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On homoclinic orbits for a class of damped vibration systems

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Abstract
In this article, we establish the new result on homoclinic orbits for a class of damped vibration systems. Some recent results in the literature are generalized and significantly improved.
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Introduction and main results

Consider the following second-order damped vibration problems

$$\ddot{u}(t) + B\dot{u}(t) - L(t)u(t) + W_u\bigl(t, u(t)\bigr) = 0, \quad t \in \mathbb{R}, \tag{VS}$$

where $u = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$, B is an antisymmetric $N \times N$ constant matrix, $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix valued function and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. As usual we say that a solution u of (VS) is homoclinic (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u \not\equiv 0$, $u(t) \to 0$, and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

When B is a zero matrix, (VS) is just the following second-order Hamiltonian systems (HSs)

$$\ddot{u}(t) - L(t)u(t) + W_u\bigl(t, u(t)\bigr) = 0, \quad t \in \mathbb{R}. \tag{HS}$$

Inspired by the excellent monographs and works [1–3], by now, the existence and multiplicity of periodic and homoclinic solutions for HSs have extensively been investigated in many articles *via* variational methods, see [4–22]. Also second-order HSs with impulses *via* variational methods have recently been considered in [23–26]. More precisely, in 1990, Rabinowitz [3] established the existence result on homoclinic orbit for the periodic second-order HS. It is well known that the periodicity is used to control the lack of compactness due to the fact that HS is set on all $\mathbb{R}$.

For the nonperiodic case, the problem is quite different from the one described in nature. Rabinowitz and Tanaka [13] introduced a type of coercive condition on the matrix L:

(L_1) $l(t) := \inf_{|x|=1} L(t)x \cdot x \to +\infty$, as $|t| \to \infty$.

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They established a compactness lemma under the nonperiodic case and obtained the existence of homoclinic orbit for the nonperiodic system (HS) under the usual Ambrosetti-Rabinowitz (AR) growth condition

$$0 < \mu W(t,u) \leq W_u\bigl(t,u(t)\bigr)u, \quad \forall t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \setminus \{0\},$$

where $\mu > 2$ is a constant. Later, Ding [7] strengthened condition (L_1) by

(L_2) there exists a constant $\alpha > 0$ such that

$$l(t)|t|^{-\alpha} \to +\infty \quad \text{as } |t| \to \infty.$$

Under the condition (L_2) and some subquadratic conditions on $W(t,u)$, Ding proved the existence and multiplicity of homoclinic orbits for the system (HS). From then on, the condition (L_1) or (L_2) are extensively used in many articles.

Compared with the case where B is a zero matrix, the case where $B \neq 0$, *i.e.*, the nonperiodic system (VS), has been considered only by a few authors, see [27–29]. Zhang and Yuan [28] studied the existence of homoclinic orbits for the nonperiodic system (VS) when W satisfies the subquadratic condition at infinity. Soon after, Wu and Zhang [27] obtained the existence and multiplicity of homoclinic orbits for the nonperiodic system (VS) when W satisfies the local (AR) growth condition

$$0 < \mu W(t,u) \leq W_u(t,u)u, \quad \forall t \in \mathbb{R} \text{ and } |u| \geq r, \tag{1}$$

where $\mu > 2$ and $r > 0$ are two constants. It is worth noticing that the matrix L is required to satisfy the condition (L_1) in the above two articles.

Inspired by [27, 28], in this article we shall replace the condition (L_1) on L by the following conditions:

(L_3) there exists a constant $\beta > 1$ such that

$$\operatorname{meas}\bigl\{t \in \mathbb{R} : |t|^{-\beta} L(t) < bI_N\bigr\} < +\infty, \quad \forall b > 0,$$

and

(L_4) there exists a constant $\gamma \geq 0$ such that

$$l(t) := \inf_{|x|=1} L(t)x \cdot x \geq -\gamma, \quad \forall t \in \mathbb{R},$$

which are first used in [20]. By using a recent critical point theorem, we prove that the nonperiodic system (VS) has at least one homoclinic orbit when W satisfies weak superquadratic at the infinity, which improve and extend the results of [27, 28].

Remark 1 In fact, there are some matrix-valued functions $L(t)$ satisfying (L_3) and (L_4), but not satisfying (L_1) or (L_2). For example,

$$L(t) = \bigl(t^4 \sin^2 t + 1\bigr)I_N.$$

We consider the following conditions:

(W_1) $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and there exist positive constants c_1 and $\nu > 2$ such that

$$c_1|u|^\nu \le W_u(t,u)u, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N.$$

(W_2) $W_u(t,u) = o(|u|)$ as $|u| \to 0$ uniformly in t.

