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Homoclinic solutions of some second-order non-periodic discrete systems

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Abstract

In this article, we discuss how to use a standard minimizing argument in critical point theory to study the existence of non-trivial homoclinic solutions of the following second-order non-autonomous discrete systems

$$\Delta^2 x_{n-1} + A\Delta x_n - L(n)x_n + \nabla W(n, x_n) = 0, \quad n \in \mathbf{Z},$$

without any periodicity assumptions. Adopting some reasonable assumptions for A and L, we establish that two new criterions for guaranteeing above systems have one non-trivial homoclinic solution. Besides that, in some particular case, for the first time the uniqueness of homoclinic solutions is also obtained.

MSC: 39A11.

Keywords: homoclinic solution, variational functional, critical point, subquadratic second-order discrete system

1. Introduction

The theory of nonlinear discrete systems has widely been used to study discrete models appearing in many fields such as electrical circuit analysis, matrix theory, control theory, discrete variational theory, etc., see for example [1,2]. Since the last decade, there have been many literatures on qualitative properties of difference equations, those studies cover many branches of difference equations, see [3-7] and references therein. In the theory of differential equations, homoclinic solutions, namely doubly asymptotic solutions, play an important role in the study of various models of continuous dynamical systems and frequently have tremendous effects on the dynamics of nonlinear systems. So, homoclinic solutions have extensively been studied since the time of Poincaré, see [8-13]. Similarly, we give the following definition: if x_n is a solution of a discrete system, x_n will be called a homoclinic solution emanating from 0 if $x_n \to 0$ as $|n| \to +\infty$. If $x_n \neq 0$, x_n is called a non-trivial homoclinic solution.

For our convenience, let $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{R}$ be the set of all natural numbers, integers, and real numbers, respectively. Throughout this article, $|\cdot|$ denotes the usual norm in $\mathbf{R}^N$ with $N \in \mathbf{N}$, $(\cdot,\cdot)$ stands for the inner product. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \ldots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \ldots, \}$ when $a \leq b$.

In this article, we consider the existence of non-trivial homoclinic solutions for the following second-order non-autonomous discrete system

$$\Delta^2 x_{n-1} + A\Delta x_n - L(n)x_n + \nabla W(n, x_n) = 0 \tag{1.1}$$

without any periodicity assumptions, where A is an antisymmetric constant matrix, $L(n) \in C^1(\mathbf{R}, \mathbf{R}^{N\times N})$ is a symmetric and positive definite matrix for all $n \in \mathbf{Z}$, $W(n, x_n) = a(n)V(x_n)$, and $a: \mathbf{R} \to \mathbf{R}^+$ is continuous and $V \in C^1(\mathbf{R}^N, \mathbf{R})$. The forward difference operator Δ is defined by $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = \Delta(\Delta x_n)$.

We may think of (1.1) as being a discrete analogue of the following second-order non-autonomous differential equation

$$x'' + Ax' - L(t)x + W_x(t, x) = 0 \tag{1.2}$$

(1.1) is the best approximations of (1.2) when one lets the step size not be equal to 1 but the variable's step size go to zero, so solutions of (1.1) can give some desirable numerical features for the corresponding continuous system (1.2). On the other hand, (1.1) does have its applicable setting as evidenced by monographs [14,15], as mentioned in which when $A = 0$, (1.1) becomes the second-order self-adjoint discrete system

$$\Delta^2 x_{n-1} - L(n)x_n + \nabla W(n, x_n) = 0, \quad n \in \mathbf{Z}, \tag{1.3}$$

which is in some way a type of the best expressive way of the structure of the solution space for recurrence relations occurring in the study of second-order linear differential equations. So, (1.3) arises with high frequency in various fields such as optimal control, filtering theory, and discrete variational theory and many authors have extensively studied its disconjugacy, disfocality, boundary value problem oscillation, and asymptotic behavior. Recently, Bin [16] studied the existence of non-trivial periodic solutions for asymptotically superquadratic and subquadratic system (1.1) when $A = 0$. Ma and Guo [17,18] gave results on existence of homoclinic solutions for similar system (1.3). In this article, we establish that two new criterions for guaranteeing the above system have one non-trivial homoclinic solution by adopting some reasonable assumptions for A and L. Besides that, in some particular case, we obtained the uniqueness of homoclinic solution for the first time.

Now we present some basic hypotheses on L and W in order to announce our first result in this article.

(H_1) $L(n) \in C^1(\mathbf{Z}, \mathbf{R}^{N\times N})$ is a symmetric and positive definite matrix and there exists a function α: $\mathbf{Z} \to \mathbf{R}^+$ such that $(L(n)x, x) \geq \alpha(n)|x|^2$ and $\alpha(n) \to +\infty$ as $|n| \to +\infty$;

(H_2) $W(n, x) = a(n)\ |x|^\gamma$, i.e., $V(x) = |x|^\gamma$, where $a: \mathbf{Z} \to \mathbf{R}$ such that $a(n_0) > 0$ for some $n_0 \in \mathbf{Z}$, $1 < \gamma < 2$ is a constant.

