for every $x_0 \in D$, where $D \subset \mathbb{R}^n$ is a region which includes the origin,
(hSV) *h-stable in variation* if system (2.3) is hS,
(GhSV) *globally h-stable in variation* if system (2.3) is GhS.

For the various definitions of stability, we refer to [20] and we obtain the following possible implications for system (2.1) among the various types of stability:

h-stability $\Rightarrow$ uniform exponential stability

$\Rightarrow$ uniform Lipschitz stability

$\Rightarrow$ uniform stability

as in [7, 34]. See [8, 26, 28, 29] for stability of nonlinear difference systems.

We consider two linear dynamic systems

$$x^{\Delta} = A(t)x \tag{2.15}$$

and

$$y^{\Delta} = B(t)y, \tag{2.16}$$

where $A, B \in C_{\mathrm{rd}}\mathcal{R}(\mathbb{T}^{\kappa}, M_n(\mathbb{R}))$.

We say that if A and B are u_{∞}-similar, then systems (2.15) and (2.16) are u_{∞}-*similar*.

Lemma 2.5 [14, Lemma 2.3] *System (2.15) is an h-system if and only if there exist a positive rd-continuous function h defined on $\mathbb{T}$ and a constant $c \geq 1$ such that*

$$\left|\Phi_A(t,t_0)\right| \leq ch(t)h(t_0)^{-1}, \quad t \in \mathbb{T}_0,$$

where Φ_A is a fundamental matrix solution of (2.15).

We obtain the following result from Lemma 2.3 in [17].

Lemma 2.6 *Assume that A and B are u_{∞}-similar. Then*

$$\Phi_B(t,t_0) = S^{-1}(t)\left[\Phi_A(t,t_0)S(t_0) + \int_{t_0}^{t} \Phi_A\big(t,\sigma(s)\big)F(s)\Phi_B(s,t_0)\Delta s\right], \quad t,t_0 \in \mathbb{T},$$

where Φ_A and Φ_B are the matrix exponential functions of (2.15) and (2.16) respectively.

Medina and Pinto [29, Theorem 3] showed that hSV implies hS. Also, they proved the converse when the condition

$$\sum_{l=n_0}^{\infty} \frac{h(l)}{h(l+1)}\left|f_x(l,n_0,x_0) - f_x(l,0)\right| < \infty, \quad n_0 \geq 0 \tag{2.17}$$

for $|x_0| \leq \delta$ holds [29, Theorem 14].

In order to establish our main results, we will introduce the following condition.

(H): $f_x(t,0)$ and $f_x(t,x(t,t_0,x_0))$ are u_{∞}-similar for $t \geq t_0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$ and $\int_{t_0}^{\infty} \frac{h(t)}{h(\sigma(t))}|F(t)|\Delta t < \infty$ with the positive rd-continuous function $h(t)$ defined on $\mathbb{T}$.

Lemma 2.7 [12, Theorem 3.4] *Assume that condition (H) is satisfied. Then variational system (2.2) is also an h-system if and only if variational system (2.3) is an h-system.*

We can obtain the same result about Lemma 2.7 by assuming that $f_x(t,0)$ and $f_x(t,x(t,t_0,x_0))$ are u_{∞}-quasisimilar for $t \geq t_0$ instead of the condition (H) in Lemma 2.7 [18, Theorem 3.3].

For nonlinear dynamic system (2.1), we can show that

$$GhSV \Leftrightarrow GhS, \qquad hS \Leftrightarrow hSV$$

by using the concept of u_{∞}-similarity.

We study the relation of h-stability between two systems (2.1) and (2.3) by assuming the condition (H) is satisfied.

Theorem 2.8 [29, Theorem 2] *Suppose that condition (H) is satisfied. If $x = 0$ of (2.1) is h-stable, then $v = 0$ of (2.2) is h-stable.*

We obtain the following result from (2.8).

Theorem 2.9 *If $z = 0$ of (2.3) is h-stable, then $x = 0$ of (2.1) is h-stable.*

We can obtain the following result by using Lemma 2.7 and Theorem 2.8.

