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Some results for the *q*-Bernoulli, *q*-Euler numbers and polynomials

Daeyeoul Kim[1] and Min-Soo Kim[2*]

* Correspondence: minsookim@kaist.ac.kr
[2]Department of Mathematics, KAIST, 373-1 Guseong-dong, Yuseong-gu, Daejeon 305-701, South Korea
Full list of author information is available at the end of the article

Abstract

The q-analogues of many well known formulas are derived by using several results of q-Bernoulli, q-Euler numbers and polynomials. The q-analogues of ζ-type functions are given by using generating functions of q-Bernoulli, q-Euler numbers and polynomials. Finally, their values at non-positive integers are also been computed.
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1. Introduction

Carlitz [1,2] introduced q-analogues of the Bernoulli numbers and polynomials. From that time on these and other related subjects have been studied by various authors (see, e.g., [3-10]). Many recent studies on q-analogue of the Bernoulli, Euler numbers, and polynomials can be found in Choi et al. [11], Kamano [3], Kim [5,6,12], Luo [7], Satoh [9], Simsek [13,14] and Tsumura [10].

For a fixed prime p, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $|\cdot|_p$ be the p-adic norm on $\mathbb{Q}$ with $|p|_p = p^{-1}$. For convenience, $|\cdot|_p$ will also be used to denote the extended valuation on $\mathbb{C}_p$.

The Bernoulli polynomials, denoted by $B_n(x)$, are defined as

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \quad n \geq 0, \tag{1.1}$$

where B_k are the Bernoulli numbers given by the coefficients in the power series

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{1.2}$$

From the above definition, we see B_k's are all rational numbers. Since $\frac{t}{e^t-1} - 1 + \frac{t}{2}$ is an even function (i.e., invariant under $x \mapsto -x$), we see that $B_k = 0$ for any odd integer k not smaller than 3. It is well known that the Bernoulli numbers can also be expressed as follows

$$B_k = \lim_{N\to\infty} \frac{1}{p^N} \sum_{a=0}^{p^N-1} a^k \tag{1.3}$$

(see [15,16]). Notice that, from the definition $B_k \in \mathbb{Q}$, and these integrals are independent of the prime p which used to compute them. The examples of (1.3) are:

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{p^N} \sum_{a=0}^{p^N-1} a &= \lim_{N\to\infty} \frac{1}{p^N} \frac{p^N(p^N-1)}{2} = -\frac{1}{2} = B_1, \\ \lim_{N\to\infty} \frac{1}{p^N} \sum_{a=0}^{p^N-1} a^2 &= \lim_{N\to\infty} \frac{1}{p^N} \frac{p^N(p^N-1)(2p^N-1)}{6} = \frac{1}{6} = B_2. \end{aligned} \tag{1.4}$$

Euler numbers E_k, $k \geq 0$ are integers given by (cf. [17-19])

$$E_0 = 1, \quad E_k = -\sum_{\substack{i=0 \\ 2|k-i}}^{k-1} \binom{k}{i} E_i \text{ for } k = 1, 2, \ldots. \tag{1.5}$$

The Euler polynomial $E_k(x)$ is defined by (see [[20], p. 25]):

$$E_k(x) = \sum_{i=0}^{k} \binom{k}{i} \frac{E_i}{2^i} \left(x - \frac{1}{2}\right)^{k-i}, \tag{1.6}$$

which holds for all nonnegative integers k and all real x, and which was obtained by Raabe [21] in 1851. Setting $x = 1/2$ and normalizing by 2^k gives the Euler numbers

$$E_k = 2^k E_k\left(\frac{1}{2}\right), \tag{1.7}$$

where $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$,.... Therefore, $E_k \neq E_k(0)$, in fact ([[19], p. 374 (2.1)])

$$E_k(0) = \frac{2}{k+1}(1 - 2^{k+1})B_{k+1}, \tag{1.8}$$

where B_k are Bernoulli numbers. The Euler numbers and polynomials (so-named by Scherk in 1825) appear in Euler's famous book, Institutiones Calculi Differentialis (1755, pp. 487-491 and p. 522).

In this article, we derive q-analogues of many well known formulas by using several results of q-Bernoulli, q-Euler numbers, and polynomials. By using generating functions of q-Bernoulli, q-Euler numbers, and polynomials, we also present the q-analogues of ζ-type functions. Finally, we compute their values at non-positive integers.

This article is organized as follows.

In Section 2, we recall definitions and some properties for the q-Bernoulli, Euler numbers, and polynomials related to the bosonic and the fermionic p-adic integral on $\mathbb{Z}_p$.

In Section 3, we obtain the generating functions of the q-Bernoulli, q-Euler numbers, and polynomials. We shall provide some basic formulas for the q-Bernoulli and q-Euler polynomials which will be used to prove the main results of this article.

In Section 4, we construct the q-analogue of the Riemann's ζ-functions, the Hurwitz ζ-functions, and the Dirichlet's L-functions. We prove that the value of their functions at non-positive integers can be represented by the q-Bernoulli, q-Euler numbers, and polynomials.

2. q-Bernoulli, q-Euler numbers and polynomials related to the Bosonic and the Fermionic p-adic integral on $\mathbb{Z}_p$

In this section, we provide some basic formulas for p-adic q-Bernoulli, p-adic q-Euler numbers and polynomials which will be used to prove the main results of this article.

