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Attracting and invariant sets of nonlinear neutral differential equations with delays

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Abstract

In this paper, we study the attracting and invariant sets for a class of nonlinear neutral differential equations with delays. By using the properties of \mathcal{M} -matrix, a new delay differential-difference inequality is established. Based on the new inequality, we get the global attracting and invariant sets and the sufficient condition ensuring the exponential stability in Lyapunov sense of nonlinear neutral differential equations with delays. Our results are independent of time delays and do not require the differentiability, boundedness of the derivative of delay functions and the boundedness of activation functions. Two examples are presented to illustrate the effectiveness of our conclusion.

Keywords: attracting set; invariant set; stability; neutral; differential-difference inequality; delays

Introduction

Delay effects exist widely in many real-world models such as the SEIRS epidemic model [1] and neural networks [2–5]. The existence of time delays may destroy a stable system and cause sustained oscillations, bifurcation or chaos and thus could be harmful. Therefore, it is of prime importance to consider the effect of delays on the dynamical behaviors of the system. Recently, there are many authors who consider the effect of delays on the stability in Lyapunov sense of the system with time delays [2–11]. In addition, another type of time delays, namely neutral-type time delays, has recently drawn much attention in research [12–21]. In fact, many practical delay systems can be modeled as differential systems of neutral type whose differential expression includes not only the derivative term of the current state but also the derivative of the past state, such as partial element equivalent circuits and transmission lines in electrical engineering, controlled constrained manipulators in mechanical engineering, neural networks models, and population dynamics (see [22] and references therein).

The works [12–22] mentioned above are focused on studying the stability in Lyapunov sense of the neutral differential equations, which requires the existence and uniqueness of equilibrium points. However, in many real physical systems, especially in nonlinear and non-autonomous dynamical systems, the equilibrium point sometimes does not exist. Therefore, an interesting subject is to discuss the stability in Lagrange sense. Basically, the goal of the study on global stability in Lagrange sense is to determine global attracting sets. Once a global attracting set is found, a rough bound of periodic states and chaotic attractors can be estimated. For this reason, some significant works have been done on the

techniques and methods of determining the invariant set and attracting set for various differential systems [23–30]. In these works mentioned before, there is only one paper [25] that considers a positive invariant set and a global attracting set for nonlinear neutral differential systems with delays, but the boundedness of activation functions is required.

It is well known that differential inequalities are very important tools for investigating the dynamical behavior of differential equations (see [11, 20, 21, 26, 28, 31–34]). Xu *et al.* developed a delay differential inequality with the impulsive initial conditions and derived some sufficient conditions to determine the invariant set and the global attracting set for a class of nonlinear non-autonomous functional differential systems with impulsive effects [28]. In [32], Eduardo Liz *et al.* developed a generalized Halanay inequality and derived some sufficient conditions for the existence and stability of almost periodic solutions for quasilinear delay systems. In [20], Xu *et al.* developed the singular impulsive delay differential inequality and transformed the n -dimensional impulsive neutral differential equation to a $2n$ -dimensional singular impulsive delay differential equation and derived some sufficient conditions ensuring the global exponential stability in Lyapunov sense of a nonlinear impulsive neutral differential equation with time-varying delays, but they assumed that the discontinuous points of the derivative of the solution belonged to the first kind. As we all know, the discontinuous points of the derivative of continuous functions may not be the first kind. In addition, we know that LMI method is another effective tool for investigating the dynamical behavior of a differential system [14, 15, 35]. The results given in the LMI form are dependent on time delays, so we must give additional constraint conditions such as differentiability or boundedness of the derivative of delay functions on the time-varying delays. However, the conditions given in the form of \mathcal{M} -matrix are usually independent of the time delays, thus, the time delays are harmless. Motivated by the before discussions, our objective in this paper is to improve the inequality established in [28] and [32] so that it is effective for neutral differential equation. By establishing a new delay differential-difference inequality, without assuming that the discontinuous points of the derivative of the solution belong to the first kind, the global attracting and invariant sets and the sufficient condition ensuring the global exponential stability in Lyapunov sense of a nonlinear neutral differential equations with delays are obtained. Our results are independent of the time delays, and do not require the differentiability, boundedness of the derivative of delay functions and the boundedness of activation functions. Two examples are presented to illustrate the effectiveness of our conclusion.

Model description and preliminaries

Throughout this paper, we use the following notations. Let R_+^n be the space of n -dimensional nonnegative real column vectors, R^n be the space of n -dimensional real column vectors, $\mathcal{N} \triangleq \{1, 2, \dots, n\}$, and $R^{m \times n}$ denote the set of $m \times n$ real matrices. Usually E denotes an $n \times n$ unit matrix. For $A, B \in R^{m \times n}$, the notation $A \geq B$ ($A > B$) means that each pair of corresponding elements of A and B satisfies the inequality ' \geq ' ($>$). Especially, $A \in R^{m \times n}$ is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$. A_r denotes the r th row vector of the matrix A .

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y . Especially, $C \triangleq C[[-\tau, 0], R^n]$ denotes the family of all continuous R^n -valued functions, where $\tau > 0$.

$PC[J, R^n] = \{\varphi : J \rightarrow R^n \text{ is continuous for all but at most a finite number of points } t \in J, \text{ and at these points } t \in J, \varphi(t^+) \text{ and } \varphi(t^-) \text{ exist, } \varphi(t^+) = \varphi(t)\}$, where $J \subset R$ is a bounded

interval, $\varphi(t^+)$ and $\varphi(t^-)$ denote the right-hand and left-hand limits of the function $\varphi(t)$, respectively. Especially, let $PC \triangleq PC[[-\tau, 0], R^n]$.