(W_3) $\tilde{W}(t,u) := \frac{1}{2}W_u(t,u)u - W(t,u) > 0$ if $u \neq 0$, and

$$\inf\left\{ \frac{\tilde{W}(t,u)}{|u|^2} : t \in \mathbb{R} \text{ with } a \le |u| < b \right\} > 0,$$

for any $a, b > 0$.

(W_4) There exist $r > 0$ and $\sigma > 1$ such that $|W_u(t,u)|^\sigma \le c\tilde{W}(t,u)|u|^\sigma$ if $|u| \ge r$.

Theorem 2 *Assume that* (L_3)*-*(L_4) *and* (W_1)*-*(W_4) *hold. Then the system (VS) has at least one homoclinic orbit.*

Remark 3 To see that our result generalizes [27] we present the following examples. These functions satisfy the weak superquadratic conditions (W_1)-(W_4), but not verify the growth condition (1).
Example:

$$W(t,u) = a(t)\left(|u|^p + (p-2)|u|^{p-\epsilon} \sin^2\left(\frac{|u|^\epsilon}{\epsilon} \right) \right),$$

where $\inf_{t\in\mathbb{R}} a(t) > 0$, and $p > 2$, $0 < \epsilon < p - 2$.

In fact it is easy to verify that (W_1)-(W_4) are satisfied. However, similar to the discussion of Remark 1.2 in [30], let $u_n = (\epsilon(n\pi + \frac{3\pi}{4}))^{\frac{1}{\epsilon}} e_1$, where $e_1 = (1, 0, \ldots, 0)$. Then for any $\mu > 2$, one has

$$\begin{aligned} W_u(t,u_n)u_n - \mu W(t,u_n) &= a(t)\big[(p-\mu)|u_n|^p \\ &\quad + (p-2)(p-\epsilon-\mu)|u_n|^{p-\epsilon} \sin^2\big(|u_n|^\epsilon/\epsilon\big) \\ &\quad + (p-2)|u_n|^p \sin 2\big(|u_n|^\epsilon/\epsilon\big)\big] \\ &= a(t)|u_n|^p \left[2 - \mu + \frac{(p-2)(p-\epsilon-\mu)\sin^2(|u_n|^\epsilon/\epsilon)}{|u_n|^\epsilon} \right] \\ &\to -\infty \quad \text{as } n \to \infty. \end{aligned}$$

That is, the condition (1) is not satisfied for any $\mu > 2$.

This article is organized as follows. In the following section, we formulate the variational setting and recall a critical point theorem required. In section 'Linking structure', we discuss linking structure of the functional. In section 'The $(C)_c$-sequence', we study the Cerami condition of the functional and give the proof of Theorem 2.

Notation Throughout the article, we shall denote by $c > 0$ various positive constants which may vary from line to line and are not essential to the problem.

Variational setting

In this section, we establish a variational setting for the system (VS). Let H be $H^1(\mathbb{R},\mathbb{R}^N)$ which is a Hilbert space with the inner product and norm given by

$$\langle u,v\rangle_H = \int_{\mathbb{R}} \big[\big(\dot u(t),\dot v(t)\big) + \big(u(t),v(t)\big)\big]\,dt$$

and

$$\|u\|_H = \left(\int_{\mathbb{R}} \big[\big|\dot u(t)\big|^2 + \big|u(t)\big|^2\big]\,dt\right)^{\frac{1}{2}}$$

for $u,v\in H$, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$. It is well known that H is continuously embedded in $L^p(\mathbb{R},\mathbb{R}^N)$ for $p\in[2,\infty)$. Define an operator $J:H\to H$ by

$$\langle Ju,v\rangle = \int_{\mathbb{R}} (Bu,\dot v)\,dt \tag{2}$$

for all $u,v\in H$. Since B is an antisymmetric $N\times N$ constant matrix, J is self-adjoint on H. Moreover, we denote by A the self-adjoint extension of the operator $-\frac{d^2}{dt^2}+L(t)+J$ with the domain $\mathcal{D}(A)\subset L^2(\mathbb{R},\mathbb{R}^N)$. Let $|\cdot|_p$ be the usual L^p-norm, and $\langle\cdot,\cdot\rangle_2$ the usual L^2-inner product. Set $E:=\mathcal{D}(|A|^{\frac{1}{2}})$, the domain of $|A|^{\frac{1}{2}}$. Define on E the inner product

$$\langle u,v\rangle_E := \big\langle |A|^{\frac{1}{2}}u, |A|^{\frac{1}{2}}v\big\rangle_2 + \langle u,v\rangle_2$$

and the norm

$$\|u\|_E = \langle u,u\rangle_E^{\frac{1}{2}}.$$

Then E is a Hilbert space and it is easy to verify that E is continuously embedded in $H^1(\mathbb{R},\mathbb{R}^N)$. Using a similar proof of Lemma 3.1 in [20], we can prove the following lemma.