Theorem 2.10 *Assume that condition (H) is satisfied. If $x = 0$ of (2.1) is h-stable, then $z = 0$ of (2.1) is h-stable in variation.*

Remark 2.9 For nonlinear dynamic system (2.1), we show that two concepts of h-stability and h-stability in variation are equivalent under the condition that two variational systems (2.2) and (2.3) are u_∞-similar.

Choi *et al.* investigated Massera type converse theorems for the nonlinear difference system $x(n + 1) = f(n, x(n))$ via n_∞-similarity in [8, Theorem 5] and [9, Theorem 2.1]. Furthermore, they characterized h-stability in variation for the nonlinear difference system by using the notion of n_∞-summable similarity in [11].

We need the following lemma to prove our main theorem.

Lemma 2.11 [36, Korollar 2.1.13] *If the delta differentiable function $h : \mathbb{T} \to \mathbb{R}$ is positive, then $\frac{h^\Delta(t)}{h(t)}$ is positively regressive, and $e_p(t, t_0)$ satisfies*

$$e_p(t, t_0) = \frac{h(t)}{h(t_0)},$$

where $p(t) = \frac{h^\Delta(t)}{h(t)}$.

We can obtain the following result that characterizes h-stability for nonlinear dynamic system (2.1) via the notions of Lyapunov functions and u_∞-similarity. It is adapted from Theorem 3.6.1 in [22] and Theorem 3.1 in [25].

Theorem 2.12 *Assume that condition (H) is satisfied. Suppose further that $h^\Delta(t)$ exists and is rd-continuous on $\mathbb{T}$. Then system (2.1) is GhS if and only if there exists a function $V(t, x)$ defined on $\mathbb{T} \times \mathbb{R}^n$ such that the following properties hold:*

(i) $|x| \le V(t, x)| \le c|x|$ for $(t, x) \in \mathbb{T} \times \mathbb{R}^n$ and a constant $c \ge 1$;
(ii) $|V(t, x_1) - V(t, x_2)| \le c|x_1 - x_2|$ for $t \in \mathbb{T}$ and $x_1, x_2 \in \mathbb{R}^n$;
(iii) $V^\Delta(t, x) \le \frac{h^\Delta(t)}{h(t)} V(t, x)$ for $(t, x) \in \mathbb{T} \times \mathbb{R}^n$;
(iv) $V(t, x)$ is continuous on $\mathbb{T} \times \mathbb{R}^n$;

$$\lim_{(\hat{t}, \hat{x}) \to (t, x), \hat{t} \ge t} \left| V(\hat{t}, \hat{x}) - V(t, x) \right| = 0.$$

Proof Necessity: Suppose that system (2.1) is GhS. Then system (2.1) is GhSV by Theorem 2.10, *i.e.*, there exist a constant $c \geq 1$ and a positive rd-continuous bounded function h defined on $\mathbb{T}$ such that for each $x \in \mathbb{R}^n$

$$\left| \Phi(t, t_0, x_0) \right| \leq c h(t) h(t_0)^{-1}, \quad t \geq t_0, \tag{2.18}$$

where Φ is a fundamental matrix solution of (2.3).

Fix $t \in \mathbb{T}$. Let $A_t := \{\tau \in \mathbb{R}_+ : t + \tau \in \mathbb{T}\}$. Then we note that A_t is nonempty from $0 \in A_t$. Define the function $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}_+$ by

$$V(t, x) = \sup_{\tau \in \mathbb{A}_t} \left| x(t + \tau, t, x) \right| h(t + \tau)^{-1} h(t),$$

where $x(t + \tau, t, x)$ is a unique solution of system (2.1) for $(t, x) \in \mathbb{T} \times \mathbb{R}^n$ with the initial value $x(t, t, x) = x$. From GhS of (2.1) we have

$$\left| x(t, t_0, x_0) \right| \leq c |x_0| h(t) h(t_0)^{-1}, \quad t \in \mathbb{T}_0, |x_0| < \infty.$$

Furthermore, we obtain

$$|x| = \left| x(t, t, x) \right| \leq \sup_{\tau \in \mathbb{A}_t} \left| x(t + \tau, t, x) \right| h(t + \tau)^{-1} h(t)$$

$$\leq c |x| h(t + \tau) h(t)^{-1} h(t + \tau)^{-1} h(t) = c |x|.$$

Thus V satisfies property (i).