Remark 1.1 From (H_1), there exists a constant $\beta > 0$ such that

$$(L(n)x, x) \geq \beta\ |\ x|^2, \quad \forall n \in \mathbf{Z}, \quad x \in \mathbf{R}^N, \tag{1.4}$$

and by (H_2), we see $V(x)$ is subquadratic as $|x| \to +\infty$ and

$$\nabla W(n, x) = \gamma a(n)\ |\ x|^{\gamma-2}x \tag{1.5}$$

In addition, we need the following estimation on the norm of A. Concretely, we suppose that (H_3) A is an antisymmetric constant matrix such that $\| A \| < \sqrt{\beta}$, where β is defined in (1.4).

Remark 1.2 In order to guarantee that (H_3) holds, it suffices to take A such that $||A||$ is small enough.

Up until now, we can state our first main result.

Theorem 1.1 If (H_1)-(H_3) are hold, then (1.1) possesses at least one non-trivial homoclinic solution.

Substitute (H_2)' by (H_2) as follows

(H_2)' $W(n, x) = a(n)V(x)$, where a: $\mathbf{Z} \to \mathbf{R}$ such that $a(n_1) > 0$ for some $n_1 \in \mathbf{Z}$ and $V \in C^1(\mathbf{R}^N, \mathbf{R})$, and $V(0) = 0$. Moreover, there exist constants $M > 0$, $M_1 > 0$, $1 < \theta < 2$ and $0 < r \leq 1$ such that

$$V(x) \geq M \mid x|^{\theta}, \quad \forall x \in \mathbf{R}^N, \quad \mid x \mid \leq r \tag{1.6}$$

and

$$\mid V'(x) \mid \leq M_1, \quad \forall x \in \mathbf{R}^N. \tag{1.7}$$

Remark 1.3 By $V(0) = 0$, $V \in C^1(\mathbf{R}^N, \mathbf{R})$ and (1.7), we have

$$\mid V(x) \mid = \left| \int_0^1 (V'(\mu x), x) d\mu \right| \leq M_1 \mid x \mid, \tag{1.8}$$

which yields that $V(x)$ is subquadratic as $|x| \to +\infty$.

We have the following theorem.

Theorem 1.2 Assume that (H_1), (H_2)' and (H_3) are satisfied, then (1.1) possesses at least one non-trivial homoclinic solution. Moreover, if we suppose that $V \in C^2(\mathbf{R}^N, \mathbf{R})$ and there exists constant ω with $0 < \omega < \beta - \sqrt{\beta} \parallel A \parallel$ such that

$$\parallel a(n)V''(x) \parallel_2 \leq \omega, \qquad \forall n \in \mathbf{Z}, \quad x \in \mathbf{R}^N, \tag{1.9}$$

then (1.1) has one and only one non-trivial homoclinic solution.

The remainder of this article is organized as follows. After introducing some notations and preliminary results in Section 2, we establish the proofs of our Theorems 1.1 and 1.2 in Section 3.

2. Variational structure and preliminary results

In this section, we are going to establish suitable variational structure of (1.1) and give some lemmas which will be fundamental importance in proving our main results. First, we state some basic notations.

Letting

$$E = \left\{ x \in S : \sum_{n \in \mathbf{Z}} [(\Delta x_n)^2 + (L(n)x_n, x_n)] < +\infty \right\},$$

where

$$S = \{x = \{x_n\} : x_n \in \mathbf{R}^N, n \in \mathbf{Z}\}$$

and

$$x = \{x_n\}_{n \in Z} = \{\ldots, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots\}.$$

According to the definition of the space E, for all $x, y \in E$ there holds

$$\begin{aligned}
&\sum_{n\in\mathbf{Z}}[(\Delta x_n, \Delta y_n) + (L(n)x_n, y_n)] \\
&\quad = \sum_{n\in\mathbf{Z}}[(\Delta x_n, \Delta y_n) + (L^{\frac{1}{2}}(n)x_n, L^{\frac{1}{2}}(n)y_n)] \\
&\quad \le \left(\sum_{n\in\mathbf{Z}}(|\Delta x_n|^2 + |L^{\frac{1}{2}}(n)x_n|^2)\right)^{\frac{1}{2}} \cdot \left(\sum_{n\in\mathbf{Z}}(|\Delta y_n|^2 + |L^{\frac{1}{2}}(n)y_n|^2)\right)^{\frac{1}{2}} < +\infty.
\end{aligned}$$