Lemma 4 *Suppose that $L(t)$ satisfies (L_3) and (L_4), then E is compactly embedded into $L^p(\mathbb{R},\mathbb{R}^N)$ for $p\in[1,+\infty]$.*

By Lemma 4, it is easy to prove that the spectrum $\sigma(A)$ has a sequence of eigenvalues (counted with their multiplicities)

$$\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots$$

with $\lambda_k\to+\infty$ as $k\to+\infty$, and corresponding eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$, $Ae_k=\lambda_k e_k$, form an orthogonal basis in $L^2(\mathbb{R},\mathbb{R}^N)$. Assume $\lambda_1,\lambda_2,\ldots,\lambda_{\ell^-}<0$, $\lambda_{\ell^-+1}=\cdots=\lambda_\ell=0$ and let $E^-:=\operatorname{span}\{e_1,\ldots,e_{\ell^-}\}$, $E^0:=\operatorname{span}\{e_{\ell^-+1},\ldots,e_\ell\}$, and $E^+:=\operatorname{cl}_E(\operatorname{span}\{e_{\ell+1},\ldots\})$. Then

$$E=E^-\oplus E^0\oplus E^+$$

is an orthogonal decomposition of E. We introduce on E the following product

$$\langle u,v\rangle := \big\langle |A|^{\frac{1}{2}}u, |A|^{\frac{1}{2}}v\big\rangle_2 + \big\langle u^0,v^0\big\rangle_2,$$

and the norm

$$\|u\| = \langle u,u\rangle^{\frac{1}{2}},$$

where $u = u^- + u^0 + u^+$, $v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+$. Then $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent (see [7]). So by Lemma 4, we see that there exists a constant $\eta_p > 0$ such that

$$|u|_p \le \eta_p \|u\|, \quad \forall u \in E, \forall p \in [1,+\infty].$$

Define the functional Φ on E by

$$\Phi(u) = \int_{\mathbb{R}} \left[\frac{1}{2}\left|\dot{u}(t)\right|^2 + \frac{1}{2}\bigl(Bu(t),\dot{u}(t)\bigr) + \frac{1}{2}\bigl(L(t)u(t),u(t)\bigr) - W\bigl(t,u(t)\bigr)\right] dt.$$

Then

$$\Phi(u) = \frac{1}{2}\bigl(\|u^+\|^2 - \|u^-\|^2\bigr) - \int_{\mathbb{R}} W\bigl(t,u(t)\bigr)\,dt, \tag{3}$$

where $u = u^- + u^0 + u^+ \in E$. Furthermore, define

$$\Psi(u) := \int_{\mathbb{R}} W(t,u)\,dt.$$

From the assumptions it follows that Φ is defined on the Banach space E and belongs to $C^1(E,\mathbb{R})$. A standard argument shows that critical points of Φ are solutions of the system (VS). Moreover, it is easy to verify that if $u \not\equiv 0$ is a solution of (VS), then $u(t) \to 0$ and $\dot{u}(t) \to 0$, as $|t| \to \infty$ (see Lemma 3.1 in [31]).

In order to study the critical points of Φ, we now recall a critical point theorem, see [32].

Let E be a Banach space. A sequence $\{u_n\} \subset E$ is said to be a $(C)_c$-sequence if

$$\Phi(u_n) \to c \quad \text{and} \quad \bigl(1 + \|u_n\|\bigr)\Phi'(u_n) \to 0.$$

Φ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence.

Theorem 5 ([32]) *Suppose* $\Phi \in C^1(E,\mathbb{R})$, $E = X \oplus Y$, *where* $\dim X < \infty$, *there exist* $R > \rho > 0$, $\kappa > 0$ *and* $e_0 \in Y \setminus \{0\}$ *such that* $\inf \Phi(Y \cap S_\rho) \ge \kappa$ *and* $\sup \Phi(\partial Q) \le 0$, *where* $S_\rho := S_\rho(0)$ *is the sphere of radius* ρ *and center* 0, *and*

$$Q = \bigl\{u = x + se_0 : s \ge 0, x \in X, \|u\| \le R\bigr\}.$$

Moreover, if Φ *satisfies the* $(C)_c$*-condition for all* $c \in [\kappa, \sup \Phi(Q)]$, *then* Φ *has a critical value in* $[\kappa, \sup \Phi(Q)]$.