Let $UD(\mathbb{Z}_p, \mathbb{C}_p)$ denote the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$. The p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{a=0}^{p^N-1} f(a)q^a = \int_{\mathbb{Z}_p} f(z)d\mu_q(z), \tag{2.1}$$

where $[x]_q = (1 - q^x)/(1 - q)$, and the limit taken in the p-adic sense. Note that

$$\lim_{q\to 1} [x]_q = x \tag{2.2}$$

for $x \in \mathbb{Z}_p$, where q tends to 1 in the region $0 <|q - 1|_p <1$ (cf. [22,5,12]). The bosonic p-adic integral on $\mathbb{Z}_p$ is considered as the limit $q \to 1$, i.e.,

$$I_1(f) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{a=0}^{p^N-1} f(a) = \int_{\mathbb{Z}_p} f(z)d\mu_1(z). \tag{2.3}$$

From (2.1), we have the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f) = \lim_{q\to -1} I_q(f) = \lim_{N\to\infty} \sum_{a=0}^{p^N-1} f(a)(-1)^a = \int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z). \tag{2.4}$$

In particular, setting $f(z) = [z]_q^k$ in (2.3) and $f(z) = [z + \frac{1}{2}]_q^k$ in (2.4), respectively, we get the following formulas for the p-adic q-Bernoulli and p-adic q-Euler numbers, respectively, if $q \in \mathbb{C}_p$ with $0 <|q - 1|_p <1$ as follows

$$B_k(q) = \int_{\mathbb{Z}_p} [z]_q^k d\mu_1(z) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{a=0}^{p^N-1} [a]_q^k, \tag{2.5}$$

$$E_k(q) = 2^k \int_{\mathbb{Z}_p} \left[z + \frac{1}{2}\right]_q^k d\mu_{-1}(z) = 2^k \lim_{N\to\infty} \sum_{a=0}^{p^N-1} \left[a + \frac{1}{2}\right]_q^k (-1)^a. \tag{2.6}$$

Remark 2.1. The q-Bernoulli numbers (2.5) are first defined by Kamano [3]. In (2.5) and (2.6), take $q \to 1$. Form (2.2), it is easy to that (see [[17], Theorem 2.5])

$$B_k(q) \to B_k = \int_{\mathbb{Z}_p} z^k d\mu_1(z), \quad E_k(q) \to E_k = \int_{\mathbb{Z}_p} (2z + 1)^k d\mu_{-1}(z).$$

For $|q - 1|_p <1$ and $z \in \mathbb{Z}_p$, we have

$$q^{iz} = \sum_{n=0}^{\infty} (q^i - 1)^n \binom{z}{n} \quad \text{and} \quad |q^i - 1|_p \le |q - 1|_p < 1, \tag{2.7}$$

where $i \in \mathbb{Z}$. We easily see that if $|q - 1|_p < 1$, then $q^x = 1$ for $x \neq 0$ if and only if q is a root of unity of order p^N and $x \in p^N\mathbb{Z}_p$ (see [16]).

By (2.3) and (2.7), we obtain

$$\begin{aligned} I_1(q^{iz}) &= \frac{1}{q^i - 1} \lim_{N\to\infty} \frac{(q^i)^{p^N} - 1}{p^N} \\ &= \frac{1}{q^i - 1} \lim_{N\to\infty} \frac{1}{p^N} \left\{ \sum_{m=0}^{\infty} \binom{p^N}{m} (q^i - 1)^m - 1 \right\} \\ &= \frac{1}{q^i - 1} \lim_{N\to\infty} \frac{1}{p^N} \sum_{m=1}^{\infty} \binom{p^N}{m} (q^i - 1)^m \\ &= \frac{1}{q^i - 1} \lim_{N\to\infty} \sum_{m=1}^{\infty} \frac{1}{m} \binom{p^N - 1}{m - 1} (q^i - 1)^m \\ &= \frac{1}{q^i - 1} \sum_{m=1}^{\infty} \frac{1}{m} \binom{-1}{m - 1} (q^i - 1)^m \\ &= \frac{1}{q^i - 1} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(q^i - 1)^m}{m} \\ &= \frac{i \log q}{q^i - 1} \end{aligned} \tag{2.8}$$

since the series log $\log(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} x^m/m$ converges at $|x|_p < 1$. Similarly, by (2.4), we obtain (see [[4], p. 4, (2.10)])

$$I_{-1}(q^{iz}) = \lim_{N\to\infty} \sum_{a=0}^{p^N - 1} (q^i)^a (-1)^a = \frac{2}{q^i + 1}. \tag{2.9}$$

From (2.5), (2.6), (2.8) and (2.9), we obtain the following explicit formulas of $B_k(q)$ and $E_k(q)$:

$$B_k(q) = \frac{\log q}{(1-q)^k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{i}{q^i - 1}, \tag{2.10}$$

$$E_k(q) = \frac{2^{k+1}}{(1-q)^k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i q^{\frac{1}{2}i} \frac{1}{q^i + 1}, \tag{2.11}$$

where $k \geq 0$ and log is the p-adic logarithm. Note that in (2.10), the term with $i = 0$ is understood to be $1/\log q$ (the limiting value of the summand in the limit $i \to 0$).

