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Periodic solutions for a class of higher-order difference equations

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Abstract

In this article, we discuss the existence of periodic solutions for the higher-order difference equation

$$x(n+k) = g(x(n)) - f(n, x(n-\tau(n))).$$

We show the existence of periodic solutions by using Schauder's fixed point theorem, and illustrate three examples.

MSC 2010: 39A10; 39A12.

Keywords: functional difference equation, periodic solution, fixed point theorem

1 Introduction and main results

Let $\mathbb{R}$ denote the set of the real numbers, $\mathbb{Z}$ the integers and $\mathbb{N}$ the positive integers. In this article, we investigate the existence of periodic solutions of the following high-order functional difference equation

$$x(n+k) = g(x(n)) - f(n, x(n-\tau(n))), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $k \in \mathbb{N}$, $\tau : \mathbb{Z} \to \mathbb{Z}$ and $\tau(n+\omega) = \tau(n)$, $f(n+\omega, u) = f(n, u)$ for any $(n, u) \in \mathbb{Z} \times \mathbb{R}$, $\omega \in \mathbb{N}$.

Difference equations have attracted the interest of many researchers in the last 20 years since they provided a natural description of several discrete models, in which the periodic solution problem is always a important topic, and the reader can consult [1-7] and the references therein. There are many good results about existence of periodic solutions for first-order functional difference equations [8-12]. Only a few article have been published on the same problem for higher-order functional difference equations. Recently, using coincidence degree theory, Liu [13] studied the second-order nonlinear functional difference equation

$$\Delta^2 x(n-1) = f(n, x(n-\tau_1(n)), x(n-\tau_2(n)), \ldots, x(n-\tau_m(n))), \tag{1.2}$$

and obtain sufficient conditions for the existence of at least one periodic solution of equation (1.2). By using fixed point theorem in a cone, Wang and Chen [14] discussed the following higher-order functional difference equation

$$x(n+m+k) - ax(n+m) - bx(n+k) + abx(n) = f(n, x(n-\tau(n))), \tag{1.3}$$

where $a \neq 1$, $b \neq 1$ are positive constants, $\tau : \mathbb{Z} \to \mathbb{Z}$ and $\tau(n+\omega) = \tau(n)$, $\omega, m, k \in \mathbb{N}$, and obtained existence theorem for single and multiple positive periodic solutions of (1.3).

Our aim of this article is to study the existence of periodic solutions for the higher-order difference equations (1.1) using the well-known Schauder's fixed point theorem. Our results extend the known results in the literature.

The main results of this article are following sufficient conditions which guarantee the existence of a periodic solution for (1.1).

Theorem 1.1. *Assume that there exist constants* $m < M$, $r > 0$ *such that* $g \in C^1[m, M]$ *with* $r \leq g'(u) \leq 1$ *for any* $u \in [m, M]$ *and* $f(n, u)$: $\mathbb{Z} \times [m, M] \to \mathbb{R}$ *is continuous in* u,

$$g(M) - M \leq f(n, u) \leq g(m) - m \tag{1.4}$$

for any $(n, u) \in \mathbb{Z} \times [m, M]$, *then* (1.1) *has at least one* ω*-periodic solution* x *with* $m \leq x \leq M$.

Theorem 1.2. *Assume that there exist constants* $m < M$ *such that* $g \in C^1[m, M]$ *with* $g'(u) \geq 1$ *for any* $u \in [m, M]$ *and* $f(n, u)$: $\mathbb{Z} \times [m, M] \to \mathbb{R}$ *is continuous in* u,

$$g(m) - m \leq f(n, u) \leq g(M) - M \tag{1.5}$$

for any $(n, u) \in \mathbb{Z} \times [m, M]$, *then* (1.1) *has at least one* ω*-periodic solution* x *with* $m \leq x \leq M$.

2 Some examples

In this section, we present three examples to illustrate our conclusions.