For $A \in R^{n \times n}$, $x \in R^n$, $\phi \in C$ and φ is a continuous function on $[t_0 - \tau, +\infty)$, we define

$$\begin{aligned} |A| &= (|a_{ij}|)_{n \times n}, & [x]^+ &= (|x_1|, \dots, |x_n|)^T, \\ [\phi]_\tau^+ &= [([\phi]^+)]_\tau, & [\phi]_\tau &= \sup_{-\tau \leq s \leq 0} \{\phi_i(t_0 + s)\}, \\ [\varphi(t)]_\tau &= ([\varphi_1(t)]_\tau, \dots, [\varphi_n(t)]_\tau)^T, & [\varphi(t)]_\tau^+ &= [[\varphi(t)]^+]_\tau, \\ [\varphi_i(t)]_\tau &= \sup_{-\tau \leq s \leq 0} \{\varphi_i(t + s)\}, & & t \geq t_0, i \in \mathcal{N}, \end{aligned}$$

and $D^+ \varphi(t)$ denotes the upper-right-hand derivative of $\varphi(t)$ at time t .

For $\varphi \in C$, we introduce the following norm:

$$\|\varphi\|_\tau = \max_{1 \leq i \leq n} \left\{ \max_{-\tau \leq s \leq 0} |\varphi_i(s)| \right\}.$$

In this paper, we consider the following nonlinear neutral differential equation with time-varying delays:

$$\begin{cases} (x_i(t) - \sum_{j=1}^n c_{ij} x_j(t - r_{ij}(t)))' = -d_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) \\ \quad + \sum_{j=1}^n b_{ij} g_j(x_j(t - \tau_{ij}(t))) + J_i, & t \geq t_0, \\ x_i(t_0 + s) = \phi_i(s), & -\tau \leq s \leq 0, i \in \mathcal{N}, \end{cases} \tag{1}$$

where τ , a_{ij} , b_{ij} , c_{ij} , d_i and J_i are constants, $\tau_{ij}(t), r_{ij}(t), f_j(t), g_j(t) \in C[R, R]$, $i, j \in \mathcal{N}$, $r_{ij}(t)$ is differentiable, and $\tau_{ij}(t), r_{ij}(t)$ satisfy

$$0 \leq \tau_{ij}(t) \leq \tau, \quad 0 < r_{ij}(t) \leq \tau, \tag{2}$$

the initial function $\phi(s) = (\phi_1(s), \dots, \phi_n(s))^T \in C$.

Throughout this paper, the solution $x(t)$ of (1) with the initial condition $\phi \in C$ is denoted by $x(t, t_0, \phi)$ or $x_t(t_0, \phi)$, where $x_t(t_0, \phi) = x(t + s, t_0, \phi)$, $s \in [-\tau, 0]$.

Definition 1 The set $S \subset C$ is called a positive invariant set of (1) if, for any initial value $\phi \in S$, we have the solution $x_t(t_0, \phi) \in S$ for $t \geq t_0$.

Definition 2 The set $S \subset C$ is called a global attracting set of (1) if, for any initial value $\phi \in C$, the solution $x_t(t_0, \phi)$ converges to S as $t \rightarrow +\infty$. That is,

$$\text{dist}(x_t(t_0, \phi), S) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

where $\text{dist}(\varphi, S) = \inf_{\psi \in S} \text{dist}(\varphi, \psi)$, $\text{dist}(\varphi, \psi) = \sup_{s \in [-\tau, 0]} |\varphi(s) - \psi(s)|$, for $\varphi \in C$.

Definition 3 The zero solution of (1) is said to be globally exponentially stable in Lyapunov sense if there exist constants $\lambda > 0$ and $M \geq 1$ such that for any solution $x(t, t_0, \phi)$ with the initial condition $\phi \in C$,

$$\|x_t(t_0, \phi)\|_\tau \leq M \|\phi\|_\tau e^{-\lambda(t-t_0)}, \quad t \geq t_0. \tag{3}$$

Definition 4 ([36]) Let the matrix $D = (d_{ij})_{n \times n}$ have non-positive off-diagonal elements (i.e., $d_{ij} \leq 0, i \neq j$), then each of the following conditions is equivalent to the statement ‘ D is a nonsingular \mathcal{M} -matrix.’

- (i) All the leading principle minors of D are positive.
- (ii) $D = C - M$ and $\rho(C^{-1}M) < 1$, where $M \geq 0, C = \text{diag}\{c_1, \dots, c_n\}$.
- (iii) The diagonal elements of D are all positive and there exists a positive vector d such that $Dd > 0$ or $D^T d > 0$.

For a nonsingular \mathcal{M} -matrix D , we denote $\Omega_M(D) \triangleq \{z \in R^n | Dz > 0, z > 0\}$.

For a nonnegative matrix $A \in R^{n \times n}$, let $\rho(A)$ be the spectral radius of A . Then $\rho(A)$ is an eigenvalue of A and its eigenspace is denoted by

$$\Omega_\rho(A) \triangleq \{z \in R^n | Az = \rho(A)z\},$$

which includes all positive eigenvectors of A provided that the nonnegative matrix A has at least one positive eigenvector (see Ref. [37]).

Lemma 1 ([36]) *If $A \geq 0$ and $\rho(A) < 1$, then*

- (a) $(E - A)^{-1} \geq 0$;
- (b) *there is a positive vector $z \in \Omega_\rho(A)$ such that $(E - A)z > 0$.*

Main results

Based on Lemma 1 in [28] and Theorem 2.1 in [32], we develop the following delay differential-difference inequality with the PC -value initial condition such that it is effective for neutral differential equation with delays.

