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Stability of a generalized quadratic functional equation in various spaces: a fixed point alternative approach

Hassan Azadi Kenary[1], Choonkil Park[2*], Hamid Rezaei[1] and Sun Young Jang[3]

* Correspondence: baak@hanyang.ac.kr
[2]Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
Full list of author information is available at the end of the article

Abstract

Using the fixed point method, we prove the Hyers-Ulam stability of the following quadratic functional equation

$$cf\left(\sum_{i=1}^{n}x_i\right)+\sum_{j=2}^{n}f\left(\sum_{i=0}^{n}x_i-(n+c-1)x_j\right)$$
$$=(n+c-1)\left(f(x_1)+c\sum_{i=2}^{n}f(x_i)+\sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1}f(x_i-x_j)\right)\right)$$

in various normed spaces.

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1. Introduction and preliminaries

In 1897, Hensel [1] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [2-5]).

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r|+|s|, \quad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. Throughout this paper, we assume that the base field is a valued field, hence call it simply a field. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 1.1. *Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $||\cdot|| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:*

(i) $||x|| = 0$ if and only if $x = 0$;
(ii) $||rx|| = |r|\ ||x||$ ($r \in K, x \in X$);
(iii) the strong triangle inequality

$$\| x + y \| \leq \max\{\| x \|, \| y \|\}, \quad \forall x, y \in X$$

holds. Then $(X, ||\cdot||)$ is called a non-Archimedean normed space.

Definition 1.2. *(i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called Cauchy if for a given $\varepsilon > 0$ there is a positive integer N such that*

$$\| x_n - x_m \| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called convergent if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\| x_n - x \| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x + y) = f(x) + f(y)$, $\langle x, y\rangle = 0$. By the Pythagorean theorem, $f(x) = ||x||^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus, orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

Pinsker [6] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [7] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation was first investigated by Gudder and Strawther [8]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [9] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [10] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of Rätz; cf. [9].

Suppose X is a real vector space with dim $X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;

(O_2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;

(O_3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O_4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of non-negative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Some interesting examples are

(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, ||.||)$ defined by $x \perp y$ if and only if $||x + \lambda y|| \geq ||x||$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly, examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [11-17]).

The stability problem of functional equations originated from the following question of Ulam [18]: *Under what condition does there exist an additive mapping near an approximately additive mapping*? In 1941, Hyers [19] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Rassias [20] extended the theorem of Hyers by considering the unbounded Cauchy difference $||f(x + y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p)$, $(\varepsilon > 0, p \in [0,1))$. The reader is referred to [21-23] and references therein for detailed information on stability of functional equations.

Ger and Sikorska [24] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $||f(x + y) - f(x) - f(y)|| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\| f(x) - g(x) \| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was Skof [25] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $||f(x + y) + f(x - y) - 2f(x) - 2f(y)|| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\| f(x) - g(x) \| \leq \frac{\varepsilon}{2}$. Cholewa [26] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by Czerwik [27] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [28-32]).

The orthogonally quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x\perp y$$

was first investigated by Vajzović [33] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljević [34], Fochi [35] and Szabó [36] generalized this result. See also [37].

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [38-51]).

Katsaras [52] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. In particular, Bag and Samanta [53], following Cheng and Mordeson [54], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [55]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [56].

Definition 1.3. (Bag and Samanta [53]) *Let* X *be a real vector space. A function* $N : X \times \mathbb{R} \to 0[1]$*is called a fuzzy norm on* X *if for all* $x, y \in X$ *and all* $s, t \in \mathbb{R}$,

($N1$) $N(x, t) = 0$ *for* $t \leq 0$*;*

($N2$) $x = 0$ *if and only if* $N(x, t) = 1$ *for all* $t > 0$*;*

($N3$) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$*if* $c \neq 0$*;*

($N4$) $N(x + y, c + t) \geq min\{N(x, s), N(y, t)\}$*;*

($N5$) $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x, t) = 1$*;*

($N6$) *for* $x \neq 0$, $N(x,.)$ *is continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space. The properties of fuzzy normed vector space and examples of fuzzy norms are given in (see [57,58]).