Then $(E, <\cdot, \cdot>)$ is an inner space with

$$< x, y > = \sum_{n\in\mathbf{Z}}[(\Delta x_n, \Delta y_n) + (L(n)x_n, y_n)], \quad \forall x, y \in E$$

and the corresponding norm

$$\| x\|^2 = \sum_{n\in\mathbf{Z}}[(\Delta x_n)^2 + (L(n)x_n, x_n)], \quad \forall x \in E.$$

Furthermore, we can get that E is a Hilbert space. For later use, given $\beta > 0$, define $l^{\beta} = \{x = \{x_n\} \in S : \sum_{n\in\mathbf{Z}} |x_n|^{\beta} < +\infty\}$ and the norm

$$\| x\|_{l^{\beta}} = \sqrt[\beta]{\sum_{n\in\mathbf{Z}} |x_n|^{\beta}} = \| x\|_{\beta}.$$

Write $l^{\infty} = \{x = \{x_n\} \in S: |x_n| < +\infty\}$ and

$$\| x\|_{l^{\infty}} = \sup_{n\in\mathbf{Z}} | x_n | .$$

Making use of Remark 1.1, there exists

$$\beta \| x \|_{l^2}^2 = \beta \sum_{n\in\mathbf{Z}} | x_n|^2 \le \sum_{n\in\mathbf{Z}}[(\Delta x_n)^2 + (L(n)x_n, x_n)] = \| x\|^2,$$

then

$$\| x\|_{l^{\infty}} \le \| x\|_{l^2} \le \beta^{-\frac{1}{2}} \| x \| \tag{2.1}$$

Lemma 2.1 Assume that L satisfies (H_1), $\{x^{(k)}\} \subset E$ such that $x^{(k)} \rightharpoonup x$. Then $x^{(k)} \rightharpoonup x$ in l^2.

Proof Without loss of generality, we assume that $x^{(k)} \rightharpoonup 0$ in E. From (H_1) we have $\alpha(n) > 0$ and $\alpha(n) \to +\infty$ as $n \to \infty$, then there exists $D > 0$ such that $|\frac{1}{\alpha(n)}| = \frac{1}{\alpha(n)} \le \varepsilon$ holds for any $\varepsilon > 0$ as $|n| > D$.

Let $I = \{n: |n| \le D, n \in \mathbf{Z}\}$ and $E_I = \{x \in E : \sum_{n\in I}[(\Delta x_n)^2 + L(n)x_n \cdot x_n] < +\infty\}$, then E_I is a $2DN$-dimensional subspace of E and clearly $x^{(k)} \rightharpoonup 0$ in E_I. This together with the uniqueness of the weak limit and the equivalence of strong convergence and weak convergence in E_I, we have $x^{(k)} \to 0$ in E_I, so there has a constant $k_0 > 0$ such that

$$\sum_{n\in I} | x_n^{(k)}|^2 \le \varepsilon, \quad \forall k \ge k_0. \tag{2.2}$$

By (H_1), there have

$$\begin{aligned}\sum_{|n|>D} |x_n^{(k)}|^2 &= \sum_{|n|>D} \frac{1}{\alpha(n)} \cdot \alpha(n) |x_n^{(k)}|^2 \\ &\leq \varepsilon \sum_{|n|>D} \alpha(n) |x_n^{(k)}|^2 \leq \varepsilon \sum_{|n|>D} (L(n)x_n^{(k)}, x_n^{(k)}) \\ &\leq \varepsilon \sum_{|n|>D} [(\Delta x_n^{(k)})^2 + (L(n)x_n^{(k)}, x_n^{(k)})] = \varepsilon \|x^{(k)}\|^2.\end{aligned}$$

Note that ε is arbitrary and $||x^{(k)}||$ is bounded, then

$$\sum_{|n|>D} |x_n^{(k)}|^2 \to 0, \tag{2.3}$$

combing with (2.2) and (2.3), $x^{(k)} \to 0$ in l^2 is true.

In order to prove our main results, we need following two lemmas.

Lemma 2.2 For any $x(j) > 0$, $y(j) > 0$, $j \in \mathbf{Z}$ there exists

$$\sum_{j\in\mathbf{Z}} x(j)y(j) \leq \left(\sum_{j\in\mathbf{Z}} x^q(j)\right)^{\frac{1}{q}} \cdot \left(\sum_{j\in\mathbf{Z}} y^s(j)\right)^{\frac{1}{s}},$$

where $q > 1$, $s > 1$, $\frac{1}{q} + \frac{1}{s} = 1$.

Lemma 2.3 [19] Let E be a real Banach space and $F \in C^1(E, \mathbf{R})$ satisfying the PS condition. If F is bounded from below, then

$$c = \inf_E F$$

is a critical point of F.