Let $(t, x_1), (t, x_2) \in \mathbb{T} \times \mathbb{R}^n$. Then we have

$$\left| V(t, x_1) - V(t, x_2) \right| = \left| \sup_{\tau \in \mathbb{A}_t} \left| x(t + \tau, t, x_1) \right| h(t + \tau)^{-1} h(t) \right.$$

$$\left. - \sup_{\tau \in \mathbb{A}_t} \left| x(t + \tau, t, x_2) \right| h(t + \tau)^{-1} h(t) \right|$$

$$\leq \sup_{\tau \in \mathbb{A}_t} \left| \left(x(t + \tau, t, x_1) - x(t + \tau, t, x_2) \right) \right| h(t + \tau)^{-1} h(t). \tag{2.19}$$

It follows from Lemma 2.2 that for each x_1 and x_2 in a convex subset D of $\mathbb{R}^n$

$$\left| x(t, t_0, x_1) - x(t, t_0, x_2) \right| \leq |x_1 - x_2| \sup_{\eta \in D} \left| \Phi(t, t_0, \eta) \right|. \tag{2.20}$$

In view of (2.18), (2.19) and (2.20), we have

$$\left| V(t, x_1) - V(t, x_2) \right| \leq |x_1 - x_2| \sup_{\tau \in \mathbb{A}_t, \eta \in D} \left| \Phi(t + \tau, t, \eta) \right| h(t + \tau)^{-1} h(t)$$

$$\leq |x_1 - x_2| c |h(t + \tau) h(t)^{-1} h(t + \tau)^{-1} h(t)$$

$$\leq c |x_1 - x_2|, \quad t \in \mathbb{T}_0, x_1, x_2 \in \mathbb{R}^n.$$

This implies that $V(t, x)$ is globally Lipschitzian in x for fixed $t \in \mathbb{T}$.

Next, we will prove property (iii). Let $x(t, t_0, x_0)$ be a unique solution of system (2.1) for each initial point $(t_0, x_0) \in \mathbb{T} \times \mathbb{R}^n$. We will consider two cases, $\sigma(t) = t$ and $\sigma(t) > t$, in the proof.

Suppose that $\sigma(t) = t$ and let $\delta \in A_t$. By the uniqueness of solutions of (2.1) and the definition of hS, we have

$$
\begin{aligned}
D^+ & V\big(t, x(t)\big) \\
&= \varlimsup_{\delta \searrow 0} \frac{1}{\delta} \Big[V\big(t + \delta, x(t + \delta, t, x)\big) - V\big(t, x(t)\big) \Big] \\
&= \varlimsup_{\delta \searrow 0} \frac{1}{\delta} \Big[\sup_{\tau \in \mathbb{A}_{t+\delta}} \big| x\big(t + \delta + \tau, t + \delta, x(t + \delta, t, x)\big) \big| h(t + \delta + \tau)^{-1} h(t + \delta) \\
&\quad - \sup_{\tau \in \mathbb{A}_t} \big| x\big(t + \tau, t, x(t)\big) \big| h(t + \tau)^{-1} h(t) \Big] \\
&= \varlimsup_{\delta \searrow 0} \frac{1}{\delta} \Big[\sup_{\tau \in \{\tau \in [\delta, \infty) : t + \tau \in \mathbb{T}\}} \big| x\big(t + \tau, t, x(t)\big) \big| h(t + \tau)^{-1} h(t + \delta) \\
&\quad - \sup_{\tau \in \mathbb{A}_t} \big| x\big(t + \tau, t, x(t)\big) \big| h(t + \tau)^{-1} h(t) \Big] \\
&\le \varlimsup_{\delta \searrow 0} \frac{1}{\delta} \Big[\sup_{\tau \in \{\tau \in \mathbb{R}_+ : t + \tau \in \mathbb{T}\}} \big| x\big(t + \tau, t, x(t)\big) \big| h(t + \tau)^{-1} h(t) \big(h(t + \delta) h(t)^{-1} - 1 \big) \Big] \\
&\le \varlimsup_{\delta \searrow 0} \frac{1}{\delta} \Big[\big(h(t + \delta) h(t)^{-1} - 1 \big) \Big] V\big(t, x(t)\big) \\
&\le \frac{h'(t)}{h(t)} V\big(t, x(t)\big).
\end{aligned}
$$