Example 2.1. Consider the difference equation

$$x(n+k) = ax(n) + q(n)\sqrt[3]{x(n-\tau(n))}, \tag{2.1}$$

$$x(n+k) = bx(n) - q(n)\sqrt[3]{x(n-\tau(n))}, \tag{2.2}$$

where $k \in \mathbb{N}$, $0 < a < 1$, $b > 1$, q is one ω-periodic function with $q(n) > 0$ for all $n \in [1, \omega]$ and $\tau : \mathbb{Z} \to \mathbb{Z}$ and $\tau(n + \omega) = \tau(n)$.

Let $m > 0$ be sufficiently small and $M > 0$ sufficiently large. It is easy to check that

$$(a-1)M \leq -q(n)\sqrt[3]{u} \leq (a-1)m,$$
$$(b-1)m \leq q(n)\sqrt[3]{u} \leq (b-1)M$$

for $n \in \mathbb{Z}$ and $u \in [m.M]$. By Theorem 1.1 (Theorem 1.2), Equation (2.1) (or (2.2)) has at least one positive ω-periodic solution x with $m \leq x \leq M$. When $k = 1$, this conclusion about (2.1) and (2.2) can been obtained from the results in [15]. Our result holds for all $k \in \mathbb{N}$.

Remark 1 Consider the difference equations

$$x(n+k) = ax(n) + q(n)f(x(n-\tau(n))), \tag{2.3}$$

$$x(n+k) = bx(n) - q(n)f(x(n-\tau(n))), \tag{2.4}$$

where $k \in \mathbb{N}$, $0 < a < 1$, $b > 1$, q is one ω-periodic function with $q(n) > 0$ for all $n \in [1, \omega]$, $\tau : \mathbb{Z} \to \mathbb{Z}$ and $\tau(n + \omega) = \tau(n)$ and $f : (0, +\infty) \to (0, +\infty)$ is continuous.

The following result generalizes the conclusion of Example 2.1.

Proposition 2.1 Assume that $f_0 = +\infty$ and $f_\infty, = 0$, here

$$f_0 = \lim_{u\to 0+}\frac{f(u)}{u},\quad f_\infty = \lim_{u\to\infty}\frac{f(u)}{u},$$

then (2.3) or (2.4) has at least one positive ω-periodic solution.

Proof Here, we only consider (2.3). From $f_0 = +\infty$ and $f_\infty = 0$, we obtain that there exist $0 < \rho_1 < \rho_2$ such that

$$f(u) \geq \frac{1-a}{\min q(n)}u,\quad 0 < u \leq \rho_1,\quad f(u) \leq \frac{1-a}{\max q(n)}u,\quad u \geq \rho_2.$$

Let $A = \min q(n) \min \{f(u): \rho_1 \leq u \leq \rho_2\}$ and $B = \max q(n) \max\{f(u): \rho_1 \leq u \leq \rho_2\}$. Choosing $\theta \in (0, 1)$ such that

$$\frac{A}{1-a} \geq \theta\rho_1,\quad \frac{B}{1-a} \leq \theta^{-1}\rho_2,$$

we obtain that

$$\begin{aligned} f(u) &\geq \frac{1-a}{\min q(n)}u \geq \frac{\theta(1-a)\rho_1}{\min q(n)},\quad \theta\rho_1 \leq u \leq \rho_1,\\ f(u) &\leq \frac{\theta^{-1}(1-a)\rho_2}{\max q(n)},\quad \rho_2 \leq u \leq \theta^{-1}\rho_2,\\ A &\leq q(n)f(u) \leq B,\quad \forall n \in \mathbb{Z}, \rho_1 \leq u \leq \rho_2. \end{aligned}$$

Using the above three inequalities, we have

$$(1-a)\theta\rho_1 \leq q(n)f(u) \leq (1-a)\theta^{-1}\rho_2,\quad \forall n \in \mathbb{Z}, \theta\rho_1 \leq u \leq \theta^{-1}\rho_2.$$

By Theorem 1.1, Equation (2.3) has at least one positive ω-periodic solution x with $\theta\rho_1 \leq x \leq \theta^{-1}\rho_2$. □

Example 2.2. Consider the difference equation

$$x(n+k) = -\frac{1}{x^\alpha(n)} + q(n), \tag{2.5}$$

where $k \in \mathbb{N}$, $\alpha > 0$, q is one ω-periodic function.