3. Proofs of main results

In order to obtain the existence of non-trivial homoclinic solutions of (1.1) by using a standard minimizing argument, we will establish the corresponding variational functional of (1.1). Define the functional $F: E \to \mathbf{R}$ as follows

$$\begin{aligned}F(x) &= \sum_{n\in\mathbf{Z}} \left[\frac{1}{2}(\Delta x_n)^2 + \frac{1}{2}(L(n)x_n, x_n) + \frac{1}{2}(Ax_n, \Delta x_n) - W(n, x_n)\right] \\ &= \frac{1}{2}\|x\|^2 + \frac{1}{2}\sum_{n\in\mathbf{Z}} (Ax_n, \Delta x_n) - \sum_{n\in\mathbf{Z}} W(n, x_n).\end{aligned} \tag{3.1}$$

Lemma 3.1 Under conditions of Theorem 1.1, we have $F \in C^1(E, \mathbf{R})$ and any critical point of F on E is a classical solution of (1.1) with $x_{\pm\infty} = 0$.

Proof We first show that $F: E \to \mathbf{R}$. By (1.4), (2.1), (H_2), and Lemma 2.2, we have

$$\begin{aligned}0 &\leq \sum_{n\in\mathbf{Z}} |W(n, x_n)| = \sum_{n\in\mathbf{Z}} |a(n)||x_n|^\gamma \\ &\leq \left(\sum_{n\in\mathbf{Z}} |a(n)|^{\frac{2}{2-\gamma}}\right)^{\frac{2-\gamma}{2}} \left(\sum_{n\in\mathbf{Z}} |x_n|^{\gamma\frac{2}{\gamma}}\right)^{\frac{\gamma}{2}} \\ &= \|a(n)\|_{\frac{2}{2-\gamma}} \|x\|_2^\gamma \leq \beta^{-\frac{\gamma}{2}} \|a(n)\|_{\frac{2}{2-\gamma}} \|x\|^\gamma \\ &< +\infty\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we show that $F: E \to \mathbf{R}$.

Next we prove $F \in C^1(E, \mathbf{R})$. Write $F_1(x) = \frac{1}{2} \| x\|^2 + \frac{1}{2} \sum_{n\in\mathbf{Z}} (Ax_n, \Delta x_n)$, $F_2(x) = \sum_{n\in\mathbf{Z}} W(n, x_n)$, it is obvious that $F(x) = F_1(x) - F_2(x)$ and $F_1(x) \in C^1(E, \mathbf{R})$. And by use of the antisymmetric property of A, it is easy to check

$$< F_1'(x), y >= \sum_{n\in\mathbf{Z}} [(\Delta x_n, \Delta y_n) + (Ax_n, \Delta y_n) + (L(n)x_n, y_n)], \quad \forall y \in E. \tag{3.3}$$

Therefore, it is sufficient to show that $F_2(x) \in C^1(E, \mathbf{R})$.

Because of $V(x) = |x|^\gamma$, i.e., $V \in C^1(\mathbf{R}^N, \mathbf{R})$, let us write $\phi(t) = F_2(x + th)$, $0 \le t \le 1$, for all $x, h \in E$, there holds

$$\begin{aligned}
\varphi'(0) &= \lim_{t\to 0} \frac{\varphi(t) - \varphi(0)}{t} \\
&= \lim_{t\to 0} \frac{F_2(x + th) - F_2(x)}{t} \\
&= \lim_{t\to 0} \frac{1}{t} \sum_{n\in\mathbf{Z}} [V(n, x_n + th_n) - V(n, x_n)] \\
&= \lim_{t\to 0} \sum_{n\in\mathbf{Z}} \nabla V(n, x_n + \theta_n th_n) \cdot h_n \\
&= \sum_{n\in\mathbf{Z}} \nabla V(n, x_n) \cdot h_n
\end{aligned}$$

where $0 <\theta_n < 1$. It follows that $F_2(x)$ is Gateaux differentiable on E.

Using (1.5) and (2.1), we get

$$\begin{aligned}
|\nabla W(n, x_n)| &=| \gamma a(n) \mid x_n|^{\gamma-2} x_n \mid = \gamma a(n) \mid x_n|^{\gamma-1} \\
&\le \gamma a(n) \parallel x \parallel_{l^\infty}^{\gamma-1} \le \gamma a(n) \beta^{-\frac{1}{2}} \parallel x\|^{\gamma-1} \\
&= da(n)
\end{aligned} \tag{3.4}$$

where $d = \gamma \beta^{-\frac{1}{2}} \parallel x\|^{\gamma-1}$ is a constant. For any $y \in E$, using (2.1), (3.4) and lemma 2.2, it follows