Linking structure

First we discuss the linking structure of Φ. By condition (W_1), one has

$$W(t,u) \ge c_1 |u|^\nu \ge 0, \tag{4}$$

for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$. Observe that if (W_4) holds, and together with (4), then if $|u| > r$, one has

$$\begin{aligned}
\left|W_u(t,u)\right|^\sigma &\le c\left(\frac{1}{2}W_u(t,u)u - W(t,u)\right)|u|^\sigma \\
&\le \frac{c}{2}W_u(t,u)u|u|^\sigma \\
&\le \frac{c}{2}\left|W_u(t,u)\right||u|^{\sigma+1},
\end{aligned}$$

and hence

$$\left|W_u(t,u)\right| \le \left(\frac{c}{2}\right)^{\frac{1}{\sigma-1}}|u|^{\frac{\sigma+1}{\sigma-1}}, \quad \text{if } |u| \ge r.$$

Let $p = 2\sigma/(\sigma-1) > 2$. Then we have

$$\left|W_u(t,u)\right| \le \left(\frac{c}{2}\right)^{\frac{1}{\sigma-1}}|u|^{p-1}, \quad \text{if } |u| \ge r. \tag{5}$$

Remark that (W_2) and (5) imply that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\left|W_u(t,u)\right| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}, \tag{6}$$

and

$$\left|W(t,u)\right| \le \varepsilon|u|^2 + C_\varepsilon|u|^p, \tag{7}$$

for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$.

Lemma 6 *Let (W_1)-(W_2) be satisfied, and assume further that (W_4) holds. Then there exists $\rho > 0$ such that $\kappa := \inf \Phi(S_\rho^+) > 0$, where $S_\rho^+ = \partial B_\rho \cap E^+$.*

Proof By (7) we have

$$\Psi(u) \le \varepsilon|u|_2^2 + C_\varepsilon|u|_p^p \le c\left(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^p\right)$$

for all $u \in E$, the lemma follows from the form of Φ (see (3)). □

Denote

$$\mathcal{H} := \mathbb{R}e_{\ell+1}, \qquad E_{\mathcal{H}} = E^- \oplus E^0 \oplus \mathcal{H}.$$

Then $E_{\mathcal{H}}$ is a finite subspace.

Lemma 7 *Under the assumptions of Theorem 2, there exists $R_{E_{\mathcal{H}}} > 0$ such that $\Phi(u) \le 0$ for all $u \in E_{\mathcal{H}}$ with $\|u\| \ge R_{E_{\mathcal{H}}}$.*

Proof It suffices to show that $\Phi(u) \to -\infty$ in $E_{\mathcal{H}}$ as $\|u\| \to \infty$. For any $u \in E_{\mathcal{H}}$, let $u = u_1^+ + u^- + u^0$, where $u_1^+ \in \mathcal{H}$, $u^- \in E^-$, $u^0 \in E^0$. Since $\dim \mathcal{H} = 1$, then

$$|u_1^+|_2^2 = \langle u_1^+, u\rangle_2 \le |u_1^+|_{\nu'} |u|_\nu \le c|u_1^+|_2 |u|_\nu,$$

where $\frac{1}{\nu'} + \frac{1}{\nu} = 1$. Thus $|u_1^+|_\nu \le c|u_1^+|_2 \le c|u|_\nu$, and together with (4), we obtain

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}} W(t, u(t))\,dt \\ &\le c|u_1^+|_\nu^2 - \frac{1}{2}\|u^-\|^2 - c|u_1^+ + u^- + u^0|_\nu^\nu \\ &\le c|u_1^+ + u^- + u^0|_\nu^2 - \frac{1}{2}\|u^-\|^2 - c|u_1^+ + u^- + u^0|_\nu^\nu,\end{aligned}$$

which shows that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$. □

As a special case we have

Lemma 8 *Assume that the assumptions of Theorem 2 are satisfied. Then, letting* $e \in \mathcal{H}$ *with* $\|e\| = 1$, *there is* $r_1 > \rho > 0$ *such that* $\sup \Phi(\partial M) \le \kappa$ *where* $M := \{u = u^- + u^0 + se : u^- + u^0 \in E^- \oplus E^0, s \ge 0, \|u\| \le r_1\}$ *and* κ *is given by Lemma 6.*

The $(C)_c$-sequence

In this section, we discuss the $(C)_c$-sequence of Φ.

