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# Sturm-Liouville BVP in Banach space

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## Abstract

We consider the existence of single and multiple positive solutions for fourth-order Sturm-Liouville boundary value problem in Banach space. The sufficient condition for the existence of single and multiple positive solutions is obtained by fixed theorem of strict set contraction operator in the frame of the ODE technique. Our results significantly extend and improve many known results including singular and nonsingular cases.

## 1 Introduction

The boundary value problems (BVPs) for ordinary differential equations play a very important role in both theory and application. They are used to describe a large number of physical, biological, and chemical phenomena. In this article, we will study the existence of positive solutions for the following fourth-order nonlinear Sturm-Liouville BVP in a real Banach space  $E$

$$\begin{cases} \frac{1}{p(t)}(p(t)u'''(t))' = f(u(t)), & 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \quad \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 u''(0) - \beta_2 \lim_{t \rightarrow 0^+} p(t)u'''(t) = 0, \\ \gamma_2 u''(1) + \delta_2 \lim_{t \rightarrow 1^-} p(t)u'''(t) = 0, \end{cases} \quad (1.1)$$

where  $\alpha_i, \beta_i, \delta_i, \gamma_i \geq 0$  ( $i = 1, 2$ ) are constants such that  $\rho_1 = \beta_1 \gamma_1 + \alpha_1 \gamma_1 + \alpha_1 \delta_1 > 0$ ,  $B(t, s) = \int_t^s \frac{d\tau}{p(\tau)}$ ,  $\rho_2 = \beta_2 \gamma_2 + \alpha_2 \gamma_2 B(0, 1) + \alpha_2 \delta_2 > 0$ , and  $p \in C^1((0, 1), (0, +\infty))$ .

Moreover  $p$  may be singular at  $t = 0$  and/or 1. BVP (1.1) is often referred to as the deformation of an elastic beam under a variety of boundary conditions, for detail, see [1-17]. For example, BVP (1.1) subject to Lidstone boundary value conditions  $u(0) = u(1) = u''(0) = u''(1) = 0$  are used to model such phenomena as the deflection of elastic beam simply supported at the endpoints, see [1,3,5,7-11,13,14]. We notice that the above articles use the completely continuous operator and require  $f$  satisfies some growth condition or assumptions of monotonicity which are essential for the technique used.

The aim of this article is to consider the existence of positive solutions for the more general Sturm-Liouville BVP by using the properties of strict set contraction operator. Here, we allow  $p$  have singularity at  $t = 0, 1$ , as far as we know, there were fewer works to be done. This article attempts to fill part of this gap in the literature.

This article is organized as follows. In Section 2, we first present some properties of the Green functions that are used to define a positive operator. Next we approximate the singular fourth-order BVP to singular second-order BVP by constructing an integral operator. In Section 3, the sufficient condition for the existence of single and multiple positive solutions for BVP (1.1) will be established. In Section 4, we give one example as the application.

## 2 Preliminaries and lemmas

In this article, we all suppose that  $(E, \|\cdot\|_1)$  is a real Banach space. A nonempty closed convex subset  $P$  in  $E$  is said to be a cone if  $\lambda P \in P$  for  $\lambda \geq 0$  and  $P \cap \{-P\} = \{\theta\}$ , where  $\theta$  denotes the zero element of  $E$ . The cone  $P$  defines a partial ordering in  $E$  by  $x \leq y$  iff  $y - x \in P$ . Recall the cone  $P$  is said to be normal if there exists a positive constant  $N$  such that  $0 \leq x \leq y$  implies  $\|x\|_1 \leq N\|y\|_1$ . The cone  $P$  is normal if every order interval  $[x, y] = \{z \in E | x \leq z \leq y\}$  is bounded in norm.

In this article, we assume that  $P \subseteq E$  is normal, and without loss of generality, we may assume that the normality of  $P$  is 1. Let  $J = [0,1]$ , and

$$C(J, E) = \{u : J \rightarrow E | u(t) \text{ continuous}\},$$

$$C^i(J, E) = \{u : J \rightarrow E | u(t) \text{ is } i\text{-order continuously differentiable}\}, i = 1, 2, \dots$$

For  $u = u(t) \in C(J, E)$ , let  $\|u\| = \max_{t \in J} \|u(t)\|_1$ , then  $C(J, E)$  is a Banach space with the norm  $\|\cdot\|$ .

