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Fixed points and quadratic equations connected with homomorphisms and derivations on non-Archimedean algebras

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Abstract

We apply the fixed point method to prove the stability of the systems of functional equations

$$\begin{cases} f(xy) = f(x)f(y); \\ f(ax+by) + f(ax-by) = 2a^2 f(x) + 2b^2 f(y), \end{cases}$$

$$\begin{cases} f(xy) = x^2 f(y) + f(x)y^2; \\ f(ax+by) + f(ax-by) = 2a^2 f(x) + 2b^2 f(y), \end{cases}$$

on non-Archimedean Banach algebras. Moreover, we give some applications of our results in non-Archimedean Banach algebras over p-adic numbers.

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1 Introduction and preliminaries

The story of the stability of functional equations dates back to 1925 when a stability result appeared in the celebrated book by Gy. Pólya and G. Szegő [1]. In 1940, Ulam [2] posed the famous Ulam stability problem which was partially solved by Hyers [3] in the framework of Banach spaces. Later, Aoki [4] considered the stability problem with unbounded Cauchy differences. In 1978, Th. M. Rassias [5] provided a generalization of the Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. Găvruta [6] obtained a generalized result of Th. M. Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function. On the other hand, J. M. Rassias [7–10] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity, a counterexample was given by Găvruta [11].

Bourgin [12] proved the stability of ring homomorphisms in two unital Banach algebras. Badora [13] gave a generalization of the Bourgin's result. The stability result concerning derivations on operator algebras was first obtained by Šemrl [14]. In [15], Badora proved the stability of functional equation $f(xy) = xf(y) + f(x)y$, where f is a mapping on normed algebra A with unit.

Let $\mathcal{A}, \mathcal{B}$ be two algebras. A mapping $f : \mathcal{A} \to \mathcal{B}$ is called a quadratic homomorphism if f is a quadratic mapping satisfying $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{A}$. For instance, let $\mathcal{A}$ be commutative. Then the mapping $f : \mathcal{A} \to \mathcal{A}$, defined by $f(x) = x^2$ ($x \in \mathcal{A}$), is a quadratic homomorphism.

A mapping $f : \mathcal{A} \to \mathcal{A}$ is called a quadratic derivation if f is a quadratic mapping satisfying $f(xy) = x^2 f(y) + f(x)y^2$ for all $x, y \in \mathcal{A}$. For instance, consider the algebra of 2×2 matrices

$$\mathcal{A} = \left\{ \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} : c_1, c_2 \in \mathbb{C} \text{ (complex field)} \right\}.$$

Then it is easy to see that the mapping $f : \mathcal{A} \to \mathcal{A}$, defined by $f\left(\left[\begin{smallmatrix} c_1 & c_2 \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & c_2^2 \\ 0 & 0 \end{smallmatrix}\right]$, is a quadratic derivation. We note that quadratic derivations and ring derivations are different.

Arriola and Beyer in [16] initiated the stability of functional equations in non-Archimedean spaces. In fact they established the stability of Cauchy functional equations over p-adic fields. After their results, some papers (see for instance [17–24]) on the stability of other equations in such spaces have been published. Although different methods are known for establishing the stability of functional equations, almost all proofs depend on the Hyers's method in [3]. In 2003, Radu [25] employed the alternative fixed point theorem, due to Diaz and Margolis [26], to prove the stability of the Cauchy additive functional equation. Subsequently, this method was applied to investigate the Hyers-Ulam stability for the Jensen functional equation [27], as well as for the Cauchy functional equation [28], by considering a general control function $\varphi(x, y)$ with suitable properties. Using such an elegant idea, several authors applied the method to investigate the stability of some functional equations, see [29–33].

Recently, Eshaghi and Khodaei [34] considered the following quadratic functional equation:

$$f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y) \tag{1.1}$$

and proved the Hyers-Ulam stability of the above functional equation in classical Banach spaces. Recently, Eshaghi [35] proved the Hyers-Ulam stability of homomorphisms and derivations on non-Archimedean Banach algebras.

