Wang and Liu *Advances in Difference Equations* 2012, **2012**:116
http://www.advancesindifferenceequations.com/content/2012/1/116

Advances in Difference Equations
a SpringerOpen Journal

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Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order

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Abstract
By using Schauder's fixed point theorem and the contraction mapping principle, we discuss the existence of solutions for nonlinear fractional differential equations with fractional anti-periodic boundary conditions. Some examples are given to illustrate the main results.

Keywords: fractional differential equations; boundary value problem; anti-periodic; fixed point theorem

1 Introduction

Fractional calculus has been recognized as an effective modeling methodology by researchers. Fractional differential equations are generalizations of classical differential equations to an arbitrary order. They have broad application in engineering and sciences such as physics, mechanics, chemistry, economics and biology, *etc.* [1–4]. For some recent development on the topic, see [5–13] and the references therein.

In [14], Ahmad *et al.* considered the following anti-periodic fractional boundary value problems:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), & t \in [0,T], T > 0, 1 < q \leq 2, \\ x(0) = -x(T), \quad {}^cD^p x(0) = -{}^cD^p x(T), & 0 < p < 1, \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, and f is a given continuous function. The results are based on some standard fixed point principles.

In recent years, there has been a great deal of research into the questions of existence and uniqueness of solutions to anti-periodic boundary value problems for differential equations. First, second and higher-order differential equations with anti-periodic boundary value conditions have been considered in papers [14–21]. The existence of solutions for anti-periodic boundary value problems for fractional differential equations was studied in [18–27].

In this paper, we investigate the existence and uniqueness of solutions for an anti-periodic fractional boundary value problem given by

$$\begin{cases} {}^cD^\alpha x(t) = f\big(t, x(t), {}^cD^q x(t)\big), & t \in [0,T], \\ x(0) = -x(T), \qquad {}^cD^p x(0) = -{}^cD^p x(T), \end{cases} \tag{2}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, T is a positive constant, $1 < \alpha \le 2$, $0 < p, q < 1$, $\alpha - q \ge 1$ and f is a given continuous function.

2 Preliminaries

Theorem 2.1 ([28]) *Let E be a closed, convex and nonempty subset of a Banach space X, let $F : E \to E$ be a continuous mapping such that FE is a relatively compact subset of X. Then F has at least one fixed point in E.*

Theorem 2.2 ([29]) *Let p and q be two positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $|f(x)|^p$ and $|g(x)|^q$ are Riemann integrable on $[a,b]$, then*

$$\left| \int_a^b f(x)g(x)\,dx \right| \le \left[\int_a^b |f(x)|^p\,dx \right]^{\frac{1}{p}} \left[\int_a^b |g(x)|^q\,dx \right]^{\frac{1}{q}}.$$

Lemma 2.1 ([14]) *For any $y \in C[0,T]$, a unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^\alpha x(t) = y(t), & t \in [0,T], T > 0, 1 < \alpha \le 2, \\ x(0) = -x(T), \qquad {}^cD^p x(0) = -{}^cD^p x(T), \end{cases} \tag{3}$$

is

$$x(t) = \int_0^T G(t,s)y(s)\,ds, \tag{4}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1} - \frac{1}{2}(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{\Gamma(2-p)(T-2t)(T-s)^{\alpha-p-1}}{2\Gamma(\alpha-p)T^{1-p}}, & s \le t, \\ -\dfrac{(T-s)^{\alpha-1}}{2\Gamma(\alpha)} + \dfrac{\Gamma(2-p)(T-2t)(T-s)^{\alpha-p-1}}{2\Gamma(\alpha-p)T^{1-p}}, & t \le s. \end{cases} \tag{5}$$

Remark 2.1 For $p \to 1^-$ the solution of the classical anti-periodic problem (${}^cD^\alpha x(t) = f(t, x(t), {}^cD^q x(t))$, $x(0) = -x(T)$, $x'(0) = -x'(T)$, $0 \le t \le T$, $1 < \alpha \le 2$, $0 < q < 1$, $\alpha - q \ge 1$) is given in [30].

3 Main results

Let $J = [0,T]$ and $C(J)$ be the space of all continuous real functions defined on J. Define the space $X = \{x(t) \in C(J) \text{ and } {}^cD^q x(t) \in C(J), 0 < q < 1\}$ endowed with the norm $\|x\| = \max_{t\in J} |x(t)| + \max_{t\in J} |{}^cD^q x(t)|$. Obviously, $(X, \|\cdot\|)$ is a Banach space.

Theorem 3.1 *Let $f : J \times R \times R \to R$ be a continuous function. Assume that*

(H_1) There exist a constant $l \in (0, \alpha - 1)$ and a real-valued function $m(t) \in L^{\frac{1}{l}}([0,T], (0,\infty))$ such that

$$\left|f(t,x,y)\right| \leq m(t) + d_1|x|^{\rho_1} + d_2|y|^{\rho_2},$$

where $d_1, d_2 \geq 0$, $0 \leq \rho_1, \rho_2 < 1$. Then the problem (2) has at least a solution on $[0,T]$.