Theorem 1 *Let $\sigma < b \leq +\infty$, and $u \in C[[\sigma, b), R_+^n], \omega \in C[[\sigma, b), R_+^p]$ satisfy*

$$\begin{cases} D^+ u(t) \leq Pu(t) + Q[u(t)]_\tau + G\omega(t) + H[\omega(t)]_\tau + \eta, \\ \omega(t) \leq Mu(t) + N[u(t)]_\tau + R[\omega(t)]_\tau + I, \quad t \in [\sigma, b), \\ u(t) = \phi(t), \quad \omega(t) = \varphi(t), \quad t \in [\sigma - \tau, \sigma), \end{cases} \tag{4}$$

where $\phi \in PC[[\sigma - \tau, \sigma], R_+^n], \varphi \in PC[[\sigma - \tau, \sigma], R_+^p], P = (p_{ij})_{n \times n}, p_{ij} \geq 0$, for $i \neq j, Q = (q_{ij})_{n \times n} \geq 0, G = (g_{ij})_{n \times p} \geq 0, H = (h_{ij})_{n \times p} \geq 0, M = (m_{ij})_{p \times n} \geq 0, N = (n_{ij})_{p \times n} \geq 0, R = (r_{ij})_{p \times p} \geq 0, \eta = (\eta_1, \dots, \eta_n)^T \geq 0$ and $I = (I_1, \dots, I_p)^T \geq 0$. Suppose that $\rho(R) < 1$ and $\Pi = -(P + Q + (G + H)(E - R)^{-1}(M + N))$ is an \mathcal{M} -matrix, then the solution of (4) has the following property:

$$\begin{cases} u(t) \leq kze^{-\lambda(t-\sigma)} + \hat{\eta}, \\ \omega(t) \leq k\tilde{z}e^{-\lambda(t-\sigma)} + \hat{I}, \quad t \in [\sigma, b), \end{cases} \tag{5}$$

provided that the initial conditions satisfy

$$\begin{cases} u(t) \leq kze^{-\lambda(t-\sigma)} + \hat{\eta}, \\ \omega(t) \leq k\tilde{z}e^{-\lambda(t-\sigma)} + \hat{I}, \quad t \in [\sigma - \tau, \sigma), \end{cases} \tag{6}$$

where

$$\begin{aligned} k &\geq 0, \quad z \in \Omega_M(\Pi), \quad \tilde{z} \triangleq (E - Re^{\lambda\tau})^{-1}(M + Ne^{\lambda\tau})z, \\ \hat{\eta} &\triangleq \Pi^{-1}\eta + \Pi^{-1}(G + H)(E - R)^{-1}I, \\ \hat{I} &\triangleq (E - R)^{-1}(M + N)\Pi^{-1}\eta + ((E - R)^{-1}(M + N)\Pi^{-1}(G + H)(E - R)^{-1} + (E - R)^{-1})I, \end{aligned}$$

and the positive constant λ is determined by the following inequalities:

$$\rho(e^{\lambda\tau}R) < 1 \quad \text{and} \quad (\lambda E + P + Qe^{\lambda\tau} + (G + He^{\lambda\tau})(E - Re^{\lambda\tau})^{-1}(M + Ne^{\lambda\tau}))z < 0. \quad (7)$$

Proof Since Π is an \mathcal{M} -matrix, there exists a vector $z \in \Omega_M(\Pi)$ such that $\Pi z > 0$, that is $(P + Q + (G + H)(E - R)^{-1}(M + N))z < 0$. By using continuity and combining with $\rho(R) < 1$, we know there exists a positive constant λ satisfying (7).

We at first shall prove that for any positive ε

$$\begin{cases} u(t) < (k + \varepsilon)ze^{-\lambda(t-\sigma)} + \hat{\eta} \triangleq \xi(t), \\ \omega(t) < (k + \varepsilon)\tilde{z}e^{-\lambda(t-\sigma)} + \hat{I} \triangleq \zeta(t), \quad t \in [\sigma, b]. \end{cases} \quad (8)$$

If inequality (8) is not true, from (6) and $u \in C[[\sigma, b), \mathbb{R}_+^n]$, $\omega \in C[[\sigma, b), \mathbb{R}_+^p]$, then there must be a constant $t^* > \sigma$ and some integer m, r such that

$$\begin{aligned} u_m(t^*) &= \xi_m(t^*), \quad D^+ u_m(t^*) \geq \xi'(t^*), \\ u_i(t) &\leq \xi_i(t), \quad t \in [\sigma - \tau, t^*], i = 1, \dots, n \end{aligned} \quad (9)$$

or

$$\omega_r(t^*) = \zeta_r(t^*), \quad \omega_j(t) \leq \zeta_j(t), \quad t \in [\sigma - \tau, t^*], j = 1, \dots, p. \quad (10)$$

By using (4), (7), (9) and (10), we have

$$\begin{aligned} D^+ u_m(t^*) &\leq P_m u(t^*) + Q_m [u(t^*)]_\tau + G_m \omega(t^*) + H_m [\omega(t^*)]_\tau + \eta_m \\ &\leq P_m [(k + \varepsilon)ze^{-\lambda(t^*-\sigma)} + \hat{\eta}] + Q_m [(k + \varepsilon)ze^{\lambda\tau} e^{-\lambda(t^*-\sigma)} + \hat{\eta}] \\ &\quad + G_m [(k + \varepsilon)\tilde{z}e^{-\lambda(t^*-\sigma)} + \hat{I}] + H_m [(k + \varepsilon)\tilde{z}e^{\lambda\tau} e^{-\lambda(t^*-\sigma)} + \hat{I}] + \eta_m \\ &= (k + \varepsilon)[P + Qe^{\lambda\tau} + (G + He^{\lambda\tau})(E - Re^{\lambda\tau})^{-1}(M + Ne^{\lambda\tau})z]_m e^{-\lambda(t^*-\sigma)} \\ &\quad + [(P + Q + (G + H)(E - R)^{-1}(M + N))\Pi^{-1}\eta]_m + \eta_m \\ &\quad + [(P + Q)\Pi^{-1}(G + H)(E - R)^{-1}I]_m + [(G + H)(E - R)^{-1}I]_m \\ &\quad + [(G + H)(E - R)^{-1}(M + N)\Pi^{-1}(G + H)(E - R)^{-1}I]_m \\ &< -\lambda(k + \varepsilon)z_m e^{-\lambda(t^*-\sigma)} + [-\Pi\Pi^{-1}\eta]_m + \eta_m + [(G + H)(E - R)^{-1}I]_m \\ &\quad + [(-\Pi - (G + H)(E - R)^{-1}(M + N))\Pi^{-1}(G + H)(E - R)^{-1}I]_m \\ &\quad + [(G + H)(E - R)^{-1}(M + N)\Pi^{-1}(G + H)(E - R)^{-1}I]_m \\ &= -\lambda(k + \varepsilon)z_m e^{-\lambda(t^*-\sigma)} = \xi'(t^*). \end{aligned} \quad (11)$$