Example 1.1. *Let* $(X, ||.||)$ *be a normed linear space and* $\alpha, \beta > 0$. *Then*

$$N(x, t) = \begin{cases} \frac{\alpha t}{\alpha t+\beta||x||} & t > 0, x \in X \\ 0 & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.4. (Bag and Samanta [53]) *Let* (X, N) *be a fuzzy normed vector space. A sequence* $\{x_n\}$ *in* X *is said to be convergent or converge if there exists an* $x \in X$ *such that* $\lim_{t\to\infty} N(x_n - x, t) = 1$ *for all* $t > 0$. *In this case,* x *is called the limit of the sequence* $\{x_n\}$ *in* X *and we denote it by* $N - \lim_{t\to\infty} x_n = x$.

Definition 1.5. (Bag and Samanta [53]) *Let* (X, N) *be a fuzzy normed vector space. A sequence* $\{x_n\}$ *in* X *is called Cauchy if for each* $\varepsilon > 0$ *and each* $t > 0$ *there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *and all* $p > 0$, *we have* $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [56]).

Definition 1.6. *Let X be a set. A function* $d : X \times X \to [0, \infty]$ *is called a generalized metric on X if d satisfies the following conditions:*

(1) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$;

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Theorem 1.1. ([59,60]) *Let* (X, d) *be a complete generalized metric space and* $J : X \to Y$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then, for all* $x \in X$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n_0 \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of J*;

(3) y^* *is the unique fixed point of J in the set* $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In this paper, we consider the following *generalized quadratic* functional equation

$$cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n + c - 1)x_j\right) = (n + c - 1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right) \quad (1)$$

and prove the Hyers-Ulam stability of the functional equation (1) in various normed spaces spaces.

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the orthogonally quadratic functional equation (1) in non-Archimedean orthogonality spaces.

In Section 3, we prove the Hyers-Ulam stability of the quadratic functional equation (1) in fuzzy Banach spaces.

2. Stability of the orthogonally quadratic functional equation (1)

Throughout this section, assume that $(X, \perp)$ is a non-Archimedean orthogonality space and that $(Y, ||.||_Y)$ is a real non-Archimedean Banach space. Assume that $|2 - n - c| \neq 0, 1$. In this section, applying some ideas from [22,24], we deal with the stability problem for the orthogonally quadratic functional equation (1) for all $x_1, ..., x_n \in X$ with $x_2 \perp x_i$ for all $i = 1, 3, ..., n$ in non-Archimedean Banach spaces.

Theorem 2.1. *Let* $\phi : X^n \to [0, \infty)$ *be a function such that there exists an* $\alpha < 1$ *with*

$$\varphi(x_1, \ldots, x_n) \leq |2 - c - n|^2 \alpha \varphi\left(\frac{x_1}{2 - c - n}, \ldots, \frac{x_n}{2 - c - n}\right) \quad (2)$$

for all $x_1, ..., x_n \in X$ *with* $x_2 \perp x_i$ $(i \neq 2)$. *Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *and satisfying*

$$\left\| cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right) - (n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right)\right\|_Y \le \varphi(x_1, \ldots, x_n) \tag{3}$$

for all $x_1, \ldots, x_n \in X$ *with* $x_2 \perp x_i$ $(i \neq 2)$ *and fixed positive real number c. Then there exists a unique orthogonally quadratic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x)\|_Y \le \frac{\varphi(0, x, 0, \ldots, 0)}{|2-c-n|^2 - |2-c-n|^2\alpha} \tag{4}$$

for all $x \in X$.