**Definition 2.1** A function  $u(t)$  is said to be a positive solution of the BVP (1.1), if  $u \in C^2([0,1], E) \cap C^3((0, 1), E)$  satisfies  $u(t) \geq 0, t \in (0, 1]$ ,  $pu''' \in C^1((0, 1), E)$  and the BVP (1.1), i.e.,  $u \in C^2([0,1], P) \cap C^3((0, 1), P)$  and  $u(t) \neq \theta, t \in J$ .

We notice that if  $u(t)$  is a positive solution of the BVP (1.1) and  $p \in C^1(0, 1)$ , then  $u \in C^4(0, 1)$ .

Now we denote that  $H(t, s)$  and  $G(t, s)$  are the Green functions for the following boundary value problem

$$\begin{cases} -u'' = 0, & 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, & \gamma_1 u(1) + \delta_1 u'(1) = 0 \end{cases}$$

and

$$\begin{cases} \frac{1}{p(t)}(p(t)v'(t))' = 0, & 0 < t < 1, \\ \alpha_2 v(0) - \lim_{t \rightarrow 0^+} \beta_2 p(t)v'(t) = 0, \\ \gamma_2 v(1) + \lim_{t \rightarrow 1^-} \delta_2 p(t)v'(t) = 0, \end{cases}$$

respectively. It is well known that  $H(t, s)$  and  $G(t, s)$  can be written by

$$H(t, s) = \frac{1}{\rho_1} \begin{cases} (\beta_1 + \alpha_1 s)(\delta_1 + \gamma_1(1 - t)), & 0 \leq s \leq t \leq 1, \\ (\beta_1 + \alpha_1 t)(\delta_1 + \gamma_1(1 - s)), & 0 \leq t \leq s \leq 1 \end{cases} \quad (2.1)$$

and

$$G(t, s) = \frac{1}{\rho_2} \begin{cases} (\beta_2 + \alpha_2 B(0, s))(\delta_2 + \gamma_2 B(t, 1)), & 0 \leq s \leq t \leq 1, \\ (\beta_2 + \alpha_2 B(0, t))(\delta_2 + \gamma_2 B(s, 1)), & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.2)$$

where  $\rho_1 = \gamma_1\beta_1 + \alpha_1\gamma_1 + \alpha_1\delta_1 > 0$ ,  $B(t, s) = \int_t^s \frac{d\tau}{p(\tau)}$ ,  $\rho_2 = \alpha_2\delta_2 + \alpha_2\gamma_2B(0, 1) + \beta_2\gamma_2 > 0$ .

It is easy to verify the following properties of  $H(t, s)$  and  $G(t, s)$

- (I)  $G(t, s) \leq G(s, s) < +\infty$ ,  $H(t, s) \leq H(s, s) < +\infty$ ;
- (II)  $G(t, s) \geq \rho G(s, s)$ ,  $H(t, s) \geq \zeta H(s, s)$ , for any  $t \in [a, b] \subset (0, 1)$ ,  $s \in [0, 1]$ , where

$$\rho = \min \left\{ \frac{\delta_2 + \gamma_2 B(b, 1)}{\delta_2 + \gamma_2 B(0, 1)}, \frac{\beta_2 + \alpha_2 B(0, a)}{\beta_2 + \alpha_2 B(0, 1)} \right\},$$

$$\xi = \min \left\{ \frac{\delta_1 + \gamma_1(1 - b)}{\delta_1 + \gamma_1}, \frac{\beta_1 + \alpha_1 a}{\beta_1 + \alpha_1} \right\}.$$

Throughout this article, we adopt the following assumptions

(H<sub>1</sub>)  $p \in C^1((0, 1), (0, +\infty))$  and satisfies

$$0 < \int_0^1 \frac{ds}{p(s)} < +\infty, \quad 0 < \lambda = \int_0^1 G(s, s)p(s)ds < +\infty.$$

(H<sub>2</sub>)  $f(u) \in C(P \setminus \{\theta\}, P)$  and there exists  $M > 0$  such that for any bounded set  $B \subset C(J, E)$ , we have

$$\alpha(f(B(t))) \leq M\alpha(B(t)), \quad 2M\lambda < 1. \tag{2.3}$$

where  $\alpha(\cdot)$  denote the Kuratowski measure of noncompactness in  $C(J, E)$ .