Suppose that $\sigma(t) > t$. Then it follows from the definition of $V^\Delta(t)$ that

$$
V_*^\Delta(t) = \frac{V(\sigma(t)) - V(t)}{\mu(t)}.
$$

Since the solution of (2.1) is unique, we have the following derivative:

$$
\begin{aligned}
V_*^\Delta(t) &= \frac{V(\sigma(t)) - V(t)}{\mu(t)} \\
&= \frac{1}{\mu(t)} \Big[\sup_{\tau \in \mathbb{A}_{\sigma(t)}} \big| x\big(\sigma(t) + \tau, \sigma(t), x(\sigma(t), t, x)\big) \big| h\big(\sigma(t) + \tau\big)^{-1} h\big(\sigma(t)\big) \\
&\quad - \sup_{\tau \in \mathbb{A}_t} \big| x\big(t + \tau, t, x(t)\big) \big| h(t + \tau)^{-1} h(t) \Big] \\
&= \frac{1}{\mu(t)} \Big[\sup_{\tau \in \mathbb{A}_{t+\mu(t)}} \big| x\big(t + \mu(t) + \tau, t + \mu(t), \\
&\quad x(t + \mu(t), t, x)\big) \big| h\big(t + \mu(t) + \tau\big)^{-1} h\big(t + \mu(t)\big) \\
&\quad - \sup_{\tau \in \mathbb{A}_t} \big| x\big(t + \tau, t, x(t)\big) \big| h(t + \tau)^{-1} h(t) \Big] \\
&= \frac{1}{\mu(t)} \Big[\sup_{\tau \in \{\tau \in [\mu(t), \infty) : t + \tau \in \mathbb{T}\}} \big| x\big(t + \tau, t, x(t)\big) \big| h(t + \tau)^{-1} h\big(t + \mu(t)\big) \\
&\quad - \sup_{\tau \in \mathbb{A}_t} \big| x\big(t + \tau, t, x(t)\big) \big| h(t + \tau)^{-1} h(t) \Big]
\end{aligned}
$$

$$\leq \frac{1}{\mu(t)}\left[h\big(t+\mu(t)\big)h(t)^{-1}-1\right]V\big(t,x(t)\big)$$

$$=\frac{h^{\Delta}(t)}{h(t)}V\big(t,x(t)\big).$$

Thus property (iii) was satisfied for two cases.

The continuity of $V(t,x)$ can be proved in a similar manner of Theorem 3.6.1 in [22] and Theorem 3.1 in [25]. It remains to show that V is continuous in the sense of (iv): let $t \in \mathbb{T}$, $x \in \mathbb{R}^n$ be fixed and choose $\varepsilon > 0$ arbitrary. Then $\delta_1 > 0$ and $\delta_2 > 0$ must be found such that

$$\left|V(\hat{t},\hat{x}) - V(t,x)\right| < \varepsilon$$

holds for all

$$\hat{t} = t + \nu, \quad \nu \in A_t, 0 \leq \nu < \delta_1 \tag{2.21}$$

and all $\hat{x} \in B_{\delta_2}(x)$, where $B_{\delta_2}(x)$ is an open ball centered on x of radius δ_2.

If $t \in \mathbb{T}$ is right scattered, then we can always choose a suitable δ_1 such that $\hat{t} = t$ is the only point satisfying condition (2.21) (see [22, Theorem 3.6.1]). Thus $V(t,x)$ is continuous in $(t,x) \in \mathbb{T} \times \mathbb{R}^n$ since V is globally Lipschitz continuous in x for fixed $t \in \mathbb{T}$.