We now move on to the p-adic q-Bernoulli and p-adic q-Euler polynomials. The p-adic q-Bernoulli and p-adic q-Euler polynomials in q^x are defined by means of the bosonic and the fermionic p-adic integral on $\mathbb{Z}_p$:

$$B_k(x, q) = \int_{\mathbb{Z}_p} [x + z]_q^k d\mu_1(z) \text{ and } E_k(x, q) = \int_{\mathbb{Z}_p} [x + z]_q^k d\mu_{-1}(z), \tag{2.12}$$

where $q \in \mathbb{C}_p$ with $0 < |q - 1|_p < 1$ and $x \in \mathbb{Z}_p$, respectively. We will rewrite the above equations in a slightly different way. By (2.5), (2.6), and (2.12), after some elementary calculations, we get

$$B_k(x,q) = \sum_{i=0}^{k} \binom{k}{i} [x]_q^{k-i} q^{ix} B_i(q) = (q^x B(q) + [x]_q)^k \tag{2.13}$$

and

$$E_k(x,q) = \sum_{i=0}^{k} \binom{k}{i} \frac{E_i(q)}{2^i} \left[x - \frac{1}{2}\right]_q^{k-i} q^{i(x-\frac{1}{2})} = \left(\frac{q^{x-\frac{1}{2}}}{2} E(q) + \left[x - \frac{1}{2}\right]_q \right)^k, \tag{2.14}$$

where the symbol $B_k(q)$ and $E_k(q)$ are interpreted to mean that $(B(q))^k$ and $(E(q))^k$ must be replaced by $B_k(q)$ and $E_k(q)$ when we expanded the one on the right, respectively, since $[x+y]_q^k = ([x]_q + q^x[y]_q)^k$ and

$$\begin{aligned} [x+z]_q^k &= \left[\frac{1}{2}\right]_q^k \left([2x-1]_{q^{\frac{1}{2}}} + q^{x-\frac{1}{2}} \left[\frac{1}{2}\right]_q^{-1} \left[z + \frac{1}{2}\right]_q \right)^k \\ &= \left[\frac{1}{2}\right]_q^k \sum_{i=0}^{k} \binom{k}{i} [2x-1]_q^{k-i} q^{(x-\frac{1}{2})i} \left[\frac{1}{2}\right]_q^{-i} \left[z + \frac{1}{2}\right]_q^i \end{aligned} \tag{2.15}$$

(cf. [4,5]). The above formulas can be found in [7] which are the q-analogues of the corresponding classical formulas in [[17], (1.2)] and [23], etc. Obviously, put $x = \frac{1}{2}$ in (2.14). Then

$$E_k(q) = 2^k E_k\left(\frac{1}{2}, q\right) \neq E_k(0,q) \quad \text{and} \quad \lim_{q \to 1} E_k(q) = E_k, \tag{2.16}$$

where E_k are Euler numbers (see (1.5) above).

Lemma 2.2 (Addition theorem).

$$B_k(x+y,q) = \sum_{i=0}^{k} \binom{k}{i} q^{iy} B_i(x,q) [y]_q^{k-i} \quad (k \geq 0),$$

$$E_k(x+y,q) = \sum_{i=0}^{k} \binom{k}{i} q^{iy} E_i(x,q) [y]_q^{k-i} \quad (k \geq 0).$$

Proof. Applying the relationship $[x+y-\frac{1}{2}]_q = [y]_q + q^y[x-\frac{1}{2}]_q$ to (2.14) for x α $x + y$, we have

$$\begin{aligned} E_k(x+y,q) &= \left(\frac{q^{x+y-\frac{1}{2}}}{2} E(q) + \left[x + y - \frac{1}{2}\right]_q \right)^k \\ &= \left(q^y \left(\frac{q^{x-\frac{1}{2}}}{2} E(q) + \left[x - \frac{1}{2}\right]_q \right) + [y]_q \right)^k \\ &= \sum_{i=0}^{k} \binom{k}{i} q^{iy} \left(\frac{q^{x-\frac{1}{2}}}{2} E(q) + \left[x - \frac{1}{2}\right]_q \right)^i [y]_q^{k-i} \\ &= \sum_{i=0}^{k} \binom{k}{i} q^{iy} E_i(x,q) [y]_q^{k-i}. \end{aligned}$$

Similarly, the first identity follows.□

Remark 2.3. From (2.12), we obtain the not completely trivial identities

$$\lim_{q\to 1} B_k(x+y,q) = \sum_{i=0}^{k} \binom{k}{i} B_i(x) y^{k-i} = (B(x)+y)^k,$$
$$\lim_{q\to 1} E_k(x+y,q) = \sum_{i=0}^{k} \binom{k}{i} E_i(x) y^{k-i} = (E(x)+y)^k,$$

where $q \in \mathbb{C}_p$ tends to 1 in $|q - 1|_p < 1$. Here $B_i(x)$ and $E_i(x)$ denote the classical Bernoulli and Euler polynomials, see [17,15] and see also the references cited in each of these earlier works.

Lemma 2.4. *Let n be any positive integer. Then*

$$\sum_{i=0}^{k} \binom{k}{i} q^i [n]_q^i B_i(x,q^n) = [n]_q^k B_k\left(x+\frac{1}{n},q^n\right),$$
$$\sum_{i=0}^{k} \binom{k}{i} q^i [n]_q^i E_i(x,q^n) = [n]_q^k E_k\left(x+\frac{1}{n},q^n\right).$$

Proof. Use Lemma 2.2, the proof can be obtained by the similar way to [[7], Lemma 2.3]. □

We note here that similar expressions to those of Lemma 2.4 are given by Luo [[7], Lemma 2.3]. Obviously, Lemma 2.4 are the q-analogues of

$$\sum_{i=0}^{k} \binom{k}{i} n^i B_i(x) = n^k B_k\left(x+\frac{1}{n}\right), \sum_{i=0}^{k} \binom{k}{i} n^i E_i(x) = n^k E_k\left(x+\frac{1}{n}\right),$$

respectively.

We can now obtain the multiplication formulas by using p-adic integrals.