In the present paper, we adopt the idea of Cădariu and Radu [28] to establish the stability of quadratic homomorphisms and quadratic derivations related to the quadratic functional equation (1.1) over non-Archimedean Banach algebras. Some applications of our results in non-Archimedean Banach algebras over p-adic numbers will be exhibited.

Now, we recall some notations and basic definitions used later on in the paper.

In 1897, Hensel [36] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. During the last three decades p-adic numbers have gained the interest of physicists for their research, in particular in problems coming from quantum physics, p-adic strings and superstrings [37, 38]. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: For $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$ (see [39, 40]).

Example 1.1 Let p be a prime number. For any nonzero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p =$

p^{-r}. Then $\|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $\|$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field. Note that if $p \geq 3$, then $|2^n| = 1$ in for each integer n.

Let $\mathbb{K}$ denote a field and function (valuation absolute) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$. A non-Archimedean valuation is a function $|\cdot|$ that satisfies the strong triangle inequality; namely, $|x+y| \leq \max\{|x|,|y|\} \leq |x|+|y|$ for all $x,y \in \mathbb{K}$. The associated field $\mathbb{K}$ is referred to as a non-Archimedean field. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|\cdot|$ is nontrivial, *i.e.*, there is a $z \in \mathbb{K}$ such that $|z| \neq 0, 1$.

Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely, $\|x+y\| \leq \max\{\|x\|,\|y\|\}$ for all $x,y \in X$. Then $(X,\|\cdot\|)$ is called a non-Archimedean space. In any such a space a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy if and only if $\{x_{n+1}-x_n\}_{n\in\mathbb{N}}$ converges to zero. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x,y \in \mathcal{A}$. For more details the reader is referred to [41, 42].

Let X be a nonempty set and $d : X \times X \to [0,\infty]$ satisfying: $d(x,y) = 0$ if and only if $x = y$, $d(x,y) = d(y,x)$ and $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ (strong triangle inequality), for all $x,y,z \in X$. Then (X,d) is called a non-Archimedean generalized metric space. (X,d) is called complete if every d-Cauchy sequence in X is d-convergent.

Using the strong triangle inequality in the proof of the main result of [26], we get to the following result:

Theorem 1.2 (Non-Archimedean Alternative Contraction Principle) *If (Ω,d) is a non-Archimedean generalized complete metric space and $T : \Omega \to \Omega$ a strictly contractive mapping (that is $d(T(x),T(y)) \leq Ld(x,y)$, for all $x,y \in T$ and a Lipschitz constant $L < 1$). Let $x \in \Omega$, then either*

(i) $d(T^n(x), T^{n+1}(x)) = \infty$ for all $n \geq 0$, or

(ii) there exists some $n_0 \geq 0$ such that $d(T^n(x), T^{n+1}(x)) < \infty$ for all $n \geq n_0$; the sequence $\{T^n(x)\}$ is convergent to a fixed point x^* of T; x^* is the unique fixed point of T in the set

$$\Lambda = \{y \in \Omega : d(T^{n_0}(x), y) < \infty\}$$

and $d(y,x^*) \leq d(y,T(y))$ for all y in this set.

2 Non-Archimedean approximately quadratic homomorphisms and quadratic derivations

Hereafter, unless otherwise stated, we will assume that $\mathcal{A}$ and $\mathcal{B}$ are two non-Archimedean Banach algebras. Also, let $|4| < 1$; and we assume that $4 \neq 0$ in $\mathbb{K}$ (*i.e.*, the characteristic of $\mathbb{K}$ is not 4).

Theorem 2.1 *Let $\ell \in \{-1,1\}$ be fixed and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0,\infty)$ such that*

$$\left\|f(ax+by)+f(ax-by)+f(zw)-2a^2f(x)-2b^2f(y)-f(z)f(w)\right\| \le \varphi(x,y,z,w) \tag{2.1}$$

for all $x,y,z,w \in \mathcal{A}$ and nonzero fixed integers a, b. If there exists an $L < 1$ such that

$$\varphi(x,y,z,w) \le |4|^{\ell(\ell+2)} L\varphi\left(\frac{x}{2^\ell},\frac{y}{2^\ell},\frac{z}{2^\ell},\frac{w}{2^\ell}\right) \tag{2.2}$$