$$\begin{aligned}
&| \sum_{n\in\mathbf{Z}} (\nabla W(n, x_n), y_n)| \le \sum_{n\in\mathbf{Z}} da(n)|y_n| \\
&= d \sum_{n\in\mathbf{Z}} a(n)|y_n| \le d\left(\sum_{n\in\mathbf{Z}} |a(n)|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbf{Z}} |y_n|^2\right)^{\frac{1}{2}} \\
&\le d\|a(n)\|_2 \left(\sum_{n\in\mathbf{Z}} \frac{1}{\beta}(L(n)y_n, y_n)\right)^{\frac{1}{2}} \\
&\le \frac{d}{\sqrt{\beta}} \|a(n)\|_2 \|y\|
\end{aligned}$$

thus the Gateaux derivative of $F_2(x)$ at x is $F_2'(x) \in E$ and

$$< F_2'(x), y >= \sum_{n\in\mathbf{Z}} (\nabla W(n, x_n), y_n), \quad \forall x, y \in E.$$

For any $y \in E$ and $\varepsilon > 0$, when $||y|| \leq \delta$, i.e., $|y| \leq \alpha^{-\frac{1}{2}}\delta$ there exists $\delta > 0$ such that

$$| \nabla W(n, x_n + y_n) - \nabla W(n, x_n) | < \varepsilon.$$

is true. Therefore,

$$\begin{aligned} | < F_2'(x+y) - F_2'(x), h > | &= |\sum_{n\in\mathbf{Z}} (\nabla W(n, x_n + y_n) - \nabla W(n, x_n), h_n)| \\ &\leq \varepsilon \sum_{n\in\mathbf{Z}} |h_n| \leq \varepsilon\beta^{-\frac{1}{2}} \|h\|, \end{aligned}$$

that is

$$\| F_2'(x+y) - F_2'(x) \| \leq \varepsilon\beta^{-\frac{1}{2}}.$$

Note that ε is arbitrary, then $F_2' : E \to E'$, $x \to F_2'(x)$ is continuous and $F_2(x) \in C^1(E, \mathbf{R})$. Hence, $F \in C^1(E, \mathbf{R})$ and for any $x, h \in E$, we have

$$\begin{aligned} < F'(x), h > &= < x, h > - \sum_{n\in\mathbf{Z}} (\nabla W(n, x_n), h_n) \\ &= \sum_{n\in\mathbf{Z}} [(-(\Delta x_{n-1})^2 + (Ax_n, \Delta x_n) + (L(n)x_n, x_n) - \nabla W(n, x_n), h_n)] \end{aligned}$$

that is

$$< F'(x), x > = \| x\|^2 - \sum_{n\in\mathbf{Z}} (\nabla W(n, x_n), x_n) \tag{3.5}$$

Computing Fréchet derivative of functional (3.1), we have

$$\frac{\partial F(x)}{\partial x(n)} = -\Delta^2 x_{n-1} - A\Delta x_n + L(n)x_n - \nabla W(n, x_n), n \in \mathbf{Z}$$

this is just (1.1). Then critical points of variational functional (3.1) corresponds to homoclinic solutions of (1.1)

Lemma 3.2 Suppose that (H_1), (H_2) in Theorem 1.1 are satisfied. Then, the functional (3.1) satisfies PS condition.

Proof Let $\{x^{(k)}\}_{k\in\mathbf{N}} \subset E$ be such that $\{F(x^{(k)})\}_{k\in\mathbf{N}}$ is bounded and $\{F'(x^{(k)})\} \to 0$ as $k \to +\infty$. Then there exists a positive constant c_1 such that

$$| F(x^{(k)}) | \leq c_1, \qquad \| F'(x^{(k)})\|_{E'} \leq c_1, \qquad \forall k \in \mathbf{N}. \tag{3.6}$$

Firstly, we will prove $\{x^{(k)}\}_{k\in\mathbf{N}}$ is bounded in E. Combining (3.1), (3.5) and remark 1.1, there holds

$$\begin{aligned} (1 - \frac{\mu}{2}) \| x^{(k)}\|^2 &= < F'(x^{(k)}), x^{(k)} > -\mu F(x^{(k)}) \\ &\quad + \sum_{n\in\mathbf{Z}} [(\nabla W(n, x_n^{(k)}), x_n^{(k)}) - \mu W(n, x_n^{(k)})] \\ &\leq < F'(x^{(k)}), x^{(k)} > -\mu F(x^{(k)}) \end{aligned}$$

together with (3.6)

$$(1-\frac{\mu}{2})\ \|x^{(k)}\|^2 \le c_1 \|x^{(k)}\| + \mu c_1. \tag{3.7}$$

Since $1 < \mu < 2$, it is not difficult to know that $\{x^{(k)}\}_{k\in \mathbf{N}}$ is a bounded sequence in E. So, passing to a subsequence if necessary, it can be assumed that $x^{(k)} \rightharpoonup x$ in E. Moreover, by Lemma 2.1, we know $x^{(k)} \rightharpoonup x$ in l^2. So for $k \to +\infty$,

$$< F'(x^{(k)}) - F'(x), x^{(k)} - x > \to 0,$$

and

$$\sum_{n\in \mathbf{Z}} (\nabla W(n, x_n^{(k)}) - \nabla W(n, x_n), x_n^{(k)} - x_n) \to 0.$$