Proof Let the condition (H_1) be valid. According to Lemma 2.1, the problem (2) is equivalent to the following integral equation:

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds \\ & + \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds. \end{aligned}$$

Define

$$\begin{aligned} (Fx)(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds \\ & + \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds, \\ B_r = {} & \bigl\{x(t) \in X, \|x\| \leq r, t \in J\bigr\}, \end{aligned}$$

where

$$\begin{aligned} r \geq {} & \max\bigl\{(3Ad_1)^{\frac{1}{1-\rho_1}}, (3Ad_2)^{\frac{1}{1-\rho_2}}, 3K\bigr\}, \\ K = {} & \frac{3MT^{\alpha-l}}{2\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l} + \frac{\Gamma(2-p)MT^{\alpha-l}}{2\Gamma(\alpha-p)}\left(\frac{1-l}{\alpha-p-l}\right)^{1-l} + \frac{M\Gamma(\alpha-l)T^{\alpha-q-l}}{\Gamma(\alpha-1)\Gamma(\alpha-q-l+1)} \\ & \times \left(\frac{1-l}{\alpha-l-1}\right)^{1-l} + \frac{M\Gamma(2-p)T^{\alpha-q-l}}{\Gamma(\alpha-p)\Gamma(2-q)}\left(\frac{1-l}{\alpha-p-l}\right)^{1-l}, \\ A = {} & \frac{T^{\alpha-q}}{\Gamma(\alpha-q+1)} + \frac{\Gamma(2-p)T^{\alpha-q}}{\Gamma(2-q)\Gamma(\alpha-p+1)} + \frac{3T^{\alpha}}{2\Gamma(\alpha+1)} + \frac{\Gamma(2-p)T^{\alpha}}{2\Gamma(\alpha-p+1)}, \end{aligned}$$

and $M = (\int_0^T (m(s))^{\frac{1}{l}}\,ds)^l$. Observe that B_r is a closed, bounded and convex subset of Banach space X. Now, we prove that $F : B_r \to B_r$. For any $x \in B_r$, by Theorem 2.2 (Hölder inequality), we have