We claim that there is a $\lambda > 0$ such that (2.5) has at least two positive ω-periodic solutions for $\min q(n) > \lambda$.

In fact, $g(x) = -x^{-\alpha}$. Let $0 < a < \sqrt[\alpha+1]{\alpha}$ be sufficiently small and $b > \sqrt[\alpha+1]{\alpha}$ be sufficiently large, then

$$\begin{aligned} \frac{\alpha}{b^{\alpha+1}} &\leq g'(x) = \frac{\alpha}{x^{\alpha+1}} \leq 1,\quad \text{for } x \in [\sqrt[\alpha+1]{\alpha}, b],\\ g'(x) &= \frac{\alpha}{x^{\alpha+1}} \geq 1,\quad \text{for } x \in [a, \sqrt[\alpha+1]{\alpha}]. \end{aligned}$$

If the following conditions are fulfilled

$$-\frac{1}{b^\alpha} - b \leq -q(n) \leq -\frac{1}{\sqrt[\alpha+1]{\alpha^\alpha}} - \sqrt[\alpha+1]{\alpha}, \forall n \in \mathbb{Z}, \tag{2.6}$$

$$-\frac{1}{a^\alpha} - a \leq -q(n) \leq -\frac{1}{\sqrt[\alpha+1]{\alpha^\alpha}} - \sqrt[\alpha+1]{\alpha}, \forall n \in \mathbb{Z}, \tag{2.7}$$

then (2.5) has at least one periodic solution $[a, \sqrt[\alpha+1]{\alpha}]$ and $[\sqrt[\alpha+1]{\alpha}, b]$ respectively. When min $q(n)$ is sufficiently large, the conditions (2.6) and (2.7) are satisfied.

Example 2.3. Consider the difference equation

$$x(n+k) = x^3(n) - 2x(n) - q(n)x^2(n - \tau(n)), \tag{2.8}$$

where $k \in \mathbb{N}$, q is one ω-periodic function with $q(n) > 0$ for all $n \in [1, \omega]$, $\tau : \mathbb{Z} \to \mathbb{Z}$ and $\tau(n + \omega) = \tau(n)$.

Let $m = 1$, $M > 3 + \max q(n)$ and $g(u) = u^3 - 2u$, $f(n, u) = q(n)u^2$. It is easy to check that $g'(u) \geq 1$ for $u \in [m, M]$, and

$$g(m) - m = -2 < f(n, u) \leq g(M) - M = M^3 - 3M, \quad \forall n \in \mathbb{Z}, u \in [m, M].$$

By Theorem 1.2, Equation (2.8) has at least one positive ω-periodic solution x with $m \leq x \leq M$.

Remark 2 Consider the difference equation

$$x(n+k) = g(x(n)) - q(n)f(x(n - \tau(n))), \tag{2.9}$$

where $k \in \mathbb{N}$, q is one ω-periodic function with $q(n) > 0$ for all $n \in [1, \omega]$, $\tau :$ ‘ → ‘ and $\tau(n + \omega) = \tau(n)$ and $f : (0, +\infty) \to (0, +\infty)$ is continuous.

Proposition 2.2 Assume that there exists $a > 0$ such that $g \in C^1([a, +\infty), R)$ with $g'(u) \geq 1$ for $u > a$, $f(u) \geq (g(a) - a)/\min q(n)$ for $u \geq a$. Further suppose that

$$\lim_{u\to+\infty} \frac{g(u) - u}{f(u)} > \max q(n), \quad \lim_{u\to+\infty} (g(u) - u) = +\infty.$$

Then (2.9) has at least one positive ω-periodic solution.