On the other hand, by direct computing, for k large enough, we have

$$\begin{aligned} &< F'(x^{(k)}) - F'(x), x^{(k)} - x > \\ &= \|x^{(k)} - x\|^2 - \sum_{n\in \mathbf{Z}} (\nabla W(n, x_n^{(k)}) - \nabla W(n, x_n), x_n^{(k)} - x_n). \end{aligned}$$

It follows that

$$\|x^{(k)} - x\| \to 0,$$

that is the functional (3.1) satisfies PS condition.

Up until now, we are in the position to give the proof of Theorem 1.1.

Proof of Theorem 1.1 By (3.1), we have, for every $m \in \mathbf{R} \setminus \{0\}$ and $x \in E \setminus \{0\}$,

$$\begin{aligned} F(mx) &= \frac{m^2}{2}\|x\|^2 + \frac{m^2}{2}\sum_{n\in \mathbf{Z}}(Ax_n, \Delta x_n) - \sum_{n\in \mathbf{Z}} W(n, mx_n) \\ &= \frac{m^2}{2}\|x\|^2 + \frac{m^2}{2}\sum_{n\in \mathbf{Z}}(Ax_n, \Delta x_n) - |m|^\gamma \sum_{n\in \mathbf{Z}} a(n)|x_n|^\gamma \\ &\ge \frac{m^2}{2}\|x\|^2 - \frac{m^2}{2}\beta^{-\frac{1}{2}}\|A\|\|x\|^2 - \beta^{-\frac{\gamma}{2}}|m|^\gamma \|a(n)\|_{\frac{2-\gamma}{2}}\|x\|^\gamma. \end{aligned} \tag{3.8}$$

Since $1 < \gamma < 2$ and $\|A\| < \sqrt{\beta}$, (3.8) implies that $F(mx) \to +\infty$ as $|m| \to +\infty$. Consequently, $F(x)$ is a functional bounded from below. By Lemma 2.3, $F(x)$ possesses a critical value $c = \inf_{x\in E} F(x)$, i.e., there is a critical point $x \in E$ such that

$$F(x) = c, \qquad F'(x) = 0.$$

On the other side, by (H_2), there exists $\delta_0 > 0$ such that $a(n) > 0$ for any $n \in [n_0 - \delta_0, n_0 + \delta_0]$. Take $c_0 \in \mathbf{R}^N \setminus \{0\}$ and let $y \in E$ be given by

$$y_n = \begin{cases} c_0 \sin[\frac{2\pi}{2\delta_0}(n - n_1)], & n \in [n_0 - \delta_0, n_0 + \delta_0] \\ 0, & n \in \mathbf{Z}\setminus[n_0 - \delta_0, n_0 + \delta_0] \end{cases}$$

Then, by (3.1), we obtain that

$$F(my) = \frac{m^2}{2}\|y\|^2 + \frac{m^2}{2}\beta^{-\frac{1}{2}}\|A\|\|y\|^2 - |m|^\gamma \sum_{n=n_0-\delta_0}^{n_0+\delta_0} a(n)|y_n|^\gamma,$$

which yields that $F(my) < 0$ for $|m|$ small enough since $1 < \gamma < 2$, i.e., the critical point $x \in E$ obtained above is non-trivial.

Although the proof of the first part of Theorem 1.2 is very similar to the proof of Theorem 1.1, for readers' convenience, we give its complete proof.

Lemma 3.3 Under the conditions of Theorem 1.2, it is easy to check that

$$< F'(x), y > = \sum_{n\in\mathbf{Z}} [(\Delta x_n, \Delta y_n) + (Ax_n, \Delta y_n) + (L(n)x_n, y_n) - (\nabla W(n, x_n), y_n)] \tag{3.9}$$

for all $x, y \in E$. Moreover, $F(x)$ is a continuously Fréchet differentiable functional defined on E, i.e., $F \in C^1(E, \mathbf{R})$ and any critical point of $F(x)$ on E is a classical solution of (1.1) with $x_{\pm\infty} = 0$.