$$\begin{aligned} \bigl|(Fx)(t)\bigr| = {} & \biggl|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds \\ & + \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\,ds\biggr| \\ \leq {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, x(s), {}^cD^q x(s)\bigr)\bigr|\,ds + \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, x(s), {}^cD^q x(s)\bigr)\bigr|\,ds \\ & + \frac{\Gamma(2-p)T^p}{2} \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} \bigl|f\bigl(s, x(s), {}^cD^q x(s)\bigr)\bigr|\,ds \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\,ds + \frac{d_1 r^{\rho_1} + d_2 r^{\rho_2}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\,ds + \int_0^T \frac{(T-s)^{\alpha-1}}{2\Gamma(\alpha)} m(s)\,ds \\
&\quad + \frac{d_1 r^{\rho_1} + d_2 r^{\rho_2}}{2\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\,ds + \frac{\Gamma(2-p)T^p}{2\Gamma(\alpha-p)} \int_0^T (T-s)^{\alpha-p-1} m(s)\,ds \\
&\quad + \frac{\Gamma(2-p)T^p(d_1 r^{\rho_1} + d_2 r^{\rho_2})}{2\Gamma(\alpha-p)} \int_0^T (T-s)^{\alpha-p-1}\,ds \\
&\le \frac{1}{\Gamma(\alpha)} \left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{1}{1-l}}\,ds \right)^{1-l} \left(\int_0^t \left(m(s)\right)^{\frac{1}{l}}\,ds \right)^{l} \\
&\quad + \frac{1}{2\Gamma(\alpha)} \left(\int_0^T \left((T-s)^{\alpha-1}\right)^{\frac{1}{1-l}}\,ds \right)^{1-l} \left(\int_0^T \left(m(s)\right)^{\frac{1}{l}}\,ds \right)^{l} \\
&\quad + \frac{\Gamma(2-p)T^p}{2\Gamma(\alpha-p)} \left(\int_0^T \left((T-s)^{\alpha-p-1}\right)^{\frac{1}{1-l}}\,ds \right)^{1-l} \left(\int_0^T \left(m(s)\right)^{\frac{1}{l}}\,ds \right)^{l} \\
&\quad + \left(\frac{3T^\alpha}{2\Gamma(\alpha+1)} + \frac{\Gamma(2-p)T^\alpha}{2\Gamma(\alpha-p+1)} \right) \left(d_1 r^{\rho_1} + d_2 r^{\rho_2}\right) \\
&\le \frac{3MT^{\alpha-l}}{2\Gamma(\alpha)} \left(\frac{1-l}{\alpha-l} \right)^{1-l} + \frac{\Gamma(2-p)MT^{\alpha-l}}{2\Gamma(\alpha-p)} \left(\frac{1-l}{\alpha-p-l} \right)^{1-l} \\
&\quad + \left(\frac{3T^\alpha}{2\Gamma(\alpha+1)} + \frac{\Gamma(2-p)T^\alpha}{2\Gamma(\alpha-p+1)} \right) \left(d_1 r^{\rho_1} + d_2 r^{\rho_2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left| {}^cD^q(Fx)(t) \right| &= \left| \int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)} (Fx)'(s)\,ds \right| \\
&= \left| \int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(\tau, x(\tau), {}^cD^q x(\tau)\right) d\tau \right.\right. \\
&\quad \left.\left. - \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\left(\tau, x(\tau), {}^cD^q x(\tau)\right) d\tau \right) ds \right| \\
&\le \int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)} \left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \left| f\left(\tau, x(\tau), {}^cD^q x(\tau)\right) \right| d\tau \right) ds \\
&\quad + \frac{\Gamma(2-p)}{T^{1-p}} \int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)} \left(\int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} \left| f\left(\tau, x(\tau), {}^cD^q x(\tau)\right) \right| d\tau \right) ds \\
&\le \frac{1}{\Gamma(1-q)\Gamma(\alpha-1)} \int_0^t (t-s)^{-q} \left(\int_0^s (s-\tau)^{\alpha-2} m(\tau)\,d\tau \right) ds \\
&\quad + \frac{d_1 r^{\rho_1} + d_2 r^{\rho_2}}{\Gamma(1-q)\Gamma(\alpha-1)} \int_0^t (t-s)^{-q} \int_0^s (s-\tau)^{\alpha-2}\,d\tau\,ds \\
&\quad + \frac{\Gamma(2-p)}{\Gamma(1-q)\Gamma(\alpha-p)T^{1-p}} \int_0^t (t-s)^{-q} \left(\int_0^T (T-\tau)^{\alpha-p-1} m(\tau)\,d\tau \right) ds \\
&\quad + \frac{(d_1 r^{\rho_1} + d_2 r^{\rho_2})\Gamma(2-p)}{\Gamma(1-q)\Gamma(\alpha-p)T^{1-p}} \int_0^t (t-s)^{-q} \int_0^T (T-\tau)^{\alpha-p-1}\,d\tau\,ds \\
&\le \frac{1}{\Gamma(1-q)\Gamma(\alpha-1)} \int_0^t (t-s)^{-q} \\
&\quad \times \left[\left(\int_0^s \left((s-\tau)^{\alpha-2}\right)^{\frac{1}{1-l}}\,d\tau \right)^{1-l} \left(\int_0^s \left(m(\tau)\right)^{\frac{1}{l}}\,d\tau \right)^{l} \right] ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Gamma(2-p)}{\Gamma(1-q)\Gamma(\alpha-p)T^{1-p}}\int_0^t (t-s)^{-q} \\
&\times\left[\left(\int_0^T \left((T-\tau)^{\alpha-p-1}\right)^{\frac{1}{1-l}}\,d\tau\right)^{1-l}\left(\int_0^T \left(m(\tau)\right)^{\frac{1}{l}}\,d\tau\right)^{l}\right]ds \\
&+\frac{d_1r^{\rho_1}+d_2r^{\rho_2}}{\Gamma(1-q)\Gamma(\alpha)}\int_0^t (t-s)^{-q}s^{\alpha-1}\,ds+\frac{(d_1r^{\rho_1}+d_2r^{\rho_2})\Gamma(2-p)T^{\alpha-q}}{\Gamma(2-q)\Gamma(\alpha-p+1)} \\
&\times\frac{M}{\Gamma(\alpha-1)\Gamma(1-q)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l}\int_0^t (t-s)^{-q}s^{\alpha-l-1}\,ds \\
&+\frac{M\Gamma(2-p)T^{\alpha-l-1}}{\Gamma(\alpha-p)\Gamma(1-q)}\left(\frac{1-l}{\alpha-p-l}\right)^{1-l}\int_0^t (t-s)^{-q}\,ds+\frac{(d_1r^{\rho_1}+d_2r^{\rho_2})T^{\alpha-q}}{\Gamma(\alpha-q+1)} \\
&+\frac{(d_1r^{\rho_1}+d_2r^{\rho_2})\Gamma(2-p)T^{\alpha-q}}{\Gamma(2-q)\Gamma(\alpha-p+1)} \\
\le{}&\frac{M\Gamma(\alpha-l)T^{\alpha-q-l}}{\Gamma(\alpha-1)\Gamma(\alpha-q-l+1)}\left(\frac{1-l}{\alpha-l-1}\right)^{1-l} \\
&+\frac{M\Gamma(2-p)T^{\alpha-q-l}}{\Gamma(\alpha-p)\Gamma(2-q)}\left(\frac{1-l}{\alpha-p-l}\right)^{1-l} \\
&+\left[\frac{T^{\alpha-q}}{\Gamma(\alpha-q+1)}+\frac{\Gamma(2-p)T^{\alpha-q}}{\Gamma(2-q)\Gamma(\alpha-p+1)}\right]\left(d_1r^{\rho_1}+d_2r^{\rho_2}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left\|(Fx)(t)\right\| &= \max_{t\in J}\left|(Fx)(t)\right|+\max_{t\in J}\left|{}^cD^q(Fx)(t)\right| \\
&\le \frac{3MT^{\alpha-l}}{2\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l}+\frac{\Gamma(2-p)MT^{\alpha-l}}{2\Gamma(\alpha-p)}\left(\frac{1-l}{\alpha-p-l}\right)^{1-l} \\
&\quad+\frac{M\Gamma(\alpha-l)T^{\alpha-q-l}}{\Gamma(\alpha-1)\Gamma(\alpha-q-l+1)} \\
&\quad\times\left(\frac{1-l}{\alpha-l-1}\right)^{1-l}+\frac{M\Gamma(2-p)T^{\alpha-q-l}}{\Gamma(\alpha-p)\Gamma(2-q)}\left(\frac{1-l}{\alpha-p-l}\right)^{1-l}+\left[\frac{T^{\alpha-q}}{\Gamma(\alpha-q+1)}\right. \\
&\quad\left.+\frac{\Gamma(2-p)T^{\alpha-q}}{\Gamma(2-q)\Gamma(\alpha-p+1)}+\frac{3T^{\alpha}}{2\Gamma(\alpha+1)}+\frac{\Gamma(2-p)T^{\alpha}}{2\Gamma(\alpha-p+1)}\right]\left(d_1r^{\rho_1}+d_2r^{\rho_2}\right) \\
&= K+\left(d_1r^{\rho_1}+d_2r^{\rho_2}\right)A\le\frac{r}{3}+\frac{r}{3}+\frac{r}{3}=r.
\end{aligned}
$$