The following lemmas play an important role in this article.

**Lemma 2.1** [17]. Let  $B \subset C[J, E]$  be bounded and equicontinuous on  $J$ , then  $\alpha(B) = \sup_{t \in J} \alpha(B(t))$ .

**Lemma 2.2** [16]. Let  $B \subset C(J, E)$  be bounded and equicontinuous on  $J$ , let  $\alpha(B)$  is continuous on  $J$  and

$$\alpha \left( \left\{ \int_J u(t)dt : u \in B \right\} \right) \leq \int_J \alpha(B(t))dt.$$

**Lemma 2.3** [16]. Let  $B \subset C(J, E)$  be a bounded set on  $J$ . Then  $\alpha(B(t)) \leq 2\alpha(B)$ .

Now we define an integral operator  $S : C(J, E) \rightarrow C(J, E)$  by

$$Sv(t) = \int_0^1 H(t, \tau)v(\tau)d\tau. \tag{2.4}$$

Then,  $S$  is linear continuous operator and by the expressed of  $H(t, s)$ , we have

$$\begin{cases} (Sv)''(t) = -v(t), & 0 < t < 1, \\ \alpha_1(Sv)(0) - \beta_1(Sv)'(0) = 0, \\ \gamma_1(Sv)(1) + \delta_1(Sv)'(1) = 0. \end{cases} \tag{2.5}$$

**Lemma 2.4.** The Sturm-Liouville BVP (1.1) has a positive solution if and only if the following integral-differential boundary value problem has a positive solution of

$$\begin{cases} \frac{1}{\rho(t)}(p(t)v'(t))' + f(Sv(t)) = 0, & 0 < t < 1, \\ \alpha_2 v(0) - \lim_{t \rightarrow 0^+} \beta_2 p(t)v'(t) = 0, \\ \gamma_2 v(1) + \lim_{t \rightarrow 1^-} \delta_2 p(t)v'(t) = 0, \end{cases} \quad (2.6)$$

where  $S$  is given in (2.4).

**Proof** In fact, if  $u$  is a positive solution of (1.1), let  $u = Sv$ , then  $v = -u''$ . This implies  $u'' = -v$  is a solution of (2.6). Conversely, if  $v$  is a positive solution of (2.6). Let  $u = Sv$ , by (2.5),  $u'' = (Sv)'' = -v$ . Thus,  $u = Sv$  is a positive solution of (1.1). This completes the proof of Lemma 2.1.

So, we only need to concentrate our study on (2.6). Now, for the given  $[a, b] \subset (0, 1)$ ,  $\rho$  as above in (II), we introduce

$$K = \{u \in C(J, P) : u(t) \geq \rho u(s), \quad t \in [a, b], \quad s \in [0, 1]\}.$$

It is easy to check that  $K$  is a cone in  $C[0,1]$ . Further, for  $u(t) \in K$ ,  $t \in [a, b]$ , we have by normality of cone  $P$  with normal constant 1 that  $\|u(t)\|_1 \geq \rho \|u\|$ .

Next, we define an operator  $T$  given by

$$Tv(t) = \int_0^1 G(t, s)\rho(s)f(Sv(s))ds, \quad t \in [0, 1], \quad (2.7)$$

Clearly,  $v$  is a solution of the BVP (2.6) if and only if  $v$  is a fixed point of the operator  $T$ .

Through direct calculation, by (II) and for  $v \in K$ ,  $t \in [a, b]$ ,  $s \in J$ , we have

$$\begin{aligned} Tv(t) &= \int_0^1 G(t, s)\rho(s)f(Sv(s))ds \\ &\geq \rho \int_0^1 G(s, s)\rho(s)f(Sv(s))ds = \rho Tv(s). \end{aligned}$$

So, this implies that  $TK \subset K$ .

**Lemma 2.5.** Assume that  $(H_1)$ ,  $(H_2)$  hold. Then  $T : K \rightarrow K$  is strict set contraction.

**Proof** Firstly, The continuity of  $T$  is easily obtained. In fact, if  $v_n, v \in K$  and  $v_n \rightarrow v$  in the sup norm, then for any  $t \in J$ , we get

$$\|Tv_n(t) - Tv(t)\|_1 \leq \|f(Sv_n(t)) - f(Sv(t))\|_1 \int_0^1 G(s, s)\rho(s)ds,$$

so, by the continuity of  $f, S$ , we have

$$\|Tv_n - Tv\| = \sup_{t \in J} \|Tv_n(t) - Tv(t)\|_1 \rightarrow 0.$$

This implies that  $Tv_n \rightarrow Tv$  in the sup norm, i.e.,  $T$  is continuous.