This contradicts the second inequality in (9), so the first inequality in (8) holds. Therefore, we have to assume that (10) holds and we shall obtain another contradiction. Next, we consider three cases.

Case 1. The elements of the M_r and N_r are not all zero. Without loss of generality, we let $m_{rl} > 0, 1 \leq l \leq n$. Then, by using (4), (10) and the first inequality in (8), we have

$$\begin{aligned}
 \omega_r(t^*) &\leq M_r u(t^*) + N_r [u(t^*)]_\tau + R_r [\omega(t^*)]_\tau + I_r \\
 &= m_{rl} u_l(t^*) + \sum_{j \neq l} m_{rj} u_j(t^*) + N_r [u(t^*)]_\tau + R_r [\omega(t^*)]_\tau + I_r \\
 &< m_{rl} \xi_l(t^*) + \sum_{j \neq l} m_{rj} \xi_j(t^*) + N_r [u(t^*)]_\tau + R_r [\omega(t^*)]_\tau + I_r \\
 &\leq M_r [(k + \varepsilon) z e^{-\lambda(t^* - \sigma)} + \hat{\eta}] + N_r [(k + \varepsilon) z e^{\lambda\tau} e^{-\lambda(t^* - \sigma)} + \hat{\eta}] \\
 &\quad + R_r [(k + \varepsilon) \tilde{z} e^{\lambda\tau} e^{-\lambda(t^* - \sigma)} + \hat{I}] + I_r \\
 &= (k + \varepsilon) [(M + N e^{\lambda\tau} + R e^{\lambda\tau} (E - R e^{\lambda\tau})^{-1} (M + N e^{\lambda\tau})) z]_r e^{-\lambda(t^* - \sigma)} \\
 &\quad + [(M + N + R(E - R)^{-1} (M + N)) \Pi^{-1} \eta]_r + [R(E - R)^{-1} I]_r + I_r \\
 &\quad + [(M + N) \Pi^{-1} (G + H) (E - R)^{-1} \\
 &\quad + R(E - R)^{-1} (M + N) \Pi^{-1} (G + H) (E - R)^{-1}] I]_r \\
 &= (k + \varepsilon) [(E - R e^{\lambda\tau})^{-1} (M + N e^{\lambda\tau}) z]_r e^{-\lambda(t^* - \sigma)} \\
 &\quad + [(E - R)^{-1} (M + N) \Pi^{-1} \eta]_r \\
 &\quad + [((E - R)^{-1} (M + N) \Pi^{-1} (G + H) (E - R)^{-1} + (E - R)^{-1}) I]_r \\
 &= (k + \varepsilon) \tilde{z}_r e^{-\lambda(t^* - \sigma)} + \hat{I}_r = \zeta_r(t^*). \tag{12}
 \end{aligned}$$

Which contradicts the first equality in (10), so under this case, the second inequality in (8) holds.

Case 2. The elements of the M_r and N_r are all zero, but the elements of the R_r are not all zero. Without loss of generality, we let $r_{rh} > 0, 1 \leq h \leq p$. Combining with $\omega \in C[[\sigma, b), R_+^p]$ and the monotonicity of $\zeta(t)$, from (10) and $[\omega(t)]_\tau = \sup_{-\tau \leq s \leq 0} \omega(t + s)$, we know there must exist $t^* - \tau \leq t_1, \dots, t_p \leq t^*$ such that

$$[\omega(t^*)]_\tau = \sup_{t^* - \tau \leq t \leq t^*} \omega(t) = (\omega_1(t_1), \dots, \omega_p(t_p))^T < (\zeta_1(t^* - \tau), \dots, \zeta_p(t^* - \tau))^T. \tag{13}$$

By using (4) and (13), we have

$$\begin{aligned}
 \omega_r(t^*) &\leq R_r [\omega(t^*)]_\tau + I_r \\
 &= r_{rh} [\omega_h(t^*)]_\tau + \sum_{j \neq h} r_{rj} [\omega_j(t^*)]_\tau + I_r \\
 &< r_{rh} \zeta_h(t^* - \tau) + \sum_{j \neq h} r_{rj} \zeta_j(t^* - \tau) + I_r \\
 &= R_r \zeta(t^* - \tau) + I_r \\
 &= R_r [(k + \varepsilon) \tilde{z} e^{\lambda\tau} e^{-\lambda(t^* - \sigma)} + \hat{I}] + I_r
 \end{aligned}$$