Notice that $(Fx)(t)$, $D^q(Fx)(t)$ are continuous on J; therefore, $F: B_r \to B_r$. In view of the continuity of f, it is easy to know that the operator F is continuous. Now, we show that F is a completely continuous operator. For each $x \in B_r$, we fix $N = \max_{t\in J}|f(t,x(t),{}^cD^q x(t))|$, for any $\varepsilon > 0$, setting

$$
\begin{aligned}
\delta = \min\Bigg\{&\frac{\Gamma(\alpha)\Gamma(\alpha-p+1)\varepsilon}{NT^{\alpha-1}(\Gamma(\alpha-p+1)+\Gamma(2-p)\Gamma(\alpha))}, \\
&\frac{1}{2}\left[\frac{\Gamma(\alpha)\Gamma(\alpha-p+1)\varepsilon}{6NT^{\alpha-1}(\Gamma(\alpha-p+1)+\Gamma(2-p)\Gamma(\alpha))}\right]^{\frac{1}{1-q}}\Bigg\}.
\end{aligned}
$$

For each $x \in B_r$, we will prove that if $t_1, t_2 \in J$ and $0 < t_2 - t_1 < \delta$, then

$$\big\|(Fx)(t_2) - (Fx)(t_1)\big\| < \varepsilon.$$

In fact,

$$\begin{aligned}
\big|(Fx)(t_2) - (Fx)(t_1)\big| &= \bigg|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s), {}^cD^q x(s)\big)\,ds \\
&\quad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s), {}^cD^q x(s)\big)\,ds \\
&\quad + \frac{(t_1-t_2)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f\big(s, x(s), {}^cD^q x(s)\big)\,ds\bigg| \\
&\le \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(s, x(s), {}^cD^q x(s)\big)\big|\,ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(s, x(s), {}^cD^q x(s)\big)\big|\,ds \\
&\quad + \frac{(t_2-t_1)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} \big|f\big(s, x(s), {}^cD^q x(s)\big)\big|\,ds \\
&\le N \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + N \int_{t_1}^{t_2} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&\quad + \frac{N(t_2-t_1)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1}\,ds \\
&= \frac{N}{\Gamma(\alpha+1)}\big(t_2^\alpha - t_1^\alpha\big) + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}(t_2-t_1).
\end{aligned}$$