Now, let  $B \subset K$  is a bounded set. It follows from the the continuity of  $S$  and  $(H_2)$  that there exists a positive number  $L$  such that  $\|f(Sv(t))\|_1 \leq L$  for any  $v \in B$ . Then, we can get

$$\|Tv(t)\|_1 \leq L\lambda < 1, \quad \forall t \in J, \quad v \in B.$$

So,  $T(B) \subset K$  is a bounded set in  $K$ .

For any  $\varepsilon > 0$ , by  $(H_1)$ , there exists a  $\delta' > 0$  such that

$$\int_0^{\delta'} G(s, s)p(s) \leq \frac{\varepsilon}{6L}, \quad \int_{1-\delta'}^1 G(s, s)p(s) \leq \frac{\varepsilon}{6L}.$$

Let  $P = \max_{t \in [\delta', 1-\delta']} p(t)$ . It follows from the continuity of  $G(t, s)$  on  $[0, 1] \times [0, 1]$  that there exists  $\delta > 0$  such that

$$|G(t, s) - G(t', s)| \leq \frac{\varepsilon}{3PL}, \quad |t - t'| < \delta, \quad t, t' \in [0, 1].$$

Consequently, when  $|t - t'| < \delta$ ,  $t, t' \in [0, 1]$ ,  $v \in B$ , we have

$$\begin{aligned} \|Tv(t) - Tv(t')\|_1 &= \left\| \int_0^1 (G(t, s) - G(t', s))p(s)f(Sv(s))ds \right\|_1 \\ &\leq \int_0^{\delta'} |G(t, s) - G(t', s)|p(s)\|f(Sv(s))\|_1 ds \\ &\quad + \int_{\delta^1}^{1-\delta'} |G(t, s) - G(t', s)|p(s)\|f(Sv(s))\|_1 ds \\ &\quad + \int_{1-\delta'}^1 |G(t, s) - G(t', s)|p(s)\|f(Sv(s))\|_1 ds \\ &\leq 2L \int_0^{\delta'} G(s, s)p(s)ds + 2L \int_{1-\delta'}^1 G(s, s)p(s)ds \\ &\quad + PL \int_0^1 |G(t, s) - G(t', s)|ds \\ &\leq \varepsilon. \end{aligned}$$

This implies that  $T(B)$  is equicontinuous set on  $J$ . Therefore, by Lemma 2.1, we have

$$\alpha(T(B)) = \sup_{t \in J} \alpha(T(B)(t)).$$

Without loss of generality, by condition  $(H_1)$ , we may assume that  $p(t)$  is singular at  $t = 0, 1$ . So, There exists  $\{a_{n_i}\}, \{b_{n_i}\} \subset (0, 1)$ ,  $\{n_i\} \subset N$  with  $\{n_i\}$  is a strict increasing sequence and  $\lim_{i \rightarrow +1} n_i = +1$  such that

$$\begin{aligned} 0 &< \dots < a_{n_i} < \dots < a_{n_1} < b_{n_1} < \dots < b_{n_i} < \dots < 1; \\ p(t) &\geq n_i, \quad t \in (0, a_{n_i}] \cup [b_{n_i}, 1), \quad p(a_{n_i}) = p(b_{n_i}) = n_i; \\ \lim_{i \rightarrow +1} a_{n_i} &= 0, \quad \lim_{i \rightarrow +1} b_{n_i} = 1. \end{aligned} \tag{2.9}$$

Next, we let

$$p_{n_i}(t) = \begin{cases} n_i, & t \in (0, a_{n_i}] \cup [b_{n_i}, 1); \\ p(t), & t \in [a_{n_i}, b_{n_i}]. \end{cases}$$

Then, from the above discussion we know that  $(p)_{n_i}$  is continuous on  $J$  for every  $i \in N$  and

$$p_{n_i}(t) \leq p(t); p_{n_i}(t) \rightarrow p(t), \quad \forall t \in (0, 1) \quad \text{as } i \rightarrow +\infty.$$