$$\begin{aligned}
 &= (k + \varepsilon) [Re^{\lambda\tau} (E - Re^{\lambda\tau})^{-1} (M + Ne^{\lambda\tau})z]_r e^{-\lambda(t^* - \sigma)} \\
 &\quad + [R(E - R)^{-1} (M + N)\Pi^{-1}\eta]_r \\
 &\quad + [R(E - R)^{-1} (M + N)\Pi^{-1} (G + H)(E - R)^{-1}I]_r + [R(E - R)^{-1}I]_r + I_r \\
 &= (k + \varepsilon) [(E - Re^{\lambda\tau})^{-1} (M + Ne^{\lambda\tau})z]_r e^{-\lambda(t^* - \sigma)} + [(E - R)^{-1} (M + N)\Pi^{-1}\eta]_r \\
 &\quad + [(E - R)^{-1} (M + N)\Pi^{-1} (G + H)(E - R)^{-1}I]_r + [(E - R)^{-1}I]_r \\
 &\quad - (k + \varepsilon) [(E - Re^{\lambda\tau})(E - Re^{\lambda\tau})^{-1} (M + Ne^{\lambda\tau})z]_r e^{-\lambda(t^* - \sigma)} \\
 &\quad - [(E - R)(E - R)^{-1} (M + N)\Pi^{-1}\eta]_r \\
 &\quad - [(E - R)(E - R)^{-1} (M + N)\Pi^{-1} (G + H)(E - R)^{-1}I]_r \\
 &= (k + \varepsilon)\tilde{z}_r e^{-\lambda(t^* - \sigma)} + \hat{I}_r = \zeta_r(t^*), \tag{14}
 \end{aligned}$$

which contradicts the first equality in (10); so under this case, the second inequality in (8) holds.

Case 3. The elements of the M_r , N_r and R_r are all zero, then the conclusion of the second inequality in (5) is trivial.

From the above analysis, we know (8) is true for all $t \in [\sigma, b)$. Letting $\varepsilon \rightarrow 0$ in (8), we can get (5).

The proof is complete. □

Remark 1 Suppose that $M = N = 0$, $R = 0$, $I = 0$ in Theorem 1, then we get Lemma 1 in [28]. Suppose that $J = 0$, $I = 0$ in Theorem 1, then we get Theorem 2.1 in [32].

For the model (1), we introduce the following assumptions:

(A₁) The functions $f_j(\cdot)$, $g_j(\cdot)$ are Lipschitz continuous, i.e., there are positive constants $k_j, l_j, j \in \mathcal{N}$ such that for all $s_1, s_2 \in R$

$$|f_j(s_1) - f_j(s_2)| \leq k_j |s_1 - s_2|, \quad |g_j(s_1) - g_j(s_2)| \leq l_j |s_1 - s_2|.$$

(A₂) Let $\|C\| < 1$ and $\hat{\Pi} = -(-2D + (D + \hat{A} + \hat{B})(E - |C|)^{-1})$ be a nonsingular \mathcal{M} -matrix, where $D = \text{diag}\{d_1, \dots, d_n\} > 0$, $\hat{A} = (|a_{ij}|k_j)_{n \times n}$, $\hat{B} = (|b_{ij}|l_j)_{n \times n}$. Let $\hat{J} = |A|[f(0)]^+ + |B|[g(0)]^+ + [J]^+$.

Theorem 2 Assume that (A₁), (A₂) hold. Then $S = \{\phi \in C[\phi]_r^+ \leq (E - |C|)^{-1}\hat{\Pi}^{-1}\hat{J}\}$ is a global attracting set of (1).

Proof Under the conditions (A₁), (A₂), from [7, 8], we know the solution $x(t, t_0, \phi)$ of (1) exists globally. We denote

$$\begin{aligned}
 u(t) &= \begin{cases} [x(t) - Cx(t - r(t))]^+, & t \geq t_0, \\ W[(E - |C|)[\phi]_r^+]^+, & t_0 - \tau \leq t \leq t_0, \end{cases} \\
 \omega(t) &= [x(t)]^+, \quad t \geq t_0 - \tau,
 \end{aligned} \tag{15}$$

where $W = \text{diag}\{w_1, \dots, w_n\} \geq 0$ such that $[\phi(t_0) - C\phi(-r(t_0))]^+ = W[(E - |C|)[\phi]_r^+]^+$.

Then, for $t \geq t_0$, from (1) and (A_1) , we calculate the upper-right-hand derivative $D^+ u(t)$ along the solutions of (1),

$$\begin{aligned}
 D^+ u_i(t) &= \operatorname{sgn} \left(x_i(t) - \sum_{j=1}^n c_{ij} x_j(t - r_{ij}(t)) \right) \left\{ -d_i \left(x_i(t) - \sum_{j=1}^n c_{ij} x_j(t - r_{ij}(t)) \right) \right. \\
 &\quad - \sum_{j=1}^n d_i c_{ij} x_j(t - r_{ij}(t)) + \sum_{j=1}^n a_{ij} [f_j(x_j(t)) - f_j(0)] \\
 &\quad \left. + \sum_{j=1}^n b_{ij} [g_j(x_j(t - \tau_{ij}(t))) - g_j(0)] + \sum_{j=1}^n (a_{ij} f_j(0) + b_{ij} g_j(0)) + J_i \right\} \\
 &\leq -d_i \left| x_i(t) - \sum_{j=1}^n c_{ij} x_j(t - r_{ij}(t)) \right| + \sum_{j=1}^n |a_{ij} k_j| |x_j(t)| + \sum_{j=1}^n |b_{ij} l_j| |x_j(t - \tau_{ij}(t))| \\
 &\quad + \sum_{j=1}^n d_i |c_{ij}| |x_j(t - r_{ij}(t))| + \sum_{j=1}^n (|a_{ij}| |f_j(0)| + |b_{ij}| |g_j(0)|) + |J_i| \\
 &\leq -d_i u_i(t) + \sum_{j=1}^n |a_{ij} k_j \omega_j(t) + \sum_{j=1}^n (|b_{ij} l_j + d_i |c_{ij}|) [\omega_j(t)]_\tau \\
 &\quad + \sum_{j=1}^n (|a_{ij}| |f_j(0)| + |b_{ij}| |g_j(0)|) + |J_i|, \quad i \in \mathcal{N}, t \geq t_0. \tag{16}
 \end{aligned}$$