By mean value theorem, we have

$$\begin{aligned}
\big|(Fx)(t_2) - (Fx)(t_1)\big| &\le \frac{N}{\Gamma(\alpha+1)}\big(t_2^\alpha - t_1^\alpha\big) + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}(t_2-t_1) \\
&\le \frac{N}{\Gamma(\alpha+1)}\alpha T^{\alpha-1}(t_2-t_1) + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}(t_2-t_1) \\
&\le \left(\frac{N}{\Gamma(\alpha)}T^{\alpha-1} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}\right)\delta < \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\big|{}^cD^q(Fx)(t_2) - {}^cD^q(Fx)(t_1)\big| \\
&\quad = \bigg|\int_0^{t_2} \frac{(t_2-s)^{-q}}{\Gamma(1-q)}(Fx)'(s)\,ds - \int_0^{t_1} \frac{(t_1-s)^{-q}}{\Gamma(1-q)}(Fx)'(s)\,ds\bigg| \\
&\quad = \bigg|\int_0^{t_1} \frac{(t_2-s)^{-q}}{\Gamma(1-q)}\bigg(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau \\
&\quad\quad - \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau\bigg)\,ds \\
&\quad\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{-q}}{\Gamma(1-q)}\bigg(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau \bigg)\,ds \\
&\quad - \int_0^{t_1} \frac{(t_1-s)^{-q}}{\Gamma(1-q)} \bigg(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau \\
&\quad - \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau \bigg)\,ds \bigg| \\
&= \bigg| \int_0^{t_1} \frac{(t_2-s)^{-q}-(t_1-s)^{-q}}{\Gamma(1-q)} \bigg(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau \\
&\quad - \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau \bigg)\,ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{-q}}{\Gamma(1-q)} \bigg(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau \\
&\quad - \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\,d\tau \bigg)\,ds \bigg| \\
&\le \int_0^{t_1} \frac{(t_2-s)^{-q}-(t_1-s)^{-q}}{\Gamma(1-q)} \bigg(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \big|f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\big|\,d\tau \bigg)\,ds \\
&\quad + \frac{\Gamma(2-p)}{T^{1-p}} \int_0^{t_1} \frac{(t_2-s)^{-q}-(t_1-s)^{-q}}{\Gamma(1-q)} \\
&\quad \times \bigg(\int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} \big|f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\big|\,d\tau \bigg)\,ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{-q}}{\Gamma(1-q)} \bigg(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \big|f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\big|\,d\tau \bigg)\,ds \\
&\quad + \frac{\Gamma(2-p)}{T^{1-p}} \int_{t_1}^{t_2} \frac{(t_2-s)^{-q}}{\Gamma(1-q)} \bigg(\int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} \big|f\big(\tau, x(\tau), {}^cD^q x(\tau)\big)\big|\,d\tau \bigg)\,ds \\
&\le \frac{N}{\Gamma(\alpha)} \int_0^{t_1} \frac{(t_2-s)^{-q}-(t_1-s)^{-q}}{\Gamma(1-q)} s^{\alpha-1}\,ds \\
&\quad + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} \int_0^{t_1} \frac{(t_2-s)^{-q}-(t_1-s)^{-q}}{\Gamma(1-q)} s^{\alpha-1}\,ds \\
&\quad + \frac{N}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{(t_2-s)^{-q}}{\Gamma(1-q)} s^{\alpha-1}\,ds + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} \int_{t_1}^{t_2} \frac{(t_2-s)^{-q}}{\Gamma(1-q)}\,ds \\
&\le \bigg(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} \bigg) \int_0^{t_1} \frac{(t_2-s)^{-q}-(t_1-s)^{-q}}{\Gamma(1-q)}\,ds \\
&\quad + \bigg(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} \bigg) \int_{t_1}^{t_2} \frac{(t_2-s)^{-q}}{\Gamma(1-q)}\,ds \\
&\le \bigg(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} \bigg) \big[2(t_2-t_1)^{1-q} + \big(t_2^{1-q} - t_1^{1-q}\big)\big].
\end{aligned}
$$

In the following, we will divide the proof into two cases.

Case 1. For $\delta \le t_1 < t_2 < T$, by mean value theorem, we have

$$
\begin{aligned}
&\big|{}^cD^q(Fx)(t_2) - {}^cD^q(Fx)(t_1)\big| \\
&\le \bigg(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} \bigg) \big[2(t_2-t_1)^{1-q} + \big(t_2^{1-q} - t_1^{1-q}\big)\big]
\end{aligned}
$$

$$\begin{aligned}
&\leq \left(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}\right)\left[2\delta^{1-q} + (1-q)\delta^{-q}(t_2-t_1)\right] \\
&< \left(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}\right)(3-q)\delta^{1-q} < \left(\frac{1}{2}\right)^{1-q}\frac{\varepsilon}{2} < \frac{\varepsilon}{2}.
\end{aligned}$$

Case 2. For $0 \leq t_1 < \delta$, $t_2 < 2\delta$, we have

$$\begin{aligned}
\left|{}^cD^q(Fx)(t_2) - {}^cD^q(Fx)(t_1)\right| &\leq \left(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}\right)\left[2(t_2-t_1)^{1-q} + \left(t_2^{1-q} - t_1^{1-q}\right)\right] \\
&\leq \left(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}\right)3t_2^{1-q} \\
&< \left(\frac{NT^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}\right)3(2\delta)^{1-q} < \frac{\varepsilon}{2}.
\end{aligned}$$

Hence,

$$\left\|(Fx)(t_2) - (Fx)(t_1)\right\| < \varepsilon.$$

Therefore, F is equicontinuous and uniformly bounded. The Arzela-Ascoli theorem implies that F is compact on B_r, so the operator F is completely continuous. Thus the conclusion of Theorem 2.1 implies that the anti-periodic boundary value problem (2) has at least one solution on $[0,T]$. This completes the proof. □

Corollary 3.1 *Let $f: J \times R \times R \to R$ be a continuous function. Assume that*

(H_2) There exist a constant $l \in (0, \alpha-1)$ and a real-valued function $m(t) \in L^{\frac{1}{l}}([0,T],(0,\infty))$ such that

$$\left|f(t,x,y)\right| \leq m(t) + d_1|x| + d_2|y|,$$

and $(d_1+d_2)A < 1$, where $d_1, d_2 \geq 0$, A is defined in the proof of Theorem 3.1. Then the problem (2) has at least a solution on $[0,T]$.