For any  $\varepsilon > 0$ , by (2.9) and  $(H_1)$ , there exists a  $i_0$  such that for any  $i > i_0$ , we have that

$$2L \int_0^{a_{n_i}} G(s, s)p(s)ds < \frac{\varepsilon}{2}, \quad 2L \int_{b_{n_i}}^1 G(s, s)p(s)ds < \frac{\varepsilon}{2}. \tag{2.10}$$

Therefore, for any bounded set  $B \subset C[J, E]$ , by (2.4), we have  $S(B) \subset B$ . In fact, if  $v \in B$ , there exists  $D > 0$  such that  $\|v\| \leq D, t \in J$ . Then by the properties of  $H(t, s)$ , we can have

$$\|Sv(t)\|_1 \leq \int_0^1 H(t, \tau)\|v(\tau)\|_1 d\tau \leq D \int_0^1 H(t, \tau) d\tau \leq D,$$

i.e.,  $S(B) \subset B$ .

Then, by Lemmas 2.2 and 2.3,  $(H_2)$ , the above discussion and note that  $p_{n_i}(t) \leq p(t), t \in (0, 1)$ , as  $t \in J, i > i_0$ , we know that

$$\begin{aligned} \alpha(T(B)(t)) &= \alpha \left( \left\{ \int_0^1 G(t, s)p(s)f(Sv(s))ds \in B \right\} \right) \\ &\leq \alpha \left( \left\{ \int_0^1 G(t, s)[p(s) - p_{n_i}(s)]f(Sv(s))ds \in B \right\} \right) \\ &\quad + \alpha \left( \left\{ \int_0^1 G(t, s)p_{n_i}(s)f(Sv(s))ds \in B \right\} \right) \\ &\leq 2L \int_0^{a_{n_i}} G(s, s)p(s)ds + 2L \int_{b_{n_i}}^1 G(s, s)p(s)ds \\ &\quad + \int_0^1 \alpha(G(t, s)p_{n_i}(s)f(Sv(s)) \in B)ds \\ &\leq \varepsilon + \int_0^1 G(s, s)p(s)\alpha(f(Sv(s)) \in B)ds \\ &\leq \varepsilon + 2M\lambda\alpha(B). \end{aligned}$$

Since the randomness of  $\varepsilon$ , we get

$$\alpha(T(B)(t)) \leq 2M\lambda\alpha(B), \quad t \in J. \tag{2.11}$$

So, it follows from (2.8) (2.11) that for any bounded set  $B \subset C[J, E]$ , we have

$$\alpha(T(B)) \leq 2M\lambda\alpha(B).$$

And note that  $2M\lambda < 1$ , we have  $T : K \rightarrow K$  is a strict set contraction. The proof is completed.

**Remark 1.** When  $E = \mathbf{R}$ , (2.3) naturally hold. In this case, we may take  $M$  as 0, consequently,

$T : K \rightarrow K$  is a completely continuous operator. So, our condition  $(H_1)$  is weaker than those of the above mention articles.

Our main tool of this article is the following fixed point theorem of cone.

**Theorem 2.1** [16]. Suppose that  $E$  is a Banach space,  $K \subset E$  is a cone, let  $\Omega_1, \Omega_2$  be two bounded open sets of  $E$  such that  $\theta \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ . Let operator  $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$  be strict set contraction. Suppose that one of the following two conditions hold,

- (i)  $\|Tx\| \leq \|x\|, \forall x \in K \cap \partial\Omega_1, \|Tx\| \geq \|x\|, \forall x \in K \cap \partial\Omega_2;$
- (ii)  $\|Tx\| \geq \|x\|, \forall x \in K \cap \partial\Omega_1, \|Tx\| \leq \|x\|, \forall x \in K \cap \partial\Omega_2.$

Then,  $T$  has at least one fixed point in  $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .

**Theorem 2.2** [16]. Suppose  $E$  is a real Banach space,  $K \subset E$  is a cone, let  $\Omega_r = \{u \in K : \|u\| \leq r\}$ . Let operator  $T : \Omega_r \rightarrow K$  be completely continuous and satisfy  $Tx \neq x, \forall x \in \partial\Omega_r$ . Then

- (i) If  $\|Tx\| \leq \|x\|, \forall x \in \partial\Omega_r$ , then  $i(T, \Omega_r, K) = 1;$
- (ii) If  $\|Tx\| \geq \|x\|, \forall x \in \partial\Omega_r$ , then  $i(T, \Omega_r, K) = 0.$

### 3 The main results

Denote

$$f_0 = \lim_{\|x\|_1 \rightarrow 0^+} \frac{\|f(x)\|_1}{\|x\|_1}, f_\infty = \lim_{\|x\|_1 \rightarrow \infty} \frac{\|f(x)\|_1}{\|x\|_1}.$$

In this section, we will give our main results.