Lemma 9 *Let* (L_3)-(L_4) *and* (W_1)-(W_4) *hold. Then any* $(C)_c$*-sequence is bounded.*

Proof Let $\{u_j\} \subset E$ be such that

$$\Phi(u_j) \to c \quad \text{and} \quad (1 + \|u_j\|)\Phi'(u_j) \to 0.$$

Then, for $C_0 > 0$,

$$C_0 \ge \Phi(u_j) - \frac{1}{2}\Phi'(u_j)u_j = \int_{\mathbb{R}} \tilde{W}(t, u_j)\,dt. \tag{8}$$

Suppose to the contrary that $\{u_j\}$ is unbounded. Setting $y_j = u_j/\|u_j\|$, then $\|y_j\| = 1$, $|y_j|_p \le c\|y_j\| = c$ for all $p \ge 2$. Passing to subsequence, $y_j \rightharpoonup y$ in E, and $y_j \to y$ in L^p for $p \ge 1$.

Note that

$$\begin{aligned} o(1) &= \Phi'(u_j)(u_j^+ - u_j^-) \\ &= \|u_j\|^2 - \int_{\mathbb{R}} W_u(t, u_j)(u_j^+ - u_j^-)\,dt \\ &= \|u_j\|^2 - \|u_j\|^2 \int_{\mathbb{R}} \frac{W_u(t, u_j)(y_j^+ - y_j^-)}{\|u_j\|}\,dt \\ &= \|u_j\|^2 \left(1 - \int_{\mathbb{R}} \frac{W_u(t, u_j)(y_j^+ - y_j^-)}{\|u_j\|}\,dt\right). \end{aligned} \tag{9}$$

From (10), we obtain

$$\int_{\mathbb{R}} \frac{W_u(t,u_j)(y_j^+ - y_j^-)}{\|u_j\|}\,dt \to 1. \tag{10}$$

Set for $s \geq 0$,

$$h(s) := \inf\bigl\{\tilde{W}(t,u) : t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \text{ with } |u| \geq s\bigr\}. \tag{11}$$

By (W_1) and (W_3), $h(s) > 0$ for all $s > 0$, and $h(s) \to \infty$ as $s \to \infty$.

For $0 \leq l < m$, let

$$C_l^m = \inf\left\{\frac{\tilde{W}(t,u)}{|u|^2} : t \in \mathbb{R} \text{ with } l \leq \bigl|u(t)\bigr| < m\right\},$$

and

$$\Omega_j(l,m) = \bigl\{t \in \mathbb{R} : l \leq \bigl|u_j(t)\bigr| < m\bigr\}. \tag{12}$$

Then by (W_3) one has $C_l^m > 0$ and

$$\tilde{W}(t,u_j) \geq C_l^m |u_j|^2 \quad \text{for all } t \in \Omega_j(l,m).$$

It follows from (8) and (12) that

$$\begin{aligned} C_0 &\geq \int_{\Omega_j(0,l)} \tilde{W}(t,u_j)\,dt + \int_{\Omega_j(l,m)} \tilde{W}(t,u_j)\,dt + \int_{\Omega_j(m,\infty)} \tilde{W}(t,u_j)\,dt \\ &\geq \int_{\Omega_j(0,l)} \tilde{W}(t,u_j)\,dt + C_l^m \int_{\Omega_j(l,m)} |u_j|^2\,dt + h(m)\bigl|\Omega_j(m,\infty)\bigr|. \end{aligned} \tag{13}$$

Using (13) we obtain

$$\bigl|\Omega_j(m,\infty)\bigr| \leq \frac{C_0}{h(m)} \to 0, \tag{14}$$

as $m \to \infty$ uniformly in j, and for any fixed $0 < l < m$,

$$\int_{\Omega_j(l,m)} |y_j|^2\,dt = \frac{1}{\|u_j\|^2} \int_{\Omega_j(l,m)} |u_j|^2\,dt \leq \frac{C_0}{C_l^m \|u_j\|^2} \to 0, \tag{15}$$

as $j \to \infty$. It follows from (14) that, for any $s \in [2,+\infty)$,

$$\int_{\Omega_j(m,\infty)} |y_j|^s\,dt \leq \left(\int_{\Omega_j(m,\infty)} |y_j|^{2s}\,dt\right)^{1/2} \cdot \bigl|\Omega_j(m,\infty)\bigr|^{1/2} \leq c\bigl|\Omega_j(m,\infty)\bigr|^{1/2} \to 0, \tag{16}$$

as $m \to \infty$ uniformly in j.