The proof of Corollary 3.1 is similar to Theorem 3.1.

Theorem 3.2 *Assume that*

(H_3) There exist a constant $r \in (0, \alpha-1)$ and a real-valued function $\mu(t) \in L^{\frac{1}{r}}([0,T],(0,\infty))$ such that

$$\left|f(t,x,y) - f(t,u,v)\right| \leq \mu(t)\left(|x-u| + |y-v|\right),$$

for any $t \in [0,T]$, $x, y, u, v \in R$, and if

$$\begin{aligned}
&\frac{3\mu^* T^{\alpha-r}}{2\Gamma(\alpha)}\left(\frac{1-r}{\alpha-r}\right)^{1-r} + \frac{\Gamma(2-p)\mu^* T^{\alpha-r}}{2\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-p-r}\right)^{1-r} + \frac{\Gamma(\alpha-r)\mu^* T^{\alpha-q-r}}{\Gamma(\alpha-1)\Gamma(\alpha-q-r+1)} \\
&\quad\times\left(\frac{1-r}{\alpha-r-1}\right)^{1-r} + \frac{\Gamma(2-p)\mu^* T^{\alpha-q-r}}{\Gamma(2-q)\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-r-p}\right)^{1-r} < 1,
\end{aligned} \tag{6}$$

where $\mu^* = (\int_0^T (\mu(s))^{\frac{1}{r}}\,ds)^r$. Then the problem (2) has a unique solution.

Proof Define the mapping $F : X \to X$ by

$$\begin{aligned}(Fx)(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\, ds \\ & - \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\, ds \\ & + \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\bigl(s, x(s), {}^cD^q x(s)\bigr)\, ds.\end{aligned}$$

For $x, y \in X$ and for each $t \in [0, T]$, by Theorem 2.2 (Hölder inequality), we obtain

$$\begin{aligned}& \bigl|(Fx)(t) - (Fy)(t)\bigr| \\ & \quad \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(s)\bigl(\bigl|x(s)-y(s)\bigr| + \bigl|{}^cD^q x(s) - {}^cD^q y(s)\bigr|\bigr)\, ds \\ & \qquad + \frac{1}{2}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \mu(s)\bigl(\bigl|x(s)-y(s)\bigr| + \bigl|{}^cD^q x(s) - {}^cD^q y(s)\bigr|\bigr)\, ds \\ & \qquad + \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \int_0^T \frac{(T-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} \mu(s)\bigl(\bigl|x(s)-y(s)\bigr| + \bigl|{}^cD^q x(s) - {}^cD^q y(s)\bigr|\bigr)\, ds \\ & \quad \le \frac{\|x-y\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\mu(s)\, ds + \frac{\|x-y\|}{2\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\mu(s)\, ds \\ & \qquad + \frac{\|x-y\|\Gamma(2-p)T^p}{2\Gamma(\alpha-p)} \int_0^T (T-s)^{\alpha-p-1}\mu(s)\, ds \\ & \quad \le \frac{\|x-y\|}{\Gamma(\alpha)} \biggl(\int_0^t \bigl((t-s)^{\alpha-1}\bigr)^{\frac{1}{1-r}}\, ds\biggr)^{1-r} \biggl(\int_0^t \bigl(\mu(s)\bigr)^{\frac{1}{r}}\, ds\biggr)^r \\ & \qquad + \frac{\|x-y\|}{2\Gamma(\alpha)} \biggl(\int_0^T \bigl((T-s)^{\alpha-1}\bigr)^{\frac{1}{1-r}}\, ds\biggr)^{1-r} \biggl(\int_0^T \bigl(\mu(s)\bigr)^{\frac{1}{r}}\, ds\biggr)^r \\ & \qquad + \frac{\|x-y\|\Gamma(2-p)T^p}{2\Gamma(\alpha-p)} \biggl(\int_0^T \bigl((T-s)^{\alpha-p-1}\bigr)^{\frac{1}{1-r}}\, ds\biggr)^{1-r} \biggl(\int_0^T \bigl(\mu(s)\bigr)^{\frac{1}{r}}\, ds\biggr)^r \\ & \quad \le \biggl[\frac{3\mu^* T^{\alpha-r}}{2\Gamma(\alpha)}\biggl(\frac{1-r}{\alpha-r}\biggr)^{1-r} + \frac{\mu^*\Gamma(2-p)T^{\alpha-r}}{2\Gamma(\alpha-p)}\biggl(\frac{1-r}{\alpha-p-r}\biggr)^{1-r}\biggr]\|x-y\|\end{aligned}$$