From (2.3), we see that

$$\begin{aligned} B_k(nx,q) &= \int_{\mathbb{Z}_p} [nx+z]_q^k d\mu_1(z) \\ &= \lim_{N\to\infty} \frac{1}{np^N} \sum_{a=0}^{np^N-1} [nx+a]_q^k \\ &= \frac{1}{n} \lim_{N\to\infty} \frac{1}{p^N} \sum_{i=0}^{n-1} \sum_{a=0}^{p^N-1} [nx+na+i]_q^k \\ &= \frac{[n]_q^k}{n} \sum_{i=0}^{n-1} \int_{\mathbb{Z}_p} \left[x+\frac{i}{n}+z\right]_{q^n}^k d\mu_1(z) \end{aligned} \tag{2.17}$$

is equivalent to

$$B_k(x,q) = \frac{[n]_q^k}{n} \sum_{i=0}^{n-1} B_k\left(\frac{x+i}{n},q^n\right). \tag{2.18}$$

If we put $x = 0$ in (2.18) and use (2.13), we find easily that

$$\begin{aligned} B_k(q) &= \frac{[n]_q^k}{n} \sum_{i=0}^{n-1} B_k\left(\frac{i}{n}, q^n\right) \\ &= \frac{[n]_q^k}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j} \left[\frac{i}{n}\right]_{q^n}^{k-j} q^{ij} B_j(q^n) \\ &= \frac{1}{n} \sum_{j=0}^{k} [n]_q^j \binom{k}{j} B_j(q^n) \sum_{i=0}^{n-1} q^{ij} [i]_q^{k-j}. \end{aligned} \tag{2.19}$$

Obviously, Equation (2.19) is the q-analogue of

$$B_k = \frac{1}{n(1-n^k)} \sum_{j=0}^{k-1} n^j \binom{k}{j} B_j \sum_{i=1}^{n-1} i^{k-j},$$

which is true for any positive integer k and any positive integer $n > 1$ (see [[24], (2)]). From (2.4), we see that

$$\begin{aligned} E_k(nx, q) &= \int_{\mathbb{Z}_p} [nx + z]_q^k d\mu_{-1}(z) \\ &= \lim_{N\to\infty} \sum_{i=0}^{n-1} \sum_{a=0}^{p^N-1} [nx + na + i]_q^k (-1)^{na+i} \\ &= [n]_q^k \sum_{i=0}^{n-1} (-1)^i \int_{\mathbb{Z}_p} \left[x + \frac{i}{n} + z\right]_{q^n}^k d\mu_{(-1)^n}(z). \end{aligned} \tag{2.20}$$

By (2.12) and (2.20), we find easily that

$$E_k(x, q) = [n]_q^k \sum_{i=0}^{n-1} (-1)^i E_k\left(\frac{x+i}{n}, q^n\right) \text{ if } n \text{ odd.} \tag{2.21}$$

From (2.18) and (2.21), we can obtain Proposition 2.5 below.

Proposition 2.5 (Multiplication formulas). *Let n be any positive integer. Then*

$$B_k(x, q) = \frac{[n]_q^k}{n} \sum_{i=0}^{n-1} B_k\left(\frac{x+i}{n}, q^n\right),$$

$$E_k(x, q) = [n]_q^k \sum_{i=0}^{n-1} (-1)^i E_k\left(\frac{x+i}{n}, q^n\right) \quad \text{if } n \textit{ odd}.$$

3. Construction generating functions of q-Bernoulli, q-Euler numbers, and polynomials

In the complex case, we shall explicitly determine the generating function $F_q(t)$ of q-Bernoulli numbers and the generating function $G_q(t)$ of q-Euler numbers:

$$F_q(t) = \sum_{k=0}^{\infty} B_k(q) \frac{t^k}{k!} = e^{B(q)t} \text{ and } G_q(t) = \sum_{k=0}^{\infty} E_k(q) \frac{t^k}{k!} = e^{E(q)t}, \tag{3.1}$$

where the symbol $B_k(q)$ and $E_k(q)$ are interpreted to mean that $(B(q))^k$ and $(E(q))^k$ must be replaced by $B_k(q)$ and $E_k(q)$ when we expanded the one on the right, respectively.

Lemma 3.1.

$$F_q(t) = e^{\frac{t}{1-q}} + \frac{t\log q}{1-q}\sum_{m=0}^{\infty} q^m e^{[m]_q t},$$

$$G_q(t) = 2\sum_{m=0}^{\infty}(-1)^m e^{2[m+\frac{1}{2}]_q t}.$$

Proof. Combining (2.10) and (3.1), $F_q(t)$ may be written as

$$\begin{aligned} F_q(t) &= \sum_{k=0}^{\infty}\frac{\log q}{(1-q)^k}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{i}{q^i-1}\frac{t^k}{k!} \\ &= 1+\log q\sum_{k=1}^{\infty}\frac{1}{(1-q)^k}\frac{t^k}{k!}\left(\frac{1}{\log q}+\sum_{i=1}^{k}\binom{k}{i}(-1)^i\frac{i}{q^i-1}\right). \end{aligned}$$