**Theorem 3.1.** Suppose that conditions  $(H_1), (H_2)$  hold. Assume that  $f$  also satisfy

$$(A_1): f(x) \geq ru^*, \xi \left\| \int_0^1 H(\tau, \tau)x(\tau)d\tau \right\|_1 \leq \|x\|_1 \leq r;$$

$$(A_2): f(x) \leq Ru^*, 0 \leq \|x\|_1 \leq R,$$

where  $u^*$  and  $u^*$  satisfy

$$\rho \left\| \int_a^b G(s, s)\rho(s)u^*(s)ds \right\|_1 \geq 1, \|u_*(s)\|_1 \int_0^1 G(s, s)\rho(s)ds \leq 1.$$

Then, the boundary value problem (1.1) has a positive solution.

**Proof of Theorem 3.1.** Without loss of generality, we suppose that  $r < R$ . For any  $u \in K$ , we have

$$\|u(t)\|_1 \geq \rho\|u\|, \quad t \in [a, b]. \tag{3.1}$$

we define two open subsets  $\Omega_1$  and  $\Omega_2$  of  $E$

$$\Omega_1 = \{u \in K : \|u\| < r\}, \quad \Omega_2 = \{u \in K : \|u\| < R\}$$

For  $u \in \partial\Omega_1$ , by (3.1), we have

$$r = \|u\| \geq \|u(s)\|_1 \geq \rho \|u\| = \rho r, \quad s \in [a, b]. \tag{3.2}$$

Then, for  $u \in K \cap \partial\Omega_1$ , by (2.4), (3.2), (II), for any  $s \in [a, b]$ ,  $u \in K \cap \partial\Omega_1$ , we have

$$\begin{aligned} r &\geq \int_0^1 H(\tau, \tau) \|u(\tau)\|_1 d\tau \geq \left\| \int_0^1 H(\tau, \tau) u(\tau) d\tau \right\|_1 \geq \|Su(s)\|_1 \\ &= \left\| \int_0^1 H(s, \tau) u(\tau) d\tau \right\|_1 \geq \xi \left\| \int_0^1 H(\tau, \tau) u(\tau) d\tau \right\|_1. \end{aligned}$$

So, for  $u \in K \cap \partial\Omega_1$ , if  $(A_1)$  holds, we have

$$\|Tu(t)\|_1 = \left\| \int_0^1 G(t, s) p(s) f(Su(s)) ds \right\|_1 \geq r \rho \left\| \int_a^b G(s, s) p(s) u^*(s) r ds \right\|_1 \geq r = \|u\|.$$

Therefore, we have

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_1. \tag{3.3}$$

On the other hand, as  $u \in K \cap \partial\Omega_2$ , by (2.4), (3.2), (II), for any  $s \in [a, b]$ ,  $u \in K \cap \partial\Omega_2$ , we have

$$R \geq \int_0^1 H(\tau, \tau) \|u(\tau)\|_1 d\tau \geq \left\| \int_0^1 H(\tau, \tau) u(\tau) d\tau \right\|_1 \geq \|Su(s)\|_1 \geq 0.$$

For  $u \in K \cap \partial\Omega_2$ , if  $(A_2)$  holds, we know

$$\begin{aligned} \|Tu(t)\|_1 &= \left\| \int_0^1 G(t, s) p(s) f(Su(s)) ds \right\|_1 \leq \left\| \int_0^1 G(t, s) p(s) u_*(s) ds \right\|_1 R \\ &\leq \int_0^1 G(t, s) p(s) \|u_*(s)\|_1 ds R \leq \int_0^1 G(s, s) p(s) ds \|u_*(s)\|_1 R \leq R = \|u\|. \end{aligned}$$

Thus

$$\|T(u)\| \leq \|u\|, \quad \forall u \in \partial\Omega_2. \tag{3.4}$$

Therefore, by (3.2), (3.3), Lemma 2.5 and  $r < R$ , we have that  $T$  has a fixed point  $v \in (\Omega_2 \setminus \bar{\Omega}_1)$ . Obviously,  $v$  is positive solution of problem (2.6).