Suppose that $t \in \mathbb{T}$ is right dense and let $\hat{t} = t + \nu$ for $\nu \in A_t$ with $\nu \geq 0$. Then we have

$$\left|V(\hat{t},\hat{x}) - V(t,x)\right| = \left|V(t+\nu,\hat{x}) - V(t,x)\right|$$

$$\leq \left|V(t+\nu,\hat{x}) - V(t+\nu,x)\right| \tag{2.22}$$

$$+ \left|V(t+\nu,x) - V\big(t+\nu,x(t+\nu,t,x)\big)\right| \tag{2.23}$$

$$+ \left|V\big(t+\nu,x(t+\nu,t,x)\big) - V(t,x)\right|. \tag{2.24}$$

Since $V(t,x)$ is Lipschitzian in x and $x(t+\nu,t,x)$ is continuous in ν, the first two terms (2.22) and (2.23) on the right-hand side of the preceding inequality are small when $|\hat{x} - x|$ and ν are small. That is, we have

$$\left|V(t+\nu,\hat{x}) - V(t+\nu,x)\right| < \frac{\varepsilon}{3}$$

for all $\hat{x} \in B_{\delta_2}(x)$ when $\delta_2 < \frac{\varepsilon}{3c}$ and

$$\left|V(t+\nu,x) - V\big(t+\nu,x(t+\nu,t,x)\big)\right|$$

$$\leq c\left|x - x(t+\nu,t,x)\right|$$

$$< \frac{\varepsilon}{3}, \quad \nu \in A_t, 0 \leq \nu < \hat{\delta}_1,$$

since it follows from $\lim_{\nu\to0,\nu\in A_t} x(t+\nu,t,x) = x$ that there exists a $\hat{\delta}_1 > 0$ such that

$$\left|x - x(t+\nu,t,x)\right| < \frac{\varepsilon}{3c}$$

for all $\nu \in A_t$ with $0 \leq \nu < \hat{\delta}_1$.

Let us consider the third term in (2.24). We note that

$$x\big(t + \nu + \tau, t + \nu, x(t + \nu, t, x)\big) = x(t + \nu + \tau, t, x).$$

Thus we have

$$
\begin{aligned}
&\Big| V\big(t + \nu, x(t + \nu, t, x)\big) - V(t, x)\Big| \\
&\quad = \Bigg| \sup_{\tau \in \mathbb{A}_{t+\nu}} \big|x\big(t + \nu + \tau, t + \nu, x(t + \nu, t, x)\big)\big| h(t + \nu + \tau)^{-1} h(t + \nu) \\
&\qquad\quad - \sup_{\tau \in \mathbb{A}_t} \big|x\big(t + \tau, t, x(t)\big)\big| h(t + \tau)^{-1} h(t) \Bigg| \\
&\quad = \Bigg| \sup_{\tau \in \{\tau \in [\nu, \infty) : t + \tau \in \mathbb{T}\}} \big|x\big(t + \tau, t, x(t)\big)\big| h(t + \tau)^{-1} h(t + \nu) \\
&\qquad\quad - \sup_{\tau \in \mathbb{A}_t} \big|x\big(t + \tau, t, x(t)\big)\big| h(t + \tau)^{-1} h(t) \Bigg| \\
&\quad = \Big| \alpha(\nu) \frac{h(t + \nu)}{h(t)} - \alpha(0) \Big|,
\end{aligned}
$$

where $\alpha(\nu) = \sup_{\tau \in \{\tau \in [\nu, \infty) : t + \tau \in \mathbb{T}\}} |x(t + \tau, t, x(t))| h(t + \tau)^{-1} h(t)$ for $\nu \in A_t$.