Here, the term with $i = 0$ is understood to be $1/\log q$ (the limiting value of the summand in the limit $i \to 0$). Specifically, by making use of the following well-known binomial identity

$$k\binom{k-1}{i-1} = i\binom{k}{i} \quad (k \geq i \geq 1).$$

Thus, we find that

$$\begin{aligned} F_q(t) &= 1+\log q\sum_{k=1}^{\infty}\frac{1}{(1-q)^k}\frac{t^k}{k!}\left(\frac{1}{\log q}+k\sum_{i=1}^{k}\binom{k-1}{i-1}(-1)^i\frac{1}{q^i-1}\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{(1-q)^k}\frac{t^k}{k!}+\log q\sum_{k=1}^{\infty}\frac{k}{(1-q)^k}\frac{t^k}{k!}\sum_{m=0}^{\infty}q^m\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i q^{mi} \\ &= e^{\frac{t}{1-q}}+\frac{\log q}{1-q}\sum_{k=1}^{\infty}\frac{k}{(1-q)^{k-1}}\frac{t^k}{k!}\sum_{m=0}^{\infty}q^m(1-q^m)^{k-1} \\ &= e^{\frac{t}{1-q}}+\frac{t\log q}{1-q}\sum_{m=0}^{\infty}q^m\sum_{k=0}^{\infty}\left(\frac{1-q^m}{1-q}\right)^k\frac{t^k}{k!}. \end{aligned}$$

Next, by (2.11) and (3.1), we obtain the result

$$\begin{aligned} G_q(t) &= \sum_{k=0}^{\infty}\frac{2^{k+1}}{(1-q)^k}\sum_{i=0}^{k}\binom{k}{i}(-1)^i q^{\frac{1}{2}i}\frac{1}{q^i+1}\frac{t^k}{k!} \\ &= 2\sum_{k=0}^{\infty}2^k\sum_{m=0}^{\infty}(-1)^m\left(\frac{1-q^{m+\frac{1}{2}}}{1-q}\right)^k\frac{t^k}{k!} \\ &= 2\sum_{m=0}^{\infty}(-1)^m\sum_{k=0}^{\infty}\left[m+\frac{1}{2}\right]_q^k\frac{(2t)^k}{k!} \\ &= 2\sum_{m=0}^{\infty}(-1)^m e^{2[m+\frac{1}{2}]_q t}. \end{aligned}$$

This completes the proof. □

Remark 3.2. The remarkable point is that the series on the right-hand side of Lemma 3.1 is uniformly convergent in the wider sense.

From (2.13)and (2.14), we define the q-Bernoulli and q-Euler polynomials by

$$F_q(t,x)=\sum_{k=0}^{\infty}B_k(x,q)\frac{t^k}{k!}=\sum_{k=0}^{\infty}\left(q^xB(q)+[x]_q\right)^k\frac{t^k}{k!}, \tag{3.2}$$

$$G_q(t,x)=\sum_{k=0}^{\infty}E_k(x,q)\frac{t^k}{k!}=\sum_{k=0}^{\infty}\left(q^{x-\frac{1}{2}}\frac{E(q)}{2}+\left[x-\frac{1}{2}\right]_q\right)^k\frac{t^k}{k!}. \tag{3.3}$$

Hence, we have

Lemma 3.3.

$$F_q(t,x)=e^{[x]_q t}F_q(q^xt)=e^{\frac{t}{1-q}}+\frac{t\log q}{1-q}\sum_{m=0}^{\infty}q^{m+x}e^{[m+x]_q t}.$$

Proof. From (3.1) and (3.2), we note that

$$\begin{aligned}F_q(t,x)&=\sum_{k=0}^{\infty}\left(q^xB(q)+[x]_q\right)^k\frac{t^k}{k!}\\&=e^{(q^xB(q)+[x]_q)t}\\&=e^{B(q)q^xt}e^{[x]_qt}\\&=e^{[x]_q t}F_q(q^xt).\end{aligned}$$

The second identity leads at once to Lemma 3.1. Hence, the lemma follows. □

Lemma 3.4.

$$G_q(t,x)=e^{[x-\frac{1}{2}]_q t}G_q\left(\frac{q^{x-\frac{1}{2}}}{2}t\right)=2\sum_{m=0}^{\infty}(-1)^m e^{[m+x]_q t}.$$

Proof. By similar method of Lemma 3.3, we prove this lemma by (3.1), (3.3), and Lemma 3.1. □

Corollary 3.5 (Difference equations).

$$\begin{aligned}&B_{k+1}(x+1,q)-B_{k+1}(x,q)=\frac{q^x\log q}{q-1}(k+1)[x]_q^k(k\geq 0),\\&E_k(x+1,q)+E_k(x,q)=2[x]_q^k\quad(k\geq 0).\end{aligned}$$

Proof. By applying (3.2) and Lemma 3.3, we obtain (3.4)

$$\begin{aligned}F_q(t,x)&=\sum_{k=0}^{\infty}B_k(x,q)\frac{t^k}{k!}\\&=1+\sum_{k=0}^{\infty}\left(\frac{1}{(1-q)^{k+1}}+(k+1)\frac{\log q}{1-q}\sum_{m=0}^{\infty}q^{m+x}[m+x]_q^k\right)\frac{t^{k+1}}{(k+1)!}.\end{aligned} \tag{3.4}$$

By comparing the coefficients of both sides of (3.4), we have $B_0(x, q) = 1$ and

$$B_k(x,q) = \frac{1}{(1-q)^k} + k\frac{\log q}{1-q}\sum_{m=0}^{\infty} q^{m+x}[m+x]_q^{k-1} \quad (k \geq 1). \tag{3.5}$$

Hence,

$$B_k(x+1,q) - B_k(x,q) = k\frac{q^x \log q}{q-1}[x]_q^{k-1} \quad (k \geq 1).$$

Similarly we prove the second part by (3.3) and Lemma 3.4. This proof is complete. □

From Lemma 2.2 and Corollary 3.5, we obtain for any integer $k \geq 0$,

$$[x]_q^k = \frac{1}{k+1}\frac{q-1}{q^x \log q}\left(\sum_{i=0}^{k+1}\binom{k+1}{i} q^i B_i(x,q) - B_{k+1}(x,q)\right),$$

$$[x]_q^k = \frac{1}{2}\left(\sum_{i=0}^{k}\binom{k}{i} q^i E_i(x,q) + E_k(x,q)\right)$$

which are the q-analogues of the following familiar expansions (see, e.g., [[7], p. 9]):

$$x^k = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i(x) \quad \text{and } x^k = \frac{1}{2}\left(\sum_{i=0}^{k}\binom{k}{i}E_i(x) + E_k(x)\right),$$

respectively.