for all $x,y,z,w \in \mathcal{A}$. Then there exists a unique quadratic homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\left\|f(x)-\mathcal{H}(x)\right\| \le \frac{L^{\frac{1-\ell}{2}}}{|4|}\psi(x) \tag{2.3}$$

for all $x \in \mathcal{A}$, where

$$\psi(x) := \max\left\{\varphi\left(\frac{x}{a},\frac{x}{b},0,0\right),\varphi\left(\frac{x}{a},0,0,0\right),\frac{1}{|2b^2|}\varphi(x,x,0,0),\right.$$
$$\left.\frac{1}{|2b^2|}\varphi(x,-x,0,0),\varphi\left(0,\frac{x}{b},0,0\right)\right\}.$$

Proof Letting $z = w = 0$ in (2.1), we get

$$\left\|f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y)\right\| \le \varphi(x,y,0,0) \tag{2.4}$$

for all $x,y \in \mathcal{A}$. Setting $y = -y$ in (2.4), we get

$$\left\|f(ax-by)+f(ax+by)-2a^2f(x)-2b^2f(-y)\right\| \le \varphi(x,-y,0,0) \tag{2.5}$$

for all $x,y \in \mathcal{A}$. It follows from (2.4) and (2.5) that

$$\left\|2b^2f(y)-2b^2f(-y)\right\| \le \max\left\{\varphi(x,y,0,0),\varphi(x,-y,0,0)\right\} \tag{2.6}$$

for all $x,y \in \mathcal{A}$. Putting $y = 0$ in (2.4), we get

$$\left\|2f(ax)-2a^2f(x)\right\| \le \varphi(x,0,0,0) \tag{2.7}$$

for all $x \in \mathcal{A}$. Setting $x = 0$ in (2.4), we get

$$\left\|f(by)+f(-by)-2b^2f(y)\right\| \le \varphi(0,y,0,0) \tag{2.8}$$

for all $y \in \mathcal{A}$. Putting $y = by$ in (2.6), we get

$$\left\|f(by)-f(-by)\right\| \le \max\left\{\frac{1}{|2b^2|}\varphi(x,by,0,0),\frac{1}{|2b^2|}\varphi(x,-by,0,0)\right\} \tag{2.9}$$

for all $x, y \in \mathcal{A}$. It follows from (2.8) and (2.9) that

$$\|2f(by) - 2b^2 f(y)\| \le \max\left\{\frac{1}{|2b^2|}\varphi(x, by, 0, 0), \frac{1}{|2b^2|}\varphi(x, -by, 0, 0), \varphi(0, y, 0, 0)\right\} \quad (2.10)$$

for all $x, y \in \mathcal{A}$. Replacing x and y by $\frac{x}{a}$ and $\frac{x}{b}$ in (2.4), respectively, we get

$$\left\|f(2x) - 2a^2 f\left(\frac{x}{a}\right) - 2b^2 f\left(\frac{x}{b}\right)\right\| \le \varphi\left(\frac{x}{a}, \frac{x}{b}, 0, 0\right) \quad (2.11)$$

for all $x \in \mathcal{A}$. Setting $x = \frac{x}{a}$ in (2.7), we get

$$\left\|2a^2 f\left(\frac{x}{a}\right) - 2f(x)\right\| \le \varphi\left(\frac{x}{a}, 0, 0, 0\right) \quad (2.12)$$

for all $x \in \mathcal{A}$. Putting $y = \frac{x}{b}$ in (2.10), we get

$$\left\|2b^2 f\left(\frac{x}{b}\right) - 2f(x)\right\| \le \max\left\{\frac{1}{|2b^2|}\varphi(x, x, 0, 0), \frac{1}{|2b^2|}\varphi(x, -x, 0, 0), \varphi\left(0, \frac{x}{b}, 0, 0\right)\right\} \quad (2.13)$$

for all $x \in \mathcal{A}$. It follows from (2.11), (2.12) and (2.13) that

$$\|f(2x) - 4f(x)\| \le \psi(x) \quad (2.14)$$

for all $x \in \mathcal{A}$. For every $g, h : \mathcal{A} \to \mathcal{B}$, define

$$d(g, h) := \inf\{C \in (0, \infty) : \|g(x) - h(x)\| \le C\psi(x), \forall x \in \mathcal{A}\}.$$