We have $\alpha(\nu) \le \alpha(0)$ for all $\nu \in A_t$ with $\nu \ge 0$. Furthermore, $\alpha(\nu)$ is a nonincreasing function in ν with

$$\lim_{\nu \to 0, \nu \in A_t} \alpha(\nu) = \alpha(0).$$

Hence, there exists a $\hat{\delta}_3 > 0$ such that

$$\Big| \alpha(\nu) \frac{h(t + \nu)}{h(t)} - \alpha(0) \Big| < \frac{\varepsilon}{3}, \quad \nu \in A_t, 0 \le \nu \le \hat{\delta}_3.$$

Now, choose $\delta_1 = \min\{\hat{\delta}_1, \hat{\delta}_3\}$. For $\hat{t} = t + \nu$ with $\nu \in A_t$ where $0 \le \nu < \delta_1$ and $\hat{x} \in B_{\delta_2}(x)$, combining all of the above estimates of the terms in (2.22)-(2.24) gives

$$\big| V(\hat{t}, \hat{x}) - V(t, x) \big| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which proves the continuity of $V(t, x)$.

Sufficiency: Assume that $V(t, x)$ satisfies the properties (i)-(iv). Let $x(t, t_0, x_0)$ be any solution of system (2.1). Then it follows from condition (iii) of $V(t, x)$ that

$$V\big(t, x(t)\big) \le V(t_0, x_0) + \int_{t_0}^{t} \frac{h^\Delta(s)}{h(s)} V\big(s, x(s)\big) \Delta s, \quad t \ge t_0.$$

From Gronwall's inequality on time scale [5] and Lemma 2.15 [14], we obtain

$$
\begin{aligned}
V\big(t, x(t)\big) &\le V(t_0, x_0) e_{p(t)}(t, t_0) \\
&\le V(t_0, x_0) h(t) h(t_0)^{-1}, \quad t \ge t_0,
\end{aligned}
\tag{2.25}
$$

where $p(t) = \frac{h^\Delta(t)}{h(t)}$. From (2.25) and property (i) of $V(t,x)$, we have

$$\left| x(t, t_0, x_0) \right| \le c|x_0| h(t) h(t_0)^{-1},$$

for each $t \ge t_0$ and $x_0 \in \mathbb{R}^n$. Hence the zero solution $x = 0$ of (2.1) is GhS. This completes the proof of the theorem. □

Remark 2.10 Assume that condition (H) is satisfied for $\mathbb{T} = \mathbb{R}$. Furthermore, suppose that $h'(t)$ exists and is continuous on $\mathbb{R}_+$. Then we can obtain Theorems 2.4 and 2.6 in [7] as a continuous version of Theorem 2.12.

Also, we can obtain the following result as a discrete version of Theorem 2.12.

Corollary 2.13 [11, Theorem 3.7] *Assume that $f_x(n, 0)$ is n_∞-summably similar to $f_x(n, x(n, n_0, x_0))$ for $n \ge n_0 \ge 0$ and every $x_0 \in \mathbb{R}^m$ with $\frac{h(n)}{h(n+1)}|F(n)| \in l_1(\mathbb{N}(n_0))$. Then system (2.1) is GhS if and only if there exists a function $V(n, z)$ defined on $\mathbb{N}(n_0) \times \mathbb{R}^m$ such that the following properties hold:*

(i) $V(n, z)$ is continuous on $\mathbb{N}(n_0) \times \mathbb{R}^m$;

(ii) $|x - y| \le V(n, x - y)| \le c|x - y|$ for $(n, x, y) \in \mathbb{N}(n_0) \times \mathbb{R}^m \times \mathbb{R}^m$;

(iii) $|V(n, z_1) - V(n, z_2)| \le c|z_1 - z_2|$ for $n \in \mathbb{N}(n_0)$, $z_1, z_2 \in \mathbb{R}^m$;

(iv) $\frac{\Delta V(n, x-y)}{V(n, x-y)} \le \frac{\Delta h(n)}{h(n)}$ for $(n, x, y) \in \mathbb{N}(n_0) \times \mathbb{R}^m \times \mathbb{R}^m$ with $x \ne y$.

Remark 2.11 Choi *et al.* [8, 9] introduced the notion of n_∞-similarity which is slightly different from n_∞-summable similarity and studied a general variational stability for a nonlinear difference system via n_∞-similarity and Lyapunov functions. We can obtain the discrete analogues [8, Theorem 5, Corollary 8] and [9, Theorem 2.1] as a discrete version of Theorem 2.12.