Corollary 3.6 (Difference equations). *Let* $k \geq 0$ *and* $n \geq 1$. *Then*

$$\begin{aligned} B_{k+1}\left(x+\frac{1}{n}, q^n\right) - B_{k+1}\left(x+\frac{1-n}{n}, q^n\right) \\ = \frac{nq^{n(x-1)+1}\log q}{q-1}\frac{k+1}{[n]_q^{k+1}}(1+q[nx-n]_q)^k, \end{aligned}$$

$$E_k\left(x+\frac{1}{n}, q^n\right) + E_k\left(x+\frac{1-n}{n}, q^n\right) = \frac{2}{[n]_q^k}(1+q[nx-n]_q)^k.$$

Proof. Use Lemma 2.4 and Corollary 3.5, the proof can be obtained by the similar way to [[7], Lemma 2.4]. □

Letting $n = 1$, Corollary 3.6 reduces to Corollary 3.5. Clearly, the above difference formulas in Corollary 3.6 become the following difference formulas when $q \to 1$:

$$B_k\left(x+\frac{1}{n}\right) - B_k\left(x+\frac{1-n}{n}\right) = k\left(x+\frac{1-n}{n}\right)^{k-1}(k \geq 1, n \geq 1), \tag{3.6}$$

$$E_k\left(x+\frac{1}{n}\right) + E_k\left(x+\frac{1-n}{n}\right) = 2\left(x+\frac{1-n}{n}\right)^{k}(k \geq 0, n \geq 1), \tag{3.7}$$

respectively (see [[7], (2.22), (2.23)]). If we now let $n = 1$ in (3.6) and (3.7), we get the ordinary difference formulas

$$B_{k+1}(x+1) - B_{k+1}(x) = (k+1)x^{k-1} \text{ and } E_k(x+1) + E_k(x) = 2x^k$$

for $k \geq 0$.

In Corollary 3.5, let $x = 0$. We arrive at the following proposition.

Proposition 3.7.

$$B_0(q) = 1,\ (qB(q)+1)^k - B_k(q) = \begin{cases} \frac{\log_p q}{q-1} & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases}$$

$$E_0(q) = 1, \quad \left(q^{-\frac{1}{2}}\frac{E(q)}{2} + \left[-\frac{1}{2}\right]_q\right)^k + \left(q^{\frac{1}{2}}\frac{E(q)}{2} + \left[\frac{1}{2}\right]_q\right)^k = 0 \quad \text{if } k \geq 1.$$

Proof. The first identity follows from (2.13). To see the second identity, setting $x = 0$ and $x = 1$ in (2.14) we have

$$E_k(0,q) = \left(\frac{q^{-\frac{1}{2}}}{2}E(q) + \left[-\frac{1}{2}\right]_q\right)^k \text{ and } E_k(1,q) = \left(\frac{q^{\frac{1}{2}}}{2}E(q) + \left[\frac{1}{2}\right]_q\right)^k.$$

This proof is complete. □

Remark 3.8. (1). We note here that quite similar expressions to the first identity of Proposition 3.7 are given by Kamano [[3], Proposition 2.4], Rim et al. [[8], Theorem 2.7] and Tsumura [[10], (1)].

(2). Letting $q \to 1$ in Proposition 3.7, the first identity is the corresponding classical formulas in [[8], (1.2)]:

$$B_0 = 1, \quad (B+1)^k - B_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

and the second identity is the corresponding classical formulas in [[25], (1.1)]:

$$E_0 = 1, \quad (E+1)^k + (E-1)^k = 0 \quad \text{if } k \geq 1.$$

4. *q*-analogues of Riemann's ζ-functions, the Hurwitz ζ-functions and the Didichlet's *L*-functions

Now, by evaluating the kth derivative of both sides of Lemma 3.1 at $t = 0$, we obtain the following

$$B_k(q) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k F_q(t)\bigg|_{t=0} = \left(\frac{1}{1-q}\right)^k - \frac{k\log q}{q-1}\sum_{m=0}^{\infty} q^m [m]_q^{k-1}, \tag{4.1}$$

$$E_k(q) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k G_q(t)\bigg|_{t=0} = 2^{k+1}\sum_{m=0}^{\infty}(-1)^m\left[m+\frac{1}{2}\right]_q^k \tag{4.2}$$

for $k \geq 0$.

Definition 4.1 (q-analogues of the Riemann's ζ-functions). For $s \in$ ≤, define

$$\zeta_q(s) = \frac{1}{s-1}\frac{1}{\left(\frac{1}{1-q}\right)^{s-1}} + \frac{\log q}{q-1}\sum_{m=1}^{\infty}\frac{q^m}{[m]_q^s},$$

$$\zeta_{q,E}(s) = \frac{2}{2^s}\sum_{m=0}^{\infty}\frac{(-1)^m}{[m+\frac{1}{2}]_q^s}.$$

Note that $\zeta_q(s)$ is a meromorphic function on ≤ with only one simple pole at $s = 1$ and $\zeta_q,E(s)$ is a analytic function on ≤.