Hence, d defines a complete generalized non-Archimedean metric on $\Omega := \{g | g : \mathcal{A} \to \mathcal{B}, g(0) = 0\}$ (see [27, 28, 33]). Let $T : \Omega \to \Omega$ be defined by $Tg(x) = \frac{1}{4^\ell} g(2^\ell x)$ for all $x \in \mathcal{A}$ and all $g \in \Omega$. If for some $g, h \in \Omega$ and $C > 0$,

$$\|g(x) - h(x)\| \le C\psi(x)$$

for all $x \in \mathcal{A}$, then

$$\|Tg(x) - Th(x)\| \le \frac{1}{|4|^\ell}\|g(2^\ell x) - h(2^\ell x)\| \le \frac{C}{|4|^\ell}\psi(2^\ell x) \le LC\psi(x)$$

for all $x \in \mathcal{A}$, so

$$d(Tg, Th) \le Ld(g, h).$$

Hence T is a strictly contractive mapping on Ω with the Lipschitz constant L. It follows from (2.14) by using (2.2) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \psi\left(\frac{x}{2}\right) \le \frac{L}{|4|}\psi(x)$$

and

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{|4|} \psi(x)$$

for all $x \in \mathcal{A}$, that is, $d(f, Tf) \leq \frac{L^{\frac{1-\ell}{2}}}{|4|} < \infty$.

Now, by the non-Archimedean alternative contraction principle, T has a unique fixed point $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, which $\mathcal{H}$ is defined by

$$\mathcal{H}(x) = \lim_{n\to\infty} T^n f(x) = \lim_{n\to\infty} \frac{1}{4^{\ell n}} f\left(2^{\ell n} x\right) \tag{2.15}$$

for all $x \in \mathcal{A}$. By (2.2),

$$\lim_{n\to\infty} \frac{1}{|4|^{\ell(\ell+2)n}} \varphi\left(2^{\ell n}x, 2^{\ell n}y, 2^{\ell n}z, 2^{\ell n}w\right) = 0 \tag{2.16}$$

for all $x, y, z, w \in \mathcal{A}$. It follows from (2.1), (2.15) and (2.16) that

$$\begin{aligned}
&\left\| \mathcal{H}(ax + by) + \mathcal{H}(ax - by) - 2a^2\mathcal{H}(x) - 2b^2\mathcal{H}(y) \right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{|4|^{\ell n}} \left\| f\left(2^{\ell n}ax + 2^{\ell n}by\right) + f\left(2^{\ell n}ax - 2^{\ell n}by\right) - 2a^2 f\left(2^{\ell n}x\right) - 2b^2 f\left(2^{\ell n}y\right) \right\| \\
&\quad \leq \lim_{n\to\infty} \frac{1}{|4|^{\ell n}} \varphi\left(2^{\ell n}x, 2^{\ell n}y, 0, 0\right) \\
&\quad \leq \lim_{n\to\infty} \frac{1}{|4|^{\ell(\ell+2)n}} \varphi\left(2^{\ell n}x, 2^{\ell n}y, 0, 0\right) = 0
\end{aligned}$$

for all $x, y \in \mathcal{A}$. This shows that $\mathcal{H}$ is quadratic. Also,

$$\begin{aligned}
\left\| \mathcal{H}(zw) - \mathcal{H}(z)\mathcal{H}(w) \right\| &= \lim_{n\to\infty} \frac{1}{|4|^{2\ell n}} \left\| f\left(4^{\ell n}zw\right) - f\left(2^{\ell n}z\right) f\left(2^{\ell n}w\right) \right\| \\
&\leq \lim_{n\to\infty} \frac{1}{|4|^{2\ell n}} \varphi\left(0, 0, 2^{\ell n}z, 2^{\ell n}w\right) \\
&\leq \lim_{n\to\infty} \frac{1}{|4|^{\ell(\ell+2)n}} \varphi\left(0, 0, 2^{\ell n}z, 2^{\ell n}w\right) = 0
\end{aligned}$$