Let $0 < \epsilon < \frac{1}{3}$. By (W_2) there is $l_\epsilon > 0$ such that

$$\bigl|W_u(t,u)\bigr| < \frac{\epsilon}{c}|u|$$

for all $|u| \leq l_\epsilon$. Consequently,

$$\int_{\Omega_j(0,l_\epsilon)} \frac{W_u(t,u_j)(y_j^+ - y_j^-)|y_j|}{|u_j|}\,dt \leq \int_{\Omega_j(0,l_\epsilon)} \frac{\epsilon}{c}\big|y_j^+ - y_j^-\big||y_j|\,dt$$
$$\leq \frac{\epsilon}{c}|y_j|_2^2 < \epsilon \tag{17}$$

for all j.

Set $\sigma' := p/2$. By (W_4), (16) and Hölder inequality, we can take $m_\epsilon \geq r$ large enough such that

$$\begin{aligned}
&\int_{\Omega_j(m_\epsilon,\infty)} \frac{W_u(t,u_j)(y_j^+ - y_j^-)|y_j|}{|u_j|}\,dt \\
&\leq \left(\int_{\Omega_j(m_\epsilon,\infty)} \frac{|W_u(t,u_j)|^\sigma}{|u_j|^\sigma}\,dt\right)^{1/\sigma} \left(\int_{\Omega_j(m_\epsilon,\infty)} \big(\big|y_j^+ - y_j^-\big||y_j|\big)^{\sigma'}\,dt\right)^{1/\sigma'} \\
&\leq \left(\int_{\Omega_j(m_\epsilon,\infty)} c_1\tilde{W}(t,u_j)\,dt\right)^{1/\sigma} \left(\int_{\mathbb{R}^N} \big(\big|y_j^+ - y_j^-\big|\big)^p\,dt\right)^{1/p} \left(\int_{\Omega_j(m_\epsilon,\infty)} |y_j|^p\,dt\right)^{1/p} \\
&\leq \epsilon
\end{aligned} \tag{18}$$

for all j. Note that there is $C = C(\epsilon) > 0$ independent of j such that $|W_u(t,u_j)| \leq C|u_j|$ for $t \in \Omega_j(l_\epsilon, m_\epsilon)$. By (15) there is j_0 such that

$$\begin{aligned}
\int_{\Omega_j(l_\epsilon,m_\epsilon)} \frac{W_u(t,u_j)(y_j^+ - y_j^-)|y_j|}{|u_j|}\,dt &\leq C\int_{\Omega_j(l_\epsilon,m_\epsilon)} \big|y_j^+ - y_j^-\big||y_j|\,dt \\
&\leq C|y_j|_2\left(\int_{\Omega_j(l_\epsilon,m_\epsilon)} |y_j|^2\,dt\right)^{1/2} \\
&\leq \epsilon
\end{aligned} \tag{19}$$

for all $j \geq j_0$. By (17)-(19), one has

$$\limsup_{j\to\infty} \int_{\mathbb{R}} \frac{W_u(t,u_j)(y_j^+ - y_j^-)}{\|u_j\|}\,dt \leq 3\epsilon < 1, \tag{20}$$

which contradicts with (10). The proof is complete. □

Lemma 10 *Under the assumptions of Theorem 2, Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous. Moreover, Ψ' is compact.*

Proof We follow the idea of [33]. Clearly, by assumptions, $\Psi(u) \geq 0$. Let $u_j \rightharpoonup u$ in E. By Lemma 10, $u_j \to u$ in $L^p(\mathbb{R})$ for $p \geq 2$, and $u_j(t) \to u(t)$ a.e. $t \in \mathbb{R}$. Hence $W(t,u_j) \to W(t,u)$ for a.e. $t \in \mathbb{R}$. Thus, it follows from Fatou's lemma that

$$\Psi(u) = \int_{\mathbb{R}} W(t,u)\,dt = \int_{\mathbb{R}} \lim_{j\to\infty} W(t,u_j)\,dt \leq \liminf_{j\to\infty} \int_{\mathbb{R}} W(t,u_j)\,dt = \liminf_{j\to\infty} \Psi(u_j),$$

which shows that the function Ψ is weakly sequentially lower semi-continuous.