Proof By (1.8) and (2.1), we have

$$\begin{aligned}
0 &\le \sum_{n\in\mathbf{Z}} | W(n, x_n) | = \sum_{n\in\mathbf{Z}} | a(n) | \cdot | V(x_n) | \le M_1 \sum_{n\in\mathbf{Z}} | a(n) | \cdot | x_n | \\
&\le M_1 \left(\sum_{n\in\mathbf{Z}} | a(n) |^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{n\in\mathbf{Z}} | x_n |^2 \right)^{\frac{1}{2}} = M_1 \| a\|_2 \| x\|_2 \\
&\le \beta^{-\frac{1}{2}} M_1 \| a\|_2 \| x \|,
\end{aligned}$$

which together with (3.1) implies that $F: E \to \mathbf{R}$. In the following, according to the proof of Lemma 3.1, it is sufficient to show that for any $y \in E$,

$$\sum_{n\in\mathbf{Z}} (\nabla W(n, x_n), y_n), \qquad \forall x \in E$$

is bounded. Moreover, By (1.8), (2.1), and Lemma 2.2, there holds

$$\begin{aligned}
| \sum_{n\in\mathbf{Z}} (\nabla W(n, x_n), y_n) | &\le \sum_{n\in\mathbf{Z}} | \nabla W(n, x_n) | \cdot | y_n | \\
&\le M_1 \sum_{n\in\mathbf{Z}} | a(n) | \cdot | x_n | \cdot | y_n | \\
&\le M_1 \| a\|_2 \| x\|_2 \| y\|_2 \\
&\le M_1 \beta^{-1} \| a\|_2 \| x \| \| y \|
\end{aligned}$$

which implies that $\sum_{n\in\mathbf{Z}} (\nabla W(n, x_n), y_n)$ is bounded for any $x, y \in E$.

Using Lemma 2.1, the remainder is similar to the proof of Lemma 3.1, so we omit the details of its proof.

Lemma 3.4 Under the conditions of Theorem 1.2, $F(x)$ satisfies the PS condition.

Proof From the proof of Lemma 3.2, we see that it is sufficient to show that for any sequence $\{x^{(k)}\}_{k\in\mathbf{N}} \subset E$ such that $\{F(x^{(k)})\}_{k\in\mathbf{N}}$ is bounded and $F'(x^{(k)}) \to 0$ as $k \to +\infty$, then $\{x^{(k)}\}_{k\in\mathbf{N}}$ is bounded in E.

In fact, since $\{F(x^{(k)})\}_{k\in\mathbf{N}}$ is bounded, there exists a constant $C_2 > 0$ such that

$$| F(x^{(k)}) | \le C_2, \quad \forall k \in \mathbf{N}. \tag{3.10}$$

Making use of (1.8), (3.1), (3.15), and Lemma 2.2, we have

$$\begin{aligned}\frac{1}{2}\parallel x^{(k)}\|^2 &= F(x^{(k)}) - \frac{1}{2}\sum_{n\in\mathbf{Z}}(Ax_n^{(k)}, \Delta x_n^{(k)}) + \sum_{n\in\mathbf{Z}} W(n, x_n^{(k)}) \\ &\leq C_2 + \frac{1}{2}\beta^{-\frac{1}{4}}\parallel A\parallel\parallel x^{(k)}\|^2 + M_1\sum_{n\in\mathbf{Z}}|a(n)\parallel x_n^{(k)}| \\ &\leq C_2 + \frac{1}{2}\beta^{-\frac{1}{2}}\parallel A\parallel\parallel x^{(k)}\|^2 + M_1\beta^{-\frac{1}{2}}\parallel a\|_2\parallel x^{(k)}\parallel,\end{aligned}$$

which implies that $\{x^{(k)}\}_{k\hat{\mathrm{I}}\mathbf{N}}$ is bounded in E, since $\parallel A\parallel< \sqrt{\beta}$.

Combining Lemma 2.1, the remainder is just the repetition of the proof of Lemma 3.2, we omit the details of its proof.

With the aid of above preparations, now we will give the proof of Theorem 1.2.

Proof of Theorem 1.2 By(1.8), (2.1), (3.1), and Lemma 2.2, we have, for every $m\in\mathbf{R}\setminus\{0\}$ and $x\in E\setminus\{0\}$,

$$\begin{aligned}F(mx) &= \frac{m^2}{2}\parallel x\|^2 + \frac{m^2}{2}\sum_{n\in\mathbf{Z}}(Ax_n, \Delta x_n) - \sum_{n\in\mathbf{Z}} W(n, mx_n) \\ &\geq \frac{m^2}{2}\parallel x\|^2 - \frac{m^2}{2}\beta^{-\frac{1}{2}}\parallel A\parallel\parallel x\|^2 - \beta^{-\frac{1}{2}}M_1\mid m\mid\parallel a(n)\|_2\parallel x\parallel,\end{aligned}$$