for all $z, w \in \mathcal{A}$. Therefore, $\mathcal{H}$ is a quadratic homomorphism. Moreover, by the non-Archimedean alternative contraction principle,

$$d(f, \mathcal{H}) \leq d(f, Tf) \leq \frac{L^{\frac{1-\ell}{2}}}{|4|}.$$

This implies the inequality (2.3) holds. □

Corollary 2.2 *Let $\mathcal{A}$, $\mathcal{B}$ be non-Archimedean Banach algebras over $\mathbb{Q}_2$, $\ell \in \{-1, 1\}$ be fixed and θ, r be non-negative real numbers such that $\ell r > 2\ell(\ell + 2)$. Suppose that a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies $f(0) = 0$ and*

$$\begin{aligned}
&\left\| f(ax + by) + f(ax - by) + f(zw) - 2a^2 f(x) - 2b^2 f(y) - f(z)f(w) \right\| \\
&\quad \leq \theta\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right)
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$, where a, b are positive fixed integers. Then there exists a unique quadratic homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\| \le 2^{\frac{\ell(4-r)+r}{2}} \theta \|x\|^r \begin{cases} 4, & \gcd(a,2) = \gcd(b,2) = 1; \\ \max\{1 + 2^{ir}, 4\}, & a = k2^i, \gcd(b,2) = 1; \\ \max\{1 + 2^{jr}, 2^{2j+2}\}, & \gcd(a,2) = 1, b = m2^j; \\ \max\{2^{ir} + 2^{jr}, 2^{2j+2}\}, & a = k2^i, b = m2^j; \end{cases}$$

for all $x \in \mathcal{A}$, where $i, j, k, m \ge 1$ are integers and $\gcd(k,2) = \gcd(m,2) = 1$.

Proof The proof follows from Theorem 2.1, by taking

$$\varphi(x, y, z, w) = \theta\big(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\big)$$

for all $x, y, z, w \in \mathcal{A}$. Then we choose $L = 2^{\ell(2\ell+4-r)}$ and we get the desired result. □

Corollary 2.3 *Let $\mathcal{A}$, $\mathcal{B}$ be non-Archimedean Banach algebras over $\mathbb{Q}_2$, $\ell \in \{-1, 1\}$ be fixed and δ, s be non-negative real numbers such that $\ell s > 2\ell(\ell + 2)$. Suppose that a mapping $f : \mathcal{A} \to \mathcal{B}$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(ax + by) + f(ax - by) + f(zw) - 2a^2 f(x) - 2b^2 f(y) - f(z)f(w)\| \\ &\quad\le \delta \max\big\{\|x\|^s, \|y\|^s, \|z\|^s, \|w\|^s\big\} \end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$, where a, b are positive fixed integers. Then there exists a unique quadratic homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\begin{aligned} &\|f(x) - \mathcal{H}(x)\| \\ &\quad\le 2^{\frac{\ell(4-s)+s}{2}} \delta \|x\|^s \begin{cases} 2, & \gcd(a,2) = \gcd(b,2) = 1; \\ \max\{2^{is}, 2\}, & a = k2^i, \gcd(b,2) = 1; \\ \max\{2^{js}, 2^{2j+1}\}, & \gcd(a,2) = 1, b = m2^j \vee a = k2^i, b = m2^j\ (j \ge i); \\ \max\{2^{is}, 2^{2j+1}\}, & a = k2^i, b = m2^j\ (i \ge j); \end{cases} \end{aligned}$$

for all $x \in \mathcal{A}$, where $i, j, k, m \ge 1$ are integers and $\gcd(k,2) = \gcd(m,2) = 1$.

Theorem 2.4 *Let $\ell \in \{-1, 1\}$ be fixed and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ if $\ell = -1$, for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying (2.2) and*

$$\begin{aligned} &\|f(ax + by) + f(ax - by) + f(zw) - 2a^2 f(x) - 2b^2 f(y) - z^2 f(w) - f(z)w^2\| \\ &\quad\le \varphi(x, y, z, w) \end{aligned} \tag{2.17}$$

for all $x, y, z, w \in \mathcal{A}$ and nonzero fixed integers a, b. Then there exists a unique quadratic derivation $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \mathcal{D}(x)\| \le \frac{L^{\frac{1-\ell}{2}}}{|4|} \psi(x) \tag{2.18}$$

for all $x \in \mathcal{A}$, where $\psi(x)$ is defined as in Theorem 2.1.