So, from (16) and (A_2) , we get

$$D^+ u(t) \leq -Du(t) + \hat{A}\omega(t) + (\hat{B} + D|C|)[\omega(t)]_\tau^+ + \hat{J}, \quad t \geq t_0. \tag{17}$$

On the other hand, we have

$$\begin{aligned}
 \omega_i(t) &= |x_i(t)| = \left| \left(x_i(t) - \sum_{j=1}^n c_{ij} x_j(t - r_{ij}(t)) \right) + \sum_{j=1}^n c_{ij} x_j(t - r_{ij}(t)) \right| \\
 &\leq \left| \left(x_i(t) - \sum_{j=1}^n c_{ij} x_j(t - r_{ij}(t)) \right) \right| + \sum_{j=1}^n |c_{ij}| |x_j(t - r_{ij}(t))| \\
 &\leq u_i(t) + \sum_{j=1}^n |c_{ij}| [\omega_j(t)]_\tau, \quad t \geq t_0. \tag{18}
 \end{aligned}$$

That is,

$$\omega(t) \leq u(t) + |C|[\omega(t)]_\tau^+, \quad t \geq t_0. \tag{19}$$

From (A_2) , Definition 4 and Lemma 1, we have $(E - |C|)^{-1} \geq 0$, $\hat{\Pi}^{-1} \geq 0$, and so

$$\varrho \triangleq \hat{\Pi}^{-1} \hat{J} \geq 0, \quad \upsilon \triangleq (E - |C|)^{-1} \hat{\Pi}^{-1} \hat{J} \geq 0. \tag{20}$$

Furthermore, for $z \in \Omega_M(\hat{\Pi})$, we have

$$(-2D + (D + \hat{A} + \hat{B})(E - |C|)^{-1})z < 0.$$

By using continuity, we can find a positive constant λ such that

$$\begin{aligned} \rho(e^{\lambda\tau}|C|) < 1 \quad \text{and} \\ (\lambda E - 2D + (D + \hat{A} + \hat{B}e^{\lambda\tau})(E - |C|e^{\lambda\tau})^{-1})z < 0 \quad \text{for } z \in \Omega_M(\hat{\Pi}), \end{aligned} \tag{21}$$

and we know

$$\tilde{z} \triangleq (E - |C|e^{\lambda\tau})^{-1}z > 0.$$

From (15) and the initial conditions in (1): $x(t_0 + s) = \phi(s)$, $s \in [-\tau, 0]$, where $\phi \in C$, we can get

$$u(t) \leq k_0z, \quad \omega(t) \leq k_0\tilde{z}, \quad k_0 = \frac{\max_{1 \leq i \leq n} \{w_i \sum_{j=1}^n \varsigma_{ij} \|\phi\|_\tau\}}{\min_{1 \leq i \leq n, 1 \leq j \leq n} \{z_i, \tilde{z}_j\}}, \quad t_0 - \tau \leq t \leq t_0, \tag{22}$$

where $(\varsigma_{ij})_{n \times n} = |E - |C||$. From (20), (22), we know

$$u(t) \leq k_0ze^{-\lambda(t-t_0)} + \varrho, \quad \omega(t) \leq k_0\tilde{z}e^{-\lambda(t-t_0)} + \upsilon, \quad t_0 - \tau \leq t \leq t_0. \tag{23}$$

From (17), (19), (23), (A_2) and Theorem 1, we get

$$u(t) \leq k_0ze^{-\lambda(t-t_0)} + \varrho, \quad \omega(t) \leq k_0\tilde{z}e^{-\lambda(t-t_0)} + \upsilon, \quad t \geq t_0. \tag{24}$$

From (24), we know the conclusion is true. The proof is complete. □

If $J = 0, f(0) = g(0) = 0$ in the model (1), then we know the model (1) has an equilibrium point zero. From Theorem 2, we get the following conclusion.

Corollary 1 *Assume that $(A_1), (A_2)$ with $\hat{J} = 0$ hold. Then the zero solution of (1) is globally exponentially stable in Lyapunov sense and the exponential convergence rate is determined by (21).*

Theorem 3 *Assume that $(A_1), (A_2)$ hold. Then $S = \{\phi \in C | [\phi]_\tau^+ \leq (E - |C|)^{-1}\hat{\Pi}^{-1}\hat{J}, [\phi(t_0) - C\phi(-r(t_0))]^+ = [(E - |C|)[\phi]_\tau^+]^+\}$ is a positive invariant set and also a global attracting set of (1).*

Proof Since $[\phi]_\tau^+ \leq (E - |C|)^{-1}\hat{\Pi}^{-1}\hat{J}$ and $[\phi(t_0) - C\phi(-r(t_0))]^+ = [(E - |C|)[\phi]_\tau^+]^+$, then from the definition of $u(t)$ and $\omega(t)$, we get

$$u(t) \leq \hat{\Pi}^{-1}\hat{J} \quad \text{and} \quad \omega(t) \leq (E - |C|)^{-1}\hat{\Pi}^{-1}\hat{J}, \quad t_0 - \tau \leq t \leq t_0. \tag{25}$$

We choose $k = 0$ in Theorem 1; the remaining proof is similar to the proof of Theorem 2, and we omit it here. So we get the conclusion. □

If we further assume that $c_{ij} = 0, i, j \in \mathcal{N}$, then the system (1) becomes

$$\begin{cases} x'_i(t) = -d_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t - \tau_{ij}(t))) + J_i, & t \geq t_0, \\ x_i(t_0 + s) = \phi_i(s), & -\tau \leq s \leq 0, i \in \mathcal{N}. \end{cases} \tag{26}$$

Therefore, we can get the following corollary.