Remark 2.12 Choi *et al.* [14, Theorem 2.16] studied h-stability for linear dynamic equations on time scales by using the unified time scale quadratic Lyapunov functions. Also, Mukdasai and Niamsup [31, Theorem 3.13] derived a sufficient condition for h-stability for a linear time-varying system with nonlinear perturbation on time scales by constructing appropriate Lyapunov functions.

We can obtain the following Massera type converse theorem for the uniform exponential asymptotic stability of linear dynamic equations on time scales as a special case of Theorem 2.12.

Corollary 2.14 [25, Theorem 3.1] *Assume that $f(t, x) = A(t)x$ is linear, where $A \in C_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R}^n \times \mathbb{R}^n)$. If system (2.1) is hS with $h(t) = e^{-\lambda t}$ on time scales $\mathbb{T}$ for a nonnegative constant λ, then there exists a function $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

(i) $|x| \le V(t, x) \le K|x|$ for all $t \in \mathbb{T}, x \in \mathbb{R}^n$.

(ii) $|V(t, x_1) - V(t, x_2)| \le K|x_1 - x_2|$ for any fixed $t \in \mathbb{T}$ and all $x_1, x_2 \in \mathbb{R}^n$.

(iii) The upper right Dini derivative of V_* exists and the estimates

$V_*^\Delta(t) \le \overline{\xi}_{-\lambda}(t) V_*(t) < 0$ hold for all $t \ge t_0$ for a positive constants K and λ. Here the

function $\overline{\xi}_\lambda : \mathbb{T} \to \mathbb{R}$ is given by

$$\overline{\xi}_\lambda(t) = \begin{cases} \frac{\exp(\mu(t)\lambda)-1}{\mu(t)}, & t < \sigma(t), \\ \lambda, & t = \sigma(t). \end{cases}$$

(iv) V is continuous from the right in $(t,x) \in \mathbb{T} \times \mathbb{R}^n$, that is,

$$\lim_{(\hat{t},\hat{x}) \to (t,x), \hat{t} \geq t} \left| V(\hat{t},\hat{x}) - V(t,x) \right| = 0.$$

3 Examples

In this section we give some examples which illustrate some results from the previous section.

To illustrate the notion of u_∞-similarity, we will give an example for scalar functions defined on time scales.

Example 3.1 Let $a, b : \mathbb{T} \to \mathbb{R}$ be scalar functions given by

$$a(t) = e_\alpha(t, t_0) - 1, \qquad b(t) = \frac{1}{2} e_\alpha(t, t_0) - 1,$$

where α is a negative regressive constant and $\lim_{t \to \infty} e_\alpha(t, t_0) = 0$ for fixed $t_0 \in \mathbb{T}$. If we put $s(t) = 1$ for each $t \in \mathbb{T}$, then $s(t)$ and $s^{-1}(t)$ are bounded and nonzero functions. Moreover, we have

$$s^\Delta(t) + s(\sigma(t))b(t) - a(t)s(t) = -\frac{1}{2} e_\alpha(t, t_0) = f(t), \quad t \in \mathbb{T}^\kappa,$$

where $f(t) = -\frac{1}{2} e_\alpha(t, t_0)$.

Thus we have

$$\int_{t_0}^\infty \left| f(t) \right| \Delta t = \int_{t_0}^\infty \frac{1}{2} e_\alpha(t, t_0) \Delta t \leq -\frac{1}{2\alpha} < \infty.$$

This implies that a and b are u_∞-similar on $\mathbb{T}$.

For the examples of nonscalar functions about t_∞-similarity on $\mathbb{T} = \mathbb{R}$ and n_∞-summable similarity on $\mathbb{T} = \mathbb{Z}$, see [7, Example] and [11, Example 2.6] respectively.

Example 3.2 To illustrate Lemma 2.5, we consider the linear dynamic system

$$x^\Delta = A(t)x = \begin{pmatrix} \frac{-e^{-t}}{2+e^{-t}} & 0 \\ 0 & 0 \end{pmatrix} x, \quad x(t_0) = x_0, t \geq t_0 \in \mathbb{T}, \tag{3.1}$$

where $A(t) = \begin{pmatrix} \frac{-e^{-t}}{2+e^{-t}} & 0 \\ 0 & 0 \end{pmatrix}$. If $\mu(t) < 2e^t + 1$ for $t \in \mathbb{T}$, then (3.1) is h-stable.