Proof There exist $\rho > 0$ such that

$$g(u) - u \geq f(u) \max q(n), \quad u \geq \rho.$$

Let $A = \min q(n) \min\{f(u): a \leq u \leq \rho\}$ and $B = \max q(n) \max\{f(u): a \leq u \leq \rho\}$. Since $\lim_{u\to+\infty}(g(u) - u) = +\infty$ and $g'(u) \geq 1$ for $u > a$, there is $M > \rho$ such that $g(M) - M > B$ and

$$f(u) \max q(n) \leq g(u) - u \leq g(M) - M, \quad \rho \leq u \leq M.$$

Thus, (2.9) has at least one ω-periodic solution x with $a \leq x \leq M$. □

3 Proof

Let X be the set of all real ω-periodic sequences. When endowed with the maximum norm $||x|| = \max_{n\in[0,\omega-1]} |x(n)|$, X is a Banach space.

Let $k \in \mathbb{N}$ and $0 < c \neq 1$, and consider the equation

$$x(n+k) = cx(n) + \gamma(n), \tag{3.1}$$

where $\gamma \in X$. Set (k, ω) is the greatest common divisor of k and ω, $h = \omega/(k, \omega)$. We obtain that if $x \in X$ satisfies (3.1), then

$$\begin{aligned} c^{-1}x(n+k) - x(n) &= c^{-1}\gamma(n), \\ c^{-2}x(n+2k) - c^{-1}x(n+k) &= c^{-2}\gamma(n+k), \\ &\cdots\cdots\cdots\cdots \\ c^{-p}x(n+hk) - c^{1-p}x(n+(h-1)k) &= c^{-p}\gamma(n+(h-1)k). \end{aligned}$$

By summing the above equations and using periodicity of x, we obtain the following result.

Lemma 3.1. *Assume that* $0 < c \neq 1$, *then* (3.1) *has a unique periodic solution*

$$x(n) = (c^{-h} - 1)^{-1} \sum_{i=1}^{h} c^{-i} \gamma(n + (i-1)k).$$

The following well-known Schauder's fixed point theorem is crucial in our arguments.

Lemma 3.2. [16]*Let* X *be a Banach space with* $D \subset X$ *closed and convex. Assume that* $T : D \to D$ *is a completely continuous map, then* T *has a fixed point in* D.

Now, we rewrite (1.1) as

$$x(n+k) = px(n) + [g(x(n)) - f(n, x(n - \tau(n)) - px(n)], \tag{3.2}$$

where $p > 0$ is a constant which is determined later. By Lemma 3.1, if x is a periodic solution of (1.1), x satisfies

$$x(n) = (p^{-h} - 1)^{-1} \sum_{i=1}^{h} p^{-i} (H_p x)(n + (i-1)k),$$

where $h = \omega/(k, \omega)$, the mapping H_p is defined as

$$(H_p x)(n) = g(x(n)) - px(n) - f(n,\ x(n - \tau(n)),\ x \in X.$$

Define a mapping T_p in X by

$$(T_p x)(n) = (p^{-h} - 1)^{-1} \sum_{i=1}^{h} p^{-i} (H_p x)(n + (i-1)k), \quad x \in X.$$

Clearly, the fixed point of T_p in X is a periodic solution of (1.1).