Corollary 2 Assume that (A_1) and (A_2) with $c_{ij} = 0, i, j \in \mathcal{N}$ hold. Then $S = \{\phi \in C|[\phi]_{\tau}^+ \leq (D - \hat{A} - \hat{B})^{-1}\hat{J}\}$ is a positive invariant set and also a global attracting set of (26).

Remark 2 The authors in [23] consider the special case of the model (26), but they require that the activation functions are continuous and monotonically nondecreasing, and the delay functions are satisfying $\frac{d\tau_{ij}(t)}{dt} \leq 0$.

Examples

Example 1 Consider the nonlinear neutral differential equation with delays

$$\begin{cases} x_1'(t) = -5x_1(t) + g_1(x_1(t - \tau_{11}(t))) - \frac{3}{4}g_2(x_2(t - \tau_{12}(t))) \\ \quad + \frac{1}{4}(1 + \frac{4}{3}\cos 4t)x_1'(t - r(t)) + J_1, \\ x_2'(t) = -4x_2(t) - \frac{1}{2}g_1(x_1(t - \tau_{21}(t))) + g_2(x_2(t - \tau_{22}(t))) \\ \quad + \frac{1}{4}(1 + \frac{4}{3}\cos 4t)x_2'(t - r(t)) + J_2, \quad t \geq 0, \end{cases} \tag{27}$$

where $g_1(s) = \frac{|s+1|-|s-1|}{2}, g_2(s) = s, 0 < r(t) = \frac{1}{2} - \frac{1}{3}\sin 4t \leq \frac{5}{6} < 1 \triangleq \tau, \tau_{ij}(t) = |\sin(i+j)t| \leq 1 \triangleq \tau$ for $i, j = 1, 2$.

By simple computation, we get

$$\begin{aligned} D &= \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}, & \hat{A} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \hat{B} &= \begin{pmatrix} 1 & \frac{3}{4} \\ \frac{1}{2} & 1 \end{pmatrix}, & |C| &= \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \\ \hat{\Pi} &= -(-2D + (D + \hat{A} + \hat{B})(E - |C|)^{-1}) = \begin{pmatrix} 2 & -1 \\ -\frac{2}{3} & \frac{4}{3} \end{pmatrix}, \\ (E - |C|)^{-1}\hat{\Pi}^{-1} &= \begin{pmatrix} \frac{8}{9} & \frac{2}{3} \\ \frac{4}{9} & \frac{4}{3} \end{pmatrix}. \end{aligned}$$

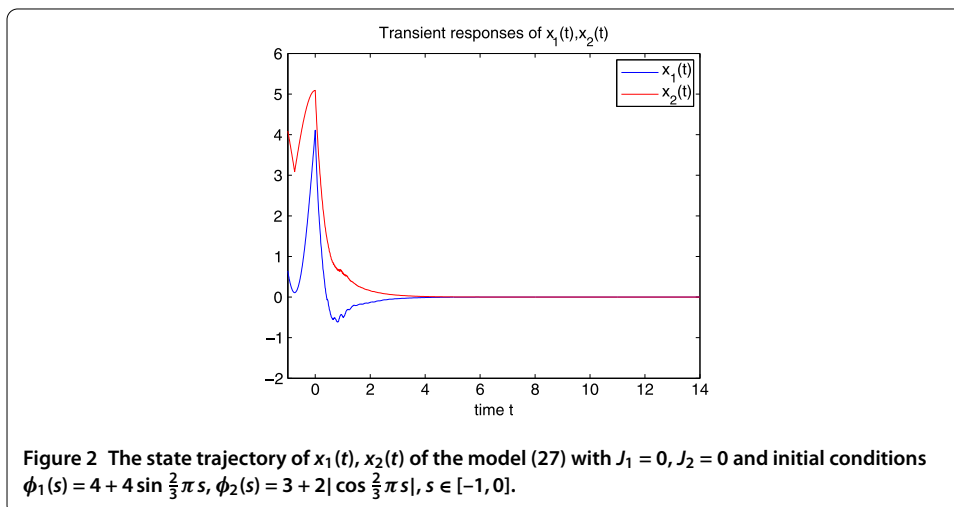
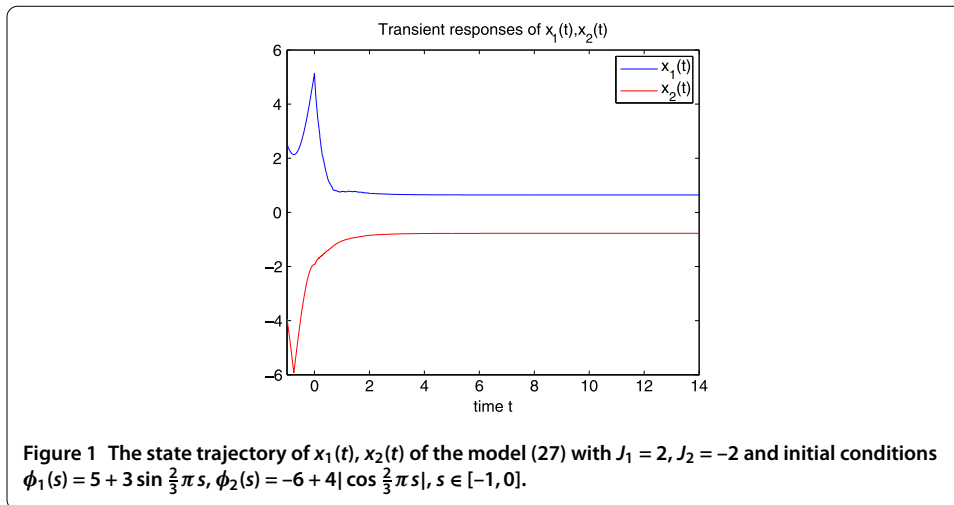
We can easily observe that $\rho(|C|) = \frac{1}{4} < 1, \hat{\Pi}$ is a nonsingular \mathcal{M} -matrix and

$$\Omega_M(\hat{\Pi}) = \left\{ (z_1, z_2)^T > 0 \mid \frac{1}{2}z_2 < z_1 < 2z_2 \right\}.$$