which yields that $F(mx)\to+\infty$ as $|m|\to+\infty$, since $\parallel A\parallel< \sqrt{\beta}$. Consequently, $F(x)$ is a functional bounded from below. By Lemmas 2.3 and 3.4, $F(x)$ possesses a critical value $c = \inf_{x\in E} F(x)$, i.e., there is a critical point $x\in E$ such that

$$F(x) = c, \qquad F'(x) = 0.$$

In the following, we show that the critical point x obtained above is non-trivial. From (H_2)', there exists $\delta_1 > 0$ such that $a(n) >0$ for any $n\in [n_1 - \delta_1, n_1 + \delta_1]$. Take $c_1\in\mathbf{R}^N$ with $0 < |c_1| = r$ where r is defined in (H_2)' and let $y\in E$ be given by

$$y_n = \begin{cases} c_1\sin[\frac{2\pi}{2\delta_1}(n-n_1)], & n\in[n_1-\delta_1, n_1+\delta_1] \\ 0, & n\in\mathbf{Z}\backslash[n_1-\delta_1, n_1+\delta_1]\end{cases}$$

Then, for every $n\in\mathbf{Z}$, $|y|\leq r\leq 1$. By (1.6), (2.1), and (3.1), we obtain that

$$F(my)\leq\frac{m^2}{2}\parallel y\|^2 + \frac{m^2}{2}\beta^{-\frac{1}{2}}\parallel A\parallel\parallel y\|^2 - M\mid m|^\theta\sum_{n=n_1-\delta_1}^{n_1+\delta_1} a(n)\mid y_n|^\theta,$$

which yields that $F(my) <0$ for $|m|$ small enough since $1 < \theta <2$, i.e., the critical point $x\in E$ obtained above is non-trivial.

Finally, we show that if (1.9) is true, then (1.1) has one and only one non-trivial homoclinic solution. On the contrary, assuming that (1.1) has at least two distinct homoclinic solutions x and y, by Lemma 3.3, we have

$$\begin{aligned}0 \quad = \quad & (F'(x) - F'(y), x-y) = \parallel x-y\|^2 - \sum_{n\in\mathbf{Z}}(Ax_n - Ay_n, \Delta x_n - \Delta y_n) \\ & + \sum_{n\in\mathbf{Z}}(\nabla W(n, x_n) - \nabla W(n, y_n), x_n - y_n).\end{aligned}$$

According to (1.9), with Lemma 2.2, we have

$$
\begin{aligned}
0 &= (F'(x) - F'(y), x - y) \\
&= \| x - y\|^2 - \sum_{n\in \mathbf{Z}} (Ax_n - Ay_n, \Delta x_n - \Delta y_n) + \sum_{n\in \mathbf{Z}} (aV'(x_n) - aV'(y_n), x_n - y_n) \\
&\geq \| x - y\|^2 - \sum_{n\in \mathbf{Z}} (Ax_n - Ay_n, \Delta x_n - \Delta y_n) - \sum_{n\in \mathbf{Z}} [a\frac{V'(x_n) - V'(y_n)}{| x_n - y_n |} \mid x_n - y_n|^2] \\
&= \| x - y\|^2 - \sum_{n\in \mathbf{Z}} (Ax_n - Ay_n, \Delta x_n - \Delta y_n) - \sum_{n\in \mathbf{Z}} aV''(z) \mid x_n - y_n|^2 \\
&\geq \| x - y\|^2 - \sum_{n\in \mathbf{Z}} (Ax_n - Ay_n, \Delta x_n - \Delta y_n) - \| aV''(z)\|_2 \| x_n - y_n \|_2^2 \\
&\geq \| x - y\|^2 - \sum_{n\in \mathbf{Z}} (Ax_n - Ay_n, \Delta x_n - \Delta y_n) - \omega\frac{1}{\beta} \| x_n - y_n\|^2 \\
&\geq \| x - y\|^2 - (\sum_{n\in \mathbf{Z}} | Ax_n - Ay_n|^2)^{\frac{1}{2}} (\sum_{n\in \mathbf{Z}} | \Delta x_n - \Delta y_n|^2)^{\frac{1}{2}} - \frac{\omega}{\beta} \| x_n - y_n\|^2 \\
&\geq \| x - y\|^2 - \frac{\| A \|}{\sqrt{\beta}} \| x - y\|^2 - \frac{\omega}{\beta} \| x_n - y_n\|^2 \\
&= \| x - y\|^2 (\frac{\beta - \sqrt{\beta} \| A \| -\omega}{\beta}),
\end{aligned}
$$

where $z \in E$ and $z \in (x, y)$, which implies that $||x - y|| = 0$, since $0 < \omega < \beta - \sqrt{\beta} \| A \|$, that is, $x \equiv y$ for all $n \in \mathbf{Z}$.

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Competing interests
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