Proof of Theorem 1.1 Let $p = r$ and $\Omega = \{x \in X : m \leq x(n) \leq M$ for $n \in$ '}, then Ω is a closed and convex set. If $r = 1$, then $g(u) = u$ on $[m, M]$. It is easy to check that any constant $c \in [m, M]$ is a periodic solution of (1.1). Set $r < 1$. Now we show that T_r satisfies all conditions of Lemma 3.2. Noting that the function $g(u)$ - ru is nondecreasing in $[m, M]$, we have for any $x \in \Omega$,

$$g(m) - rm \leq g(x(n)) - rx(n) \leq g(M) - rM, \quad \forall n \in \mathbb{Z}.$$

Let (1.4) be fulfilled. For any $x \in \Omega$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
(H_r x)(n) &= g(x(n)) - px(n) - f(n, x(n - \tau(n)) \\
&\leq g(M) - rM - (g(M) - M) \\
&= (1 - r)M, \\
(H_r x)(n) &= g(x(n)) - px(n) - f(n, x(n - \tau(n)) \\
&\geq g(m) - rm - (g(m) - m) \\
&= (1 - r)m.
\end{aligned}$$

Hence, for any $x \in \Omega$ and $n \in \mathbb{Z}$,

$$\begin{aligned}(T_r x)(n) &= (r^{-h}-1)^{-1}\sum_{i=1}^{h} r^{-i}(H_p x)(n+(i-1)k) \\ &\le (r^{-h}-1)^{-1}\sum_{i=1}^{h} r^{-i}(1-r)M = M, \\ (T_r x)(n) &= (r^{-h}-1)^{-1}\sum_{i=1}^{h} r^{-i}(H_p x)(n+(i-1)k) \\ &\ge (r^{-h}-1)^{-1}\sum_{i=1}^{h} r^{-i}(1-r)m = m.\end{aligned}$$

Hence, $T_r(\Omega) \subseteq \Omega$.

Since X is finite-dimensional and $g(u)$, $f(n, u)$ are continuous in u, one easily show that T_r is completely continuous in Ω. Therefore, T_r has a fixed point $x \in \Omega$ by Lemma 3.2, which is a ω = periodic solution of (1.1). The proof is complete. □

Proof of Theorem 1.2 Since $g \in C^1[m, M]$, $\max\{g'(u): m \le u \le M\}$ exists and $\max\{g'(u): m \le u \le M\} \ge 1$. Let $p = \max\{g'(u): m \le u \le M\}$. If $p = 1$, then $g(u) \equiv u$ on $[m, M]$. It is easy to check that any constant $c \in [m, M]$ is a periodic solution of (1.1). Next, we assume that $p > 1$. Set $\Omega = \{x \in X : m \le x(n) \le M \text{ for } n \in \mathbb{Z}\}$. Noting that the function $g(u) - pu$ is nonincreasing in $[m, M]$, we have for any $x \in \Omega$,

$$g(M) - pM \le g(x(n)) - px(n) \le g(m) - pm, \quad \forall n \in \mathbb{Z}.$$

For any $x \in \Omega$ and $n \in \mathbb{Z}$,

$$\begin{aligned}(H_p x)(n) &= g(x(n)) - px(n) - f(n, x(n-\tau(n)) \\ &\le g(m) - pm - (g(m) - m) \\ &= (1-p)m, \\ (H_p x)(n) &= g(x(n)) - px(n) - f(n, x(n-\tau(n)) \\ &\ge g(M) - pM - (g(M) - M) \\ &= (1-p)M.\end{aligned}$$

Hence, for any $x \in \Omega$ and $n \in \mathbb{Z}$,

$$\begin{aligned}(T_p x)(n) &= (p^{-h}-1)^{-1}\sum_{i=1}^{h} p^{-i}(H_p x)(n+(i-1)k) \\ &\ge (p^{-h}-1)^{-1}\sum_{i=1}^{h} p^{-i}(1-p)m = m, \\ (T_p x)(n) &= (p^{-h}-1)^{-1}\sum_{i=1}^{h} p^{-i}(H_p x)(n+(i-1)k) \\ &\le (p^{-h}-1)^{-1}\sum_{i=1}^{h} p^{-i}(1-p)M = M.\end{aligned}$$

Hence, $T_p(\Omega) \subseteq \Omega$. T_p has a fixed point $x \in \Omega$. The proof is complete. □

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Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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