and

$$\begin{aligned}& \bigl|{}^cD^q(Fx)(t) - {}^cD^q(Fy)(t)\bigr| \\ & \quad = \biggl|\int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)}(Fx)'(s)\, ds - \int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)}(Fy)'(s)\, ds\biggr| \\ & \quad = \biggl|\int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)}\biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(\tau, x(\tau), {}^cD^q x(\tau)\bigr)\, d\tau \\ & \qquad - \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\bigl(\tau, x(\tau), {}^cD^q x(\tau)\bigr)\, d\tau\biggr) \\ & \qquad - \int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)}\biggl(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(\tau, y(\tau), {}^cD^q y(\tau)\bigr)\, d\tau \\ & \qquad - \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)} f\bigl(\tau, y(\tau), {}^cD^q y(\tau)\bigr)\, d\tau\biggr)\biggr|\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f\left(\tau,x(\tau),{}^cD^q x(\tau)\right)-f\left(\tau,y(\tau),{}^cD^q y(\tau)\right)\right|d\tau\right)ds \\
&\quad+\frac{\Gamma(2-p)}{T^{1-p}}\int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)} \\
&\quad\times\left(\int_0^T \frac{(T-\tau)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left|f\left(\tau,x(\tau),{}^cD^q x(\tau)\right)-f\left(\tau,y(\tau),{}^cD^q y(\tau)\right)\right|d\tau\right)ds \\
&\leq \frac{\|x-y\|}{\Gamma(\alpha-1)\Gamma(1-q)}\int_0^t (t-s)^{-q}\left(\int_0^s (s-\tau)^{\alpha-2}\mu(\tau)\,d\tau\right)ds \\
&\quad+\frac{\|x-y\|\Gamma(2-p)}{T^{1-p}\Gamma(1-q)\Gamma(\alpha-p)}\int_0^t (t-s)^{-q}\left(\int_0^T (s-\tau)^{\alpha-p-1}\mu(\tau)\,d\tau\right)ds \\
&\leq \frac{\|x-y\|\mu^*}{\Gamma(\alpha-1)\Gamma(1-q)}\left(\frac{1-r}{\alpha-r-1}\right)^{1-r}\int_0^t (t-s)^{-q}s^{a-r-1}\,ds \\
&\quad+\frac{\|x-y\|\mu^*\Gamma(2-p)T^{\alpha-r-1}}{\Gamma(1-q)\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-r-p}\right)^{1-r}\int_0^t (t-s)^{-q}\,ds \\
&\leq \frac{\|x-y\|\mu^* T^{\alpha-q-r}\Gamma(\alpha-r)}{\Gamma(\alpha-1)\Gamma(\alpha-q-r+1)}\left(\frac{1-r}{\alpha-r-1}\right)^{1-r} \\
&\quad+\frac{\|x-y\|\mu^*\Gamma(2-p)T^{\alpha-q-r}}{\Gamma(2-q)\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-r-p}\right)^{1-r} \\
&\leq \left[\frac{\mu^* T^{\alpha-q-r}\Gamma(\alpha-r)}{\Gamma(\alpha-1)\Gamma(\alpha-q-r+1)}\left(\frac{1-r}{\alpha-r-1}\right)^{1-r}\right. \\
&\quad\left.+\frac{\mu^*\Gamma(2-p)T^{\alpha-q-r}}{\Gamma(2-q)\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-r-p}\right)^{1-r}\right]\|x-y\|.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\|Fx-Fy\| \leq{}& \left[\frac{3\mu^* T^{\alpha-r}}{2\Gamma(\alpha)}\left(\frac{1-r}{\alpha-r}\right)^{1-r}+\frac{\Gamma(2-p)\mu^* T^{\alpha-r}}{2\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-p-r}\right)^{1-r}\right. \\
&+\frac{\Gamma(\alpha-r)\mu^* T^{\alpha-q-r}}{\Gamma(\alpha-1)\Gamma(\alpha-q-r+1)} \\
&\left.\times\left(\frac{1-r}{\alpha-r-1}\right)^{1-r}+\frac{\Gamma(2-p)\mu^* T^{\alpha-q-r}}{\Gamma(2-q)\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-r-p}\right)^{1-r}\right]\|x-y\|.
\end{aligned}
$$

From the assumption (6), it follows that F is a contraction mapping. Therefore, the Banach fixed point theorem yields that F has a unique fixed point which is the unique solution of the problem (2). □