Now, by Lemma 2.4 we see that  $u = Sv$  is a position solution of BVP (1.1). The proof of Theorem 3.1 is complete.

**Theorem 3.2.** Suppose that conditions  $(H_1)$ ,  $(H_2)$  and  $(A_1)$  in Theorem 3.1 hold. Assume that  $f$  also satisfy

$$(A_3): f_0 = 0;$$

$$(A_4): f_\infty = 0.$$

Then, the boundary value problem (1.1) have at least two solutions.



**Proof of Theorem 3.2.** First, by condition  $(A_3)$ , (2.4) and the property of limits, we can have

$\lim_{\|u\|_1 \rightarrow 0^+} \|f(Su)\|_1 / \|u\|_1 = 0$ . Then, for any  $m > 0$  such that  $m \int_a^b G(s,s)p(s)ds \leq 1$ , there exists a constant  $\rho_* \in (0, r)$  such that

$$\|f(Su)\|_1 \leq m\|u\|_1, \quad 0 < \|u\|_1 \leq \rho_*, \quad u \neq 0. \tag{3.5}$$

Set  $\Omega_{\rho_*} = \{u \in K : \|u\|_1 < \rho_*\}$ , for any  $u \in K \cap \partial\Omega_{\rho_*}$ , by (3.4), we have

$$\|f(Su)\|_1 \leq m\|u\|_1 \leq m\rho_*.$$

For  $u \in K \cap \partial\Omega_{\rho_*}$ , we have

$$\begin{aligned} \|Tu(t)\|_1 &= \left\| \int_0^1 G(t,s)p(s)f(Su(s))ds \right\|_1 \leq \int_0^1 G(t,s)p(s)\|f(Su(s))\|_1 ds \\ &\leq \int_a^b G(t,s)p(s)m\rho_* ds \leq \rho_* m \int_a^b G(s,s)p(s)ds \leq \rho_* = \|u\|. \end{aligned}$$

Therefore, we can have

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}.$$

Then by Theorem 2.2, we have

$$i(T, \Omega_{\rho_*}, K) = 1. \tag{3.6}$$

Next, by condition  $(A_4)$ , (2.4) and the property of limits, we can have  $\lim_{\|u\|_1 \rightarrow 1} \|f(Su)\|_1 / \|u\|_1 = 0$ . Then, for any  $\bar{m} > 0$  such that  $\bar{m} \int_a^b G(s,s)p(s)ds \leq 1$ , there exists a constant  $\rho_0 > 0$  such that

$$\|f(Su)\|_1 \leq \bar{m}\|u\|_1, \quad \|u\|_1 > \rho_0. \tag{3.7}$$

We choose a constant  $\rho^* > \max\{r, \rho_0\}$ , obviously,  $\rho_* < r < \rho^*$ . Set  $\Omega_{\rho^*} = \{u \in K : \|u\|_1 < \rho^*\}$ , for any  $u \in K \cap \partial\Omega_{\rho^*}$ , by (3.6), we have

$$\|f(Su)\|_1 \leq \bar{m}\|u\|_1 \leq \bar{m}\rho^*.$$

For  $u \in K \cap \partial\Omega_{\rho^*}$ , we have

$$\begin{aligned} \|Tu(t)\|_1 &= \left\| \int_0^1 G(t,s)p(s)g(s)f(Su(s))ds \right\|_1 \leq \int_0^1 G(t,s)p(s)g(s)\|f(Su(s))\|_1 ds \\ &\leq \int_a^b G(t,s)p(s)\bar{m}\rho^* ds \leq \rho^* \bar{m} \int_a^b G(s,s)p(s)ds \leq \rho^* = \|u\|. \end{aligned}$$

Therefore, we can have

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{\rho^*}.$$

Then by Theorem 2.2, we have

$$i(T, \Omega_{\rho^*}, K) = 1. \tag{3.8}$$

Finally, set  $\Omega_r = \{u \in K : \|u\| < r\}$ , For any  $u \in \partial\Omega_r$ , by  $(A_2)$ , Lemma 2.2 and also similar to the latter proof of Theorem 3.1, we can also have