Let $z = (1, 1)^T \in \Omega_M(\hat{\Pi})$, and $\lambda = 0.11$, which satisfies the inequalities

$$\begin{aligned} \rho(e^{\lambda\tau}|C|) &= 0.2791 < 1, \\ (\lambda E - 2D + (D + \hat{A} + \hat{B}e^{\lambda\tau})(E - |C|e^{\lambda\tau})^{-1})z &= (-0.2448, -0.0190)^T < \mathbf{0}. \end{aligned}$$

Case 1 Let $J_1 = 2, J_2 = -2$, so by Theorem 2, we know $S = \{\phi \in C|[\phi]_{\tau}^+ \leq (E - |C|)^{-1}\hat{\Pi}^{-1}\hat{J} = (\frac{28}{9}, \frac{32}{9})^T\}$ is a global attracting set of (27), and by Theorem 3, we know $S^* = \{\phi \in C|[\phi]_{\tau}^+ \leq$

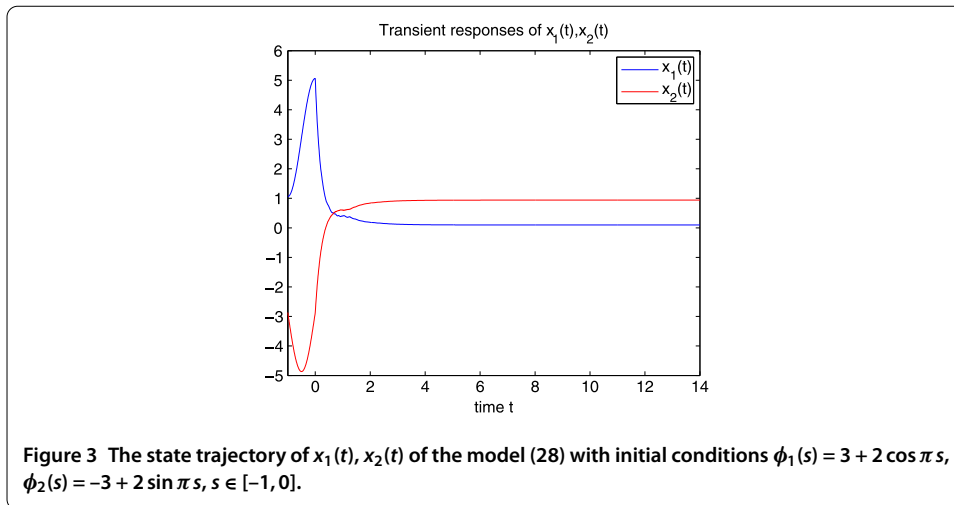


$(\frac{28}{9}, \frac{32}{9})^T$ and $\phi(0) = \phi(-\frac{1}{2}) = \pm[\phi]_{\tau}^+$ is a positive invariant and global attracting set of (27). (See Figure 1.)

Remark 3 The authors in [25] considered the global attracting set of neutral type system, but the boundedness of activation functions is required, so the Theorem 1 in [25] is ineffective for the model (27).

Case 2 If $J_1 = J_2 = 0$, from Corollary 1, we know the zero solution of (27) is globally exponentially stable in Lyapunov sense and the exponential convergence rate is equal to 0.11. (See Figure 2.)

Remark 4 It is evident that the delay functions $r(t) = \frac{1}{2} - \frac{1}{3} \sin 4t, \tau_{ij}(t) = |\sin(i + j)t|$ do not satisfy the condition $\sup_{t \in \mathbb{R}} \dot{r}(t) < 1, \sup_{t \in \mathbb{R}} \dot{\tau}_{ij}(t) < 1, i, j = 1, 2$, so the results in [14, 35] are invalid for the model (27).



Example 2 Consider the nonlinear differential equation with delays

$$\begin{cases} x_1'(t) = -5x_1(t) + g_1(x_1(t - \tau_{11}(t))) - \frac{3}{4}g_2(x_2(t - \tau_{12}(t))) + 1, \\ x_2'(t) = -4x_2(t) - \frac{1}{2}g_1(x_1(t - \tau_{21}(t))) + g_2(x_2(t - \tau_{22}(t))) + 3, \quad t \geq 0, \end{cases} \quad (28)$$

where $g_1(s) = \frac{|s+1|-|s-1|}{2}$, $g_2(s) = \sin s$, $\tau_{ij}(t) = |\sin(i+j)t| \leq 1 \triangleq \tau$ for $i, j = 1, 2$.

Similarly to the computation of Example 1, from Corollary 2, we can get the set $S = \{\phi \in C | [\phi]_{\tau}^+ \leq (D - \hat{A} - \hat{B})^{-1} \hat{J} = (\frac{14}{31}, \frac{100}{93})^T\}$ is an invariant and global attracting set of the model (28). (See Figure 3.)

Remark 5 It is evident that the activation function $g_2(s) = \sin s$ is not monotonically non-decreasing and the delay functions $\tau_{ij}(t) = |\sin(i+j)t|$ do not satisfy $\frac{d\tau_{ij}(t)}{dt} \leq 0$, $i, j = 1, 2$, so the results in [23] are invalid for the model (28).

Competing interests

The author declares that they have no competing interests.

Author's contributions

SJL carried out the main proof of the theorems and examples in this paper alone. The author approved the final manuscript.

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