4 Examples

Example 4.1 Let $\alpha = \frac{3}{2}$, $p = q = \frac{1}{2}$, $T = 1$. Consider the following anti-periodic fractional boundary value problem:

$$
\begin{cases}
{}^cD^{\frac{3}{2}}x(t) = f\left(t,x(t),{}^cD^{\frac{1}{2}}x(t)\right), & t \in [0,1], \\
x(0) = -x(1), \qquad {}^cD^{\frac{1}{2}}x(0) = -{}^cD^{\frac{1}{2}}x(1).
\end{cases}
\tag{7}
$$

We have

$$f\bigl(t,x(t),{}^cD^{\frac{1}{2}}x(t)\bigr) = m(t) + \left(t-\frac{1}{2}\right)^4\bigl[\bigl(x(t)\bigr)^{\rho_1} + \bigl({}^cD^{\frac{1}{2}}x(t)\bigr)^{\rho_2}\bigr],$$

$m(t) \in L^4([0,1],(0,\infty))$, $0 \le \rho_1, \rho_2 \le 1$.

Since

$$\begin{aligned}\bigl|f\bigl(t,x(t),{}^cD^{\frac{1}{2}}x(t)\bigr)\bigr| &\le \bigl|m(t)\bigr| + \left(t-\frac{1}{2}\right)^4\bigl|x(t)\bigr|^{\rho_1} + \left(t-\frac{1}{2}\right)^4\bigl|D^{\frac{1}{2}}x(t)\bigr|^{\rho_2} \\ &\le \bigl|m(t)\bigr| + \frac{1}{16}\bigl|x(t)\bigr|^{\rho_1} + \frac{1}{16}\bigl|{}^cD^{\frac{1}{2}}x(t)\bigr|^{\rho_2},\end{aligned}$$

therefore, by Theorem 3.1, the problem (7) has at least a solution on $[0,1]$.

Example 4.2 Consider the following anti-periodic fractional boundary value problem:

$$\begin{cases}{}^cD^{\frac{3}{2}}x(t) = \dfrac{1}{(t+4)^2}\left(\dfrac{|x+{}^cD^{\frac{1}{2}}x|}{1+|x+{}^cD^{\frac{1}{2}}x|}+5t^2\right), \\ x(0) = -x(1), \qquad {}^cD^{\frac{1}{2}}x(0) = -{}^cD^{\frac{1}{2}}x(1).\end{cases} \tag{8}$$

We have

$$\bigl|f\bigl(t,x,{}^cD^{\frac{1}{2}}x\bigr) - f\bigl(t,y,{}^cD^{\frac{1}{2}}y\bigr)\bigr| \le \frac{1}{16}\bigl(|x-y| + \bigl|{}^cD^{\frac{1}{2}}x - {}^cD^{\frac{1}{2}}y\bigr|\bigr).$$

Obviously, $\mu(t) \equiv \frac{1}{16} \in L^4([0,1],(0,\infty))$, $r = \frac{1}{4}$ and $\mu^* = (\int_0^T(\mu(s))^{\frac{1}{r}}\,ds)^r = (\int_0^1(\frac{1}{16})^4\,ds)^{\frac{1}{4}} = \frac{1}{16}$. Note that $\Gamma(\frac{3}{2}) \approx 0.8862$, $\Gamma(\frac{7}{4}) \approx 0.9191$, $\Gamma(\frac{5}{4}) \approx 0.9064$, we have

$$\begin{aligned}&\frac{3\mu^* T^{\alpha-r}}{2\Gamma(\alpha)}\left(\frac{1-r}{\alpha-r}\right)^{1-r} + \frac{\Gamma(2-p)\mu^* T^{\alpha-r}}{2\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-p-r}\right)^{1-r} + \frac{\Gamma(\alpha-r)\mu^* T^{\alpha-q-r}}{\Gamma(\alpha-1)\Gamma(\alpha-q-r+1)} \\ &\quad\times\left(\frac{1-r}{\alpha-r-1}\right)^{1-r} + \frac{\Gamma(2-p)\mu^* T^{\alpha-q-r}}{\Gamma(2-q)\Gamma(\alpha-p)}\left(\frac{1-r}{\alpha-r-p}\right)^{1-r} \\ &\quad= \frac{3(\frac{3}{5})^{\frac{3}{4}}}{32\Gamma(\frac{3}{2})} + \frac{\Gamma(\frac{3}{2})}{32} + \frac{\Gamma(\frac{5}{4})3^{\frac{3}{4}}}{16\Gamma(\frac{1}{2})\Gamma(\frac{7}{4})} + \frac{1}{16} \\ &\quad\approx 0.7212 + 0.0277 + 0.0793 + 0.0625 = 0.8907 < 1.\end{aligned}$$

Therefore, (8) has a unique solution on $[0,1]$ by Theorem 3.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The author Zhenhai Liu contributed to each part of this study equally and read and approved the final version of the manuscript.

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Acknowledgement
The authors are highly grateful for the referee's careful reading and comments on this note.

Received: 8 April 2012 Accepted: 30 June 2012 Published: 20 July 2012

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doi:10.1186/1687-1847-2012-116
Cite this article as: Wang and Liu: Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order. *Advances in Difference Equations* 2012 **2012**:116.