Proof A matrix exponential function $\Phi_A(t, t_0)$ of (3.1) is given by

$$\Phi_A(t, t_0) = \begin{pmatrix} e_p(t, t_0) & 0 \\ 0 & 1 \end{pmatrix},$$

where $p(t) = \frac{-e^{-t}}{2+e^{-t}}$ and $e_p(t, t_0) = \exp \int_{t_0}^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau$. Here the cylinder transformation $\xi_\mu(z)$ is given by

$$\xi_\mu(z) = \begin{cases} \frac{1}{\mu} \operatorname{Log}(1 + \mu z) & \text{if } \mu \neq 0, \\ z & \text{if } \mu = 0. \end{cases}$$

It follows that

$$0 < e_p(t, t_0) = \begin{cases} \exp(\int_{t_0}^t \frac{1}{\mu(\tau)} \operatorname{Log}(1 + \mu(\tau)p(\tau))\Delta\tau) & \text{if } \mu \neq 0, \\ \exp(\int_{t_0}^t p(\tau)\,d\tau) & \text{if } \mu = 0 \end{cases}$$

$$\leq \begin{cases} \exp(\int_{t_0}^t \frac{1}{\mu(\tau)} \operatorname{Log}(1 + \mu(\tau)|p(\tau)|)\Delta\tau) & \text{if } \mu \neq 0, \\ \exp(\int_{t_0}^t |p(\tau)|\Delta\tau) & \text{if } \mu = 0 \end{cases}$$

$$= \exp \int_{t_0}^t |p(\tau)|\Delta\tau = \exp \int_{t_0}^t \frac{e^{-t}}{2 + e^{-t}} \Delta\tau$$

$$\leq M$$

for each $t, t_0 \in \mathbb{T}$ with $t \geq t_0$, where M is a positive constant. Thus we obtain

$$\left|\Phi_A(t, t_0)\right| = \left| \begin{pmatrix} e_p(t, t_0) & 0 \\ 0 & 1 \end{pmatrix} \right| \leq ch(t)h(t_0)^{-1}, \quad t \geq t_0,$$

where $h(t) = e_p(t, t_1)$ is a positive bounded rd-continuous function for a fixed point $t_1 \in \mathbb{T}$, and $c := M + 1$ is a positive constant. Hence system (3.1) is h-stable by Lemma 2.5. □

To illustrate that the converse of Theorem 2.8 does not hold in general, we give the following example.

Example 3.3 [11, Example 5.2] Let $\mathbb{T}$ be the unbounded above time scales with $\mu(t) < 2$ for each $t \in \mathbb{T}$. We consider the nonlinear dynamic equation

$$x^\Delta = f(t, x) = -\frac{1}{2}x + x^2, \quad x(t_0) = x_0 = 1 \tag{3.2}$$

and its variational dynamic equation

$$v^\Delta(t) = f_x(t, 0)v(t) = -\frac{1}{2}v(t), \quad v(t_0) = v_0 \neq 0, \tag{3.3}$$

where $f_x(t, x) = -\frac{1}{2} + 2x$. Then $v = 0$ of (3.3) is h-stable, but $x = 0$ of (3.2) is not h-stable.

Proof Since the fundamental solution is $\phi(t) = e_{-\frac{1}{2}}(t, t_0)v_0$ for each $t \geq t_0$, Eq. (3.3) is h-stable with a positive bounded function $h(t) = e_{-\frac{1}{2}}(t, \hat{t}_0)$ for a fixed point $\hat{t}_0 \in \mathbb{T}$. But (3.2) is not h-stable because there exists an unbounded solution $x(t, \hat{t}_0, 1)$ of (3.2) such that

$$x(t, \hat{t}_0, 1) = x(t) > t, \quad t \in \mathbb{T}_0. \tag{□}$$

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