Also, we have

$$\lim_{q\to 1}\zeta_q(s) = \sum_{m=1}^{\infty}\frac{1}{m^s} = \zeta(s) \quad \text{and} \quad \lim_{q\to 1}\zeta_{q,E}(s) = 2\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)^s} = \zeta_E(s). \tag{4.3}$$

(In [[26], p. 1070], our $\zeta_E(s)$ is denote $\varphi(s)$.)

The values of $\zeta_q(s)$ and $\zeta_q,E(s)$ at non-positive integers are obtained by the following proposition.

Proposition 4.2. *For $k \geq 1$, we have*

$$\zeta_q(1-k) = -\frac{B_k(q)}{k} \quad \textit{and} \quad \zeta_{q,E}(1-k) = E_{k-1}(q).$$

Proof. It is clear by (4.1) and (4.2). □

We can investigate the generating functions $F_q(t, x)$ and $G_q(t, x)$ by using a method similar to the method used to treat the q-analogues of Riemann's ζ-functions in Definition 4.1.

Definition 4.3 (q-analogues of the Hurwitz ζ-functions). For $s \in$ ≤ and $0 < x \leq 1$, define

$$\zeta_q(s,\ x) = \frac{1}{s-1}\frac{1}{\left(\frac{1}{1-q}\right)^{s-1}} + \frac{\log q}{q-1}\sum_{m=0}^{\infty}\frac{q^{m+x}}{[m+x]_q^s},$$

$$\zeta_{q,E}(s,\ x) = 2\sum_{m=0}^{\infty}\frac{(-1)^m}{[m+x]_q^s}.$$

Note that $\zeta_q(s, x)$ is a meromorphic function on ≤ with only one simple pole at $s = 1$ and $\zeta_q,E(s, x)$ is a analytic function on ≤.

The values of $\zeta_q(s, x)$ and $\zeta_q,E(s, x)$ at non-positive integers are obtained by the following proposition.

Proposition 4.4. *For $k \geq 1$, we have*

$$\zeta_q(1-k,\ x) = -\frac{B_k(x,q)}{k} \quad \textit{and} \quad \zeta_{q,E}(1-k,\ x) = E_{k-1}(x,\ q).$$

Proof. From Lemma 3.3 and Definition 4.3, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k F_q(t,\ x)\Bigg|_{t=0} = -k\zeta_q(1-k,\ x)$$

for $k \geq 1$. We obtain the desired result by (3.2). Similarly the second form follows by Lemma 3.4 and (3.3). □

Proposition 4.5. *Let d be any positive integer. Then*

$$F_q(t,\ x) = \frac{1}{d}\sum_{i=0}^{d-1}F_{q^d}\left([d]_q t,\ \frac{x+i}{d}\right),$$

$$G_q(t,\ x) = \sum_{i=0}^{d-1}(-1)^i G_{q^d}\left([d]_q t,\ \frac{x+i}{d}\right) \quad \textit{if } d \textit{ odd}.$$

Proof. Substituting $m = nd + i$ with $n = 0, 1, \ldots$ and $i = 0, \ldots, d - 1$ into Lemma 3.3, we have

$$\begin{aligned} F_q(t, x) &= e^{\frac{t}{1-q}} + \frac{t \log q}{1-q} \sum_{m=0}^{\infty} q^{m+x} e^{[m+x]_q t} \\ &= e^{\frac{[d]_q t}{1-q^d}} + \frac{1}{d} \sum_{i=0}^{d-1} \frac{[d]_q t \log q^d}{1-q^d} \sum_{n=0}^{\infty} q^{nd+x+i} e^{[nd+x+i]_q t} \\ &= \frac{1}{d} \sum_{i=0}^{d-1} \left(e^{\frac{[d]_q t}{1-q^d}} + \frac{[d]_q t \log q^d}{1-q^d} \sum_{n=0}^{\infty} (q^d)^{n+\frac{x+i}{d}} e^{[n+\frac{x+i}{d}]_{q^d} [d]_q t} \right), \end{aligned}$$

where we use $[n + (x + i)/d]_{q^d} [d]_q = [nd + x + i]_q$. So we have the first form. Similarly the second form follows by Lemma 3.4. □

From (3.2), (3.3), Propositions 4.4 and 4.5, we obtain the following:

Corollary 4.6. *Let d and k be any positive integer. Then*

$$\zeta_q(1-k, x) = \frac{[d]_q^k}{d} \sum_{i=0}^{d-1} \zeta_{q^d} \left(1-k, \frac{x+i}{d}\right),$$

$$\zeta_{q,E}(-k, x) = [d]_q^k \sum_{i=0}^{d-1} (-1)^i \zeta_{q^d,E} \left(-k, \frac{x+i}{d}\right) \quad \textit{if } d \textit{ odd}.$$

Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. We define the generating function $F_{q,\chi}(x, t)$ and $G_{q,\chi}(x, t)$ of the generalized q-Bernoulli and q-Euler polynomials as follows:

$$\begin{aligned} F_{q,\chi}(t, x) &= \sum_{k=0}^{\infty} B_{k,\chi}(x, q) \frac{t^k}{k!} \\ &= \frac{1}{f} \sum_{a=1}^{f} \chi(a) F_{q^f} \left([f]_q t, \frac{a+x}{f}\right) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} G_{q,\chi}(t, x) &= \sum_{k=0}^{\infty} E_{k,\chi}(x, q) \frac{t^k}{k!} \\ &= \sum_{a=1}^{f} (-1)^a \chi(a) G_{q^f} \left([f]_q t, \frac{a+x}{f}\right) \quad \text{if } f \text{ odd}, \end{aligned} \tag{4.5}$$

where $B_{k,\chi}(x, q)$ and $E_{k,\chi}(x, q)$ are the generalized q-Bernoulli and q-Euler polynomials, respectively. Clearly (4.4) and (4.5) are equal to

$$F_{q,\chi}(t, x) = \frac{t \log q}{1-q} \sum_{m=0}^{\infty} \chi(m) q^{m+x} e^{[m+x]_q t}, \tag{4.6}$$

$$G_{q,\chi}(t, x) = 2 \sum_{k=0}^{\infty} (-1)^m \chi(m) e^{[m+x]_q t} \quad \text{if } f \text{ odd}, \tag{4.7}$$

respectively. As $q \to 1$ in (4.6) and (4.7), we have $F_{q,\chi}(t, x) \to F_\chi(t, x)$ and $G_{q,\chi}(t, x) \to G_\chi(t, x)$, where $F_\chi(t, x)$ and $G_\chi(t, x)$ are the usual generating function of generalized Bernoulli and Euler numbers, respectively, which are defined as follows [13]:

$$F_\chi(t,\ x) = \sum_{a=1}^{f} \frac{\chi(a)te^{(a+x)t}}{e^{ft}-1} = \sum_{k=0}^{\infty} B_{k,\chi}(x)\frac{t^k}{k!}, \tag{4.8}$$

$$G_\chi(t,\ x) = 2\sum_{a=1}^{f} \frac{(-1)^a\chi(a)e^{(a+x)t}}{e^{ft}+1} = \sum_{k=0}^{\infty} G_{k,\chi}(x)\frac{t^k}{k!} \quad \text{if } f \text{ odd.} \tag{4.9}$$

From (3.2), (3.3), (4.4) and (4.5), we can easily see that

$$B_{k,\chi}(x,\ q) = \frac{[f]_q^k}{f}\sum_{a=1}^{f}\chi(a)B_k\left(\frac{a+x}{f},\ q^f\right), \tag{4.10}$$

$$E_{k,\chi}(x,\ q) = [f]_q^k\sum_{a=1}^{f}(-1)^a\chi(a)E_k\left(\frac{a+x}{f},\ q^f\right) \quad \text{if } f \text{ odd.} \tag{4.11}$$

By using the definitions of $\zeta_q(s, x)$ and $\zeta_{q,E}(s, x)$, we can define the q-analogues of Dirichlet's L-function.

Definition 4.7 (q-analogues of the Dirichlet's L-functions). For $s \in \mathbb{C}$ and $0 < x \le 1$,

$$L_q(s,\ x,\ \chi) = \frac{\log q}{q-1}\sum_{m=0}^{\infty}\frac{\chi(m)q^{m+x}}{[m+x]_q^s},$$

$$\ell_q(s,\ x,\ \chi) = 2\sum_{m=0}^{\infty}\frac{(-1)^m\chi(m)}{[m+x]_q^s}.$$

Similarly, we can compute the values of $L_q(s, x, \chi)$ at non-positive integers.

Theorem 4.8. *For* $k \ge 1$, *we have*

$$L_q(1-k,\ x,\ \chi) = -\frac{B_{k,\chi}(x,q)}{k} \quad \textit{and} \quad \ell_q(1-k,\ x,\ \chi) = E_{k-1,\chi}(x,\ q).$$

Proof. Using Lemma 3.3 and (4.4), we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}B_{k,\chi}(x,\ q)\frac{t^k}{k!} &= \frac{1}{f}\sum_{a=1}^{f}\chi(a)\left(e^{\frac{[f]_q t}{1-q^f}} + \frac{[f]_q t\log q^f}{1-q^f}\sum_{n=0}^{\infty}(q^f)^{n+\frac{x+a}{f}}e^{[n+\frac{x+a}{f}]_{q^f}[f]_q t}\right)\\ &= \frac{t\log q}{1-q}\sum_{m=0}^{\infty}\chi(m)q^{m+x}e^{[m+x]_q t},\end{aligned}$$

where we use $[n + (a+x)/f]_{q^f}[f]_q = [nf+a+x]_q$ and $\sum_{a=1}^{f}\chi(a) = 0$. Therefore, we obtain

$$\begin{aligned}B_{k,\chi}(x,\ q) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k\left(\sum_{k=0}^{\infty}B_{k,\chi}(x,\ q)\frac{t^k}{k!}\right)\Bigg|_{t=0}\\ &= \frac{k\log q}{1-q}\sum_{m=0}^{\infty}\chi(m)q^{m+x}[m+x]_q^{k-1}.\end{aligned}$$

Hence for $k \geq 1$

$$\begin{aligned} -\frac{B_{k,\chi}(x,q)}{k} &= \frac{\log q}{q-1}\sum_{m=0}^{\infty}\chi(m)q^{m+x}[m+x]_q^{k-1} \\ &= L_q(1-k,\ x,\ \chi). \end{aligned}$$

Similarly the second identity follows. This completes the proof. □

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Author details
[1]National Institute for Mathematical Sciences, Doryong-dong, Yuseong-gu Daejeon 305-340, South Korea [2]Department of Mathematics, KAIST, 373-1 Guseong-dong, Yuseong-gu, Daejeon 305-701, South Korea

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