Proof By (2.2), if $\ell = 1$, we obtain $\varphi(0,0,0,0) = 0$. Letting $x = y = z = w = 0$ in (2.17), we get $f(0) \leq \varphi(0,0,0,0)$. So $f(0) = 0$ for $\ell = 1$.

By the same reasoning as that in the proof of Theorem 2.1, there exists a unique quadratic mapping $\mathcal{D}: \mathcal{A} \to \mathcal{A}$ satisfying (2.18). The mapping $\mathcal{D}: \mathcal{A} \to \mathcal{A}$ is given by $\mathcal{D}(x) = \lim_{n\to\infty} T^n f(x) = \lim_{n\to\infty} \frac{1}{4^{\ell n}} f(2^{\ell n}x)$ for all $x \in \mathcal{A}$. It follows from (2.17) that

$$\begin{aligned}
&\left\|\mathcal{D}(zw) - z^2\mathcal{D}(w) - \mathcal{D}(z)w^2\right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{|4|^{2\ell n}} \left\|f\left(4^{\ell n}zw\right) - \left(2^{\ell n}z\right)^2 f\left(2^{\ell n}w\right) - f\left(2^{\ell n}z\right)\left(2^{\ell n}w\right)^2\right\| \\
&\quad \leq \lim_{n\to\infty} \frac{1}{|4|^{2\ell n}} \varphi\left(0,0,2^{\ell n}z, 2^{\ell n}w\right) \\
&\quad \leq \lim_{n\to\infty} \frac{1}{|4|^{\ell(\ell+2)n}} \varphi\left(0,0,2^{\ell n}z, 2^{\ell n}w\right) = 0
\end{aligned}$$

for all $z, w \in \mathcal{A}$. Therefore $\mathcal{D}: \mathcal{A} \to \mathcal{A}$ is a quadratic derivation satisfying (2.18). □

Very recently, J. M. Rassias [43] considered the Cauchy difference controlled by the product and sum of powers of norms, that is, $\theta\{\|x\|^p\|y\|^p + (\|x\|^{2p} + \|y\|^{2p})\}$.

Corollary 2.5 *Let $\mathcal{A}$ be a non-Archimedean Banach algebra over $\mathbb{Q}_2$, $\ell \in \{-1, 1\}$ be fixed and ε, p, q be non-negative real numbers such that $\ell(p+q) > 2\ell(\ell+2)$. Suppose that a mapping $f: \mathcal{A} \to \mathcal{A}$ satisfies*

$$\begin{aligned}
&\left\|f(ax+by) + f(ax-by) + f(zw) - 2a^2f(x) - 2b^2f(y) - z^2f(w) - f(z)w^2\right\| \\
&\quad \leq \varepsilon\left(\|x\|^p\|y\|^q + \|z\|^p\|w\|^q\right)
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$, where a, b are positive fixed integers. Then there exists a unique quadratic derivation $\mathcal{D}: \mathcal{A} \to \mathcal{A}$ such that

$$\begin{aligned}
&\left\|f(x) - \mathcal{D}(x)\right\| \\
&\quad \leq 2^{\frac{\ell(4-p-q)+p+q}{2}} \varepsilon\|x\|^{p+q} \begin{cases} 2, & \gcd(a,2) = \gcd(b,2) = 1; \\ \max\{2^{ip}, 2\}, & a = k2^i, \gcd(b,2) = 1; \\ \max\{2^{jq}, 2^{2j+1}\}, & \gcd(a,2) = 1, b = m2^j; \\ \max\{2^{ip+jq}, 2^{2j+1}\}, & a = k2^i, b = m2^j; \end{cases}
\end{aligned}$$

for all $x \in \mathcal{A}$, where $i, j, k, m \geq 1$ are integers and $\gcd(k,2) = \gcd(m,2) = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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