Now we show that Ψ' is compact. It is clear that, for any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\Psi'(u_j)\varphi = \int_{\mathbb{R}} W_u(t,u_j)\varphi\,dt \to \int_{\mathbb{R}} W_u(t,u)\varphi\,dt = \Psi'(u)\varphi. \tag{21}$$

Since $C_0^\infty(\mathbb{R})$ is dense in E, for any $v \in E$, we take $\varphi_n \in C_0^\infty(\mathbb{R})$ such that

$$\|\varphi_n - v\| \to 0 \quad \text{as } j \to \infty.$$

By (6), one has

$$\begin{aligned}
\left|\Psi'(u_j)v - \Psi'(u)v\right| &\le \left|\left(\Psi'(u_j) - \Psi'(u)\right)\varphi_n\right| + \left|\left(\Psi'(u_j) - \Psi'(u)\right)(v - \varphi_n)\right| \\
&\le \left|\left(\Psi'(u_j) - \Psi'(u)\right)\varphi_n\right| \\
&\quad + c\int_{\mathbb{R}} \left(|u| + |u_j| + |u|^{p-1} + |u_j|^{p-1}\right)|v - \varphi_n| \\
&\le \left|\left(\Psi'(u_j) - \Psi'(u)\right)\varphi_n\right| + c\|v - \varphi_n\|.
\end{aligned}$$

For any $\epsilon > 0$, fix n so that $\|v - \varphi_n\| < \epsilon/2c$. By (21) there exists j_0 such that

$$\left|\left(\Psi'(u_j) - \Psi'(u)\right)\varphi_n\right| < \epsilon/2 \quad \text{for all } j \ge j_0.$$

Then $|(\Psi'(u_j) - \Psi'(u))\varphi_n| < \epsilon$ for all $j \ge j_0$, which proves the weakly sequentially continuity. Therefore, Ψ' is compact by the weakly continuity of Ψ' since E is a Hilbert space. □

Lemma 10 implies that Φ' is weakly sequentially continuous, *i.e.*, if $u_j \rightharpoonup u$ in E, then $\Phi'(u_j) \to \Phi'(u)$. Let $\{u_j\}$ be an arbitrary $(C)_c$-sequence, by Lemma 9, it is bounded, up to a subsequence, we may assume $u_j \rightharpoonup u$ in E. Plainly, u is a critical point of Φ.

Lemma 11 *Under the assumptions of Lemma 9, Φ satisfies $(C)_c$-condition.*

Proof Let $\{u_j\}$ be any $(C)_c$-sequence. By Lemmas 4, 9, and 10, one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \left(W_u(t,u_j) - W_u(t,u)\right)\left(u_j^+ - u^+\right)dt\right| \\
&\quad \le \left(\int_{\mathbb{R}} \left|W_u(t,u_j) - W_u(t,u)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left|u_j^+ - u^+\right|^2 dt\right)^{\frac{1}{2}} \\
&\quad \le c\left(\int_{\mathbb{R}} \left|u_j^+ - u^+\right|^2 dt\right)^{\frac{1}{2}} \to 0
\end{aligned}$$

and

$$\begin{aligned}
o(1) &= \left(\Phi'(u_j) - \Phi'(u), u_j^+ - u^+\right) \\
&= \left\|u_j^+ - u^+\right\|^2 + \int_{\mathbb{R}} \left(W_u(t,u_j) - W_u(t,u)\right)\left(u_j^+ - u^+\right)dt \\
&= \left\|u_j^+ - u^+\right\|^2 + o(1).
\end{aligned}$$

So $u_j^+ \to u^+$ as $j \to \infty$. Since $\dim(E^- \oplus E^0) < \infty$, we have $u_j^- + u_j^0 \to u^- + u^0$, and therefore $u_j \to u$ as $j \to \infty$ in E. □

Proof of the theorem

Proof of Theorem 2 Lemma 8 shows that Φ possesses the linking structure of Theorem 5, and Lemma 11 implies that Φ satisfies the $(C)_c$-condition. Therefore, by Theorem 5 Φ has at least one critical point u. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, JS, JN and MO contributed to each part of this study equally and read and approved the final version of the manuscript.