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_r.$$

Then by Theorem 2.2, we have

$$i(T, \Omega_r, K) = 0. \tag{3.9}$$

Therefore, by (3.5), (3.7), (3.8), and  $\rho_* < r < \rho^*$ , we have

$$i(T, \Omega_r \setminus \bar{\Omega}_{\rho_*}, k) = -1, i(T, \Omega_{\rho^*} \setminus \bar{\Omega}_r, k) = 1.$$

Then  $T$  have fixed point  $v_1 \in \Omega_r \setminus \bar{\Omega}_{\rho_*}$ , and fixed point  $v_2 \in \Omega_{\rho^*} \setminus \bar{\Omega}_r$ . Obviously,  $v_1, v_2$  are all positive solutions of problem (2.6).

Now, by Lemma 2.4 we see that  $u_1 = Sv_1, u_2 = Sv_2$  are two position solutions of BVP (1.1). The proof of Theorem 3.2 is complete.

#### 4 Application

In this section, in order to illustrate our results, we consider some examples.

Now, we consider the following concrete second-order singular BVP (SBVP)

**Example 4.1.** Consider the following SBVP

$$\begin{cases} \frac{3}{\sqrt[3]{t}} \left( \frac{1}{3} \sqrt[3]{tu}'''(t) \right)' + 160 \left[ u^{\frac{1}{2}} + u^{\frac{1}{3}} \right] = \theta, & 0 < t < 1, \\ u(0) - 3u'(0) = \theta, & u(1) + 2u'(1) = \theta, \\ 3u''(0) - \lim_{t \rightarrow 0^+} \frac{1}{3} \sqrt[3]{tu}'''(t) = \theta, & u''(1) + \lim_{t \rightarrow 1^-} \frac{1}{3} \sqrt[3]{tu}'''(t) = \theta, \end{cases} \tag{4.1}$$

where

$$\alpha_1 = \gamma_1 = 1, \beta_1 = 3, \delta_1 = 2, \beta_2 = \gamma_2 = \delta_2 = 1, \alpha_2 = 3, \\ p(t) = \frac{1}{3} \sqrt[3]{t}, f(u) = 160(u^{\frac{1}{2}} + u^{\frac{1}{3}}).$$

Then obviously,

$$\int_0^1 \frac{1}{p(t)} dt = \frac{3}{2}, f_\infty = 0, f_0 = 0,$$

By computing, we know that the Green's function are

$$H(t, s) = \frac{1}{6} \begin{cases} (3+s)(3-t), & 0 \leq s \leq t \leq 1, \\ (3+t)(3-s), & 0 \leq t \leq s \leq 1. \end{cases} \\ G(t, s) = \frac{1}{7} \begin{cases} (1+3s)(2-t), & 0 \leq s \leq t \leq 1, \\ (1+3t)(2-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

It is easy to note that  $0 \leq G(s, s) \leq 1$  and conditions  $(H_1), (H_2), (A_3), (A_4)$  hold.

Next, by computing, we know that  $\rho = 0.44, \zeta = 0.8$ . We choose  $r = 3, u^* = 104$ , as

$$1.05 = \rho \zeta r \leq \|u\| = \max\{u(t), t \in J\} \leq 3 \text{ and } \rho \left\| \int_a^b G(s, s) p(s) u^*(s) ds \right\| = 1.3 > 1,$$

because of the monotone increasing of  $f(u)$  on  $[0, \infty)$ , then

$$f(u) \geq f(1.05) = 326.4, \quad 1.05 \leq \|u\| \leq 3.$$

Therefore, as

$$\rho \xi r = \rho \xi \left\| \int_0^1 H(\tau, \tau) \|u\| d\tau \right\| \leq \xi \left\| \int_0^1 H(\tau, \tau) u(\tau) d\tau \right\|,$$

so we have

$$f(u) \geq ru^*, \xi \left\| \int_0^1 H(\tau, \tau) u(\tau) d\tau \right\| \leq \|u\| \leq r,$$

then conditions  $(A_1)$  holds. Then by Theorem 3.2, SBVP (4.1) has at least two positive solutions  $u_1, u_2$  and  $0 < \|u_1\| < 3 < \|u_2\|$ .

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All authors contributed equally to the manuscript and read and approved the final manuscript.

#### Competing interests

We declare that we have no significant competing financial, professional, or personal interests that might have influenced the performance or presentation of the work described in this manuscript.

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