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References

1. Coti-Zelati, V, Rabinowitz, PH: Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials. J. Am. Math. Soc. **4**, 693-727 (1991)
2. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Applied Mathematical Sciences, vol. 74. Springer, New York (1989)
3. Rabinowitz, PH: Homoclinic orbits for a class of Hamiltonian systems. Proc. R. Soc. Edinb. A **114**, 33-38 (1990)
4. Ambrosetti, A, Coti-Zelati, V: Multiple homoclinic orbits for a class of conservative systems. Rend. Semin. Mat. Univ. Padova **89**, 177-194 (1993)
5. Chen, G, Ma, S: Periodic solutions for Hamiltonian systems without Ambrosetti-Rabinowitz condition and spectrum. J. Math. Anal. Appl. **379**, 842-851 (2011)
6. Ding, J, Xu, J, Zhang, F: Existence of homoclinic orbits for Hamiltonian systems with superquadratic potentials. Abstr. Appl. Anal. **2009**, Art. ID 128624 (2009)
7. Ding, Y: Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems. Nonlinear Anal. **25**, 1095-1113 (1995)
8. Izydorek, M, Janczewska, J: Homoclinic solutions for a class of second order Hamiltonian systems. J. Differ. Equ. **219**, 375-389 (2005)
9. Kim, Y: Existence of periodic solutions for planar Hamiltonian systems at resonance. J. Korean Math. Soc. **48**, 1143-1152 (2011)
10. Omana, W, Willem, M: Homoclinic orbits for a class of Hamiltonian systems. Differ. Integral Equ. **5**, 1115-1120 (1992)
11. Paturel, E: Multiple homoclinic orbits for a class of Hamiltonian sytems. Calc. Var. Partial Differ. Equ. **12**, 117-143 (2001)
12. Rabinowitz, PH: Variational methods for Hamiltonian systems. In: Handbook of Dynamical Systems, vol. 1, Part 1, Chapter 14, pp. 1091-1127. North-Holland, Amsterdam (2002)
13. Rabinowitz, PH, Tanaka, K: Some results on connecting orbits for a class of Hamiltonian systems. Math. Z. **206**, 473-499 (1991)
14. Séré, E: Existence of infinitely many homoclinic orbits in Hamiltonian systems. Math. Z. **209**, 133-160 (1992)
15. Sun, J, Chen, H, Nieto, JJ: Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems. J. Math. Anal. Appl. **373**, 20-29 (2011)
16. Sun, J, Chen, H, Nieto, JJ: Homoclinic orbits for a class of first-order nonperiodic asymptotically quadratic Hamiltonian systems with spectrum point zero. J. Math. Anal. Appl. **378**, 117-127 (2011)
17. Sun, J, Chen, H, Nieto, JJ: On ground state solutions for some non-autonomous Schrödinger systems. J. Differ. Equ. **252**, 3365-3380 (2012)
18. Tang, X, Xiao, L: Homoclinic solutions for non-autonomous second-order Hamiltonian systems with a coercive potential. J. Math. Anal. Appl. **351**, 586-594 (2009)
19. Tang, X, Lin, X: Homoclinic solutions for a class of second order Hamiltonian systems. J. Math. Anal. Appl. **354**, 539-549 (2009)
20. Wan, L, Tang, C: Existence and multiplicity of homoclinic orbits for second order Hamiltonian systems without (AR) condition. Discrete Contin. Dyn. Syst., Ser. B **15**, 255-271 (2011)

21. Zhang, P, Tang, C: Infinitely many periodic solutions for nonautonomous sublinear second-order Hamiltonian systems. Abstr. Appl. Anal. **2010**, Art. ID 620438 (2010)
22. Zhang, Q, Liu, C: Infinitely many homoclinic solutions for second order Hamiltonian systems. Nonlinear Anal. **72**, 894-903 (2010)
23. Nieto, JJ: Variational formulation of a damped Dirichlet impulsive problem. Appl. Math. Lett. **23**, 940-942 (2010)
24. Sun, J, Chen, H, Nieto, JJ: Infinitely many solutions for second-order Hamiltonian system with impulsive effects. Math. Comput. Model. **54**, 544-555 (2011)
25. Sun, J, Chen, H, Nieto, JJ, Otero-Novoa, M: The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects. Nonlinear Anal. **72**, 4575-4586 (2010)
26. Xiao, J, Nieto, JJ: Variational approach to some damped Dirichlet nonlinear impulsive differential equations. J. Franklin Inst. **348**, 369-377 (2011)
27. Wu, X Zhang, W: Existence and multiplicity of homoclinic solutions for a class of damped vibration problems. Nonlinear Anal. **74**, 4392-4398 (2011)
28. Zhang, Z, Yuan, R: Homoclinic solutions for some second-order non-autonomous systems. Nonlinear Anal. **71**, 5790-5798 (2009)
29. Zhu, W: Existence of homoclinic solutions for second order systems. Nonlinear Anal. **75**, 2455-2463 (2012)
30. Willem, M, Zou, W: On a Schrödinger equation with periodic potential and spectrum point zero. Indiana Univ. Math. J. **52**, 109-132 (2003)
31. Wang, J, Xu, J, Zhang, F: Homoclinic orbits for a class of Hamiltonian systems with superquadratic or asymptotically quadratic potentials. Commun. Pure Appl. Anal. **10**, 269-286 (2011)
32. Ding, Y, Szulkin, A: Bound states for semilinear Schrödinger equations with sign changing potential. Calc. Var. Partial Differ. Equ. **29**, 397-419 (2007)
33. Ding, Y: Multiple homoclinics in a Hamiltonian system with asymptotically or super linear terms. Commun. Contemp. Math. **8**, 453-480 (2006)

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