Proof. Putting $x_2 = x$ and $x_1 = x_3 = \cdots = x_n = 0$ in (3), we get

$$\| f((2-c-n)x) - (2-c-n)^2 f(x)\|_Y \le \varphi(0, x, 0, \ldots, 0) \tag{5}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| \frac{f((2-c-n)x)}{(2-c-n)^2} - f(x)\right\|_Y \le \frac{\varphi(0, x, 0, \ldots, 0)}{|2-c-n|^2} \tag{6}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y; h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \| g(x) - h(x)\|_Y \le \mu\varphi(0, x, 0, \ldots, 0), \ \forall x \in X\},$$

where, as usual, $\inf \varphi = +\infty$. It is easy to show that (S, d) is complete (see [61]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{(2-c-n)^2} g((2-c-n)x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then,

$$\| g(x) - h(x)\|_Y \le \varepsilon\varphi(0, x, 0, \ldots, 0)$$

for all $x \in X$. Hence,

$$\| Jg(x) - Jh(x)\|_Y = \left\| \frac{g((2-c-n)x)}{(2-c-n)^2} - \frac{h((2-c-n)x)}{(2-c-n)^2}\right\|_Y \le \alpha \in \varphi(0, x, 0, \ldots, 0)$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le \alpha\varepsilon$. This means that

$$d(Jg, Jh) \le \alpha d(g, h)$$

for all $g, h \in S$.

It follows from (6) that $d(f, Jf) \le \frac{1}{|2-c-n|^2}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q((2-c-n)x) = (2-c-n)^2 Q(x) \tag{7}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : \mathrm{d}(h, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\| f(x) - Q(x)\|_Y \leq \mu\varphi(0, x, 0, \ldots, 0)$$

for all $x \in X$;

(2) $\mathrm{d}(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{m\to\infty} \frac{1}{(2-c-n)^{2m}} g((2-c-n)^m x) = Q(x)$$

for all $x \in X$;

(3) $\mathrm{d}(f, Q) \leq \frac{1}{1-\alpha}\mathrm{d}(f, Jf)$, which implies the inequality

$$\mathrm{d}(f, Q) \leq \frac{1}{|2-c-n|^2 - |2-c-n|^2\alpha}.$$

This implies that the inequality (4) holds.

It follows from (2) and (3) that

$$\begin{aligned}
&\left\| cQ\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} Q\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right)\right. \\
&\left. - (n+c-1)\left(Q(x_1) + c\sum_{i=2}^{n} Q(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} Q(x_i - x_j)\right)\right)\right\|_Y \\
&= \lim_{n\to\infty} \frac{1}{|2-c-n|^{2m}} \left\| cf\left(\sum_{i=1}^{n}(2-c-n)^m x_i\right)\right. \\
&\quad + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n}(2-c-n)^m x_i - (n+c-1)(2-c-n)^m x_j\right) \\
&\quad - (n+c-1)\left(f((2-c-n)^m x_1) + c\sum_{i=2}^{n} f((2-c-n)^m x_i)\right. \\
&\quad \left.\left. + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f((2-c-n)^m (x_i - x_j))\right)\right)\right\|_Y \\
&\leq \lim_{m\to\infty} \frac{\varphi((2-c-n)^m x_1, \ldots, (2-c-n)^m x_n)}{|2-c-n|^{2m}} \\
&\leq \lim_{m\to\infty} \frac{|2-c-n|^{2m}\alpha^m}{|2-c-n|^{2m}} \varphi(x_1, \ldots, x_m) = 0
\end{aligned}$$

for all $x_1, \ldots, x_n \in X$ with $x_2 \perp x_i$. So Q satisfies (1) for all $x_1, \ldots, x_n \in X$ with $x_2 \perp x_i$. Hence, $Q : X \to Y$ is a unique orthogonally quadratic mapping satisfying (1), as desired. □

From now on, in corollaries, assume that $(X, \perp)$ is a non-Archimedean orthogonality normed space.

Corollary 2.1. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying*

$$\begin{aligned} &\left\| cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right)\right. \\ &\left. -(n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right)\right\|_Y \le \theta\left(\sum_{i=1}^{n} \| x_i\|^p\right) \end{aligned} \tag{8}$$

for all $x_1, ..., x_n \in X$ with $x_2 \perp x_i$. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| = \begin{cases} \frac{|2-c-n|^p \theta \|x\|^p}{|2-c-n|^{2+p} - |2-c-n|^3} & \text{if } |2-c-n| < 1 \\ \frac{\theta \|x\|^p}{|2-c-n|^2 - |2-c-n|^{p+1}} & \text{if } |2-c-n| > 1 \end{cases}.$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x_1, \ldots, x_n) = \theta(\sum_{i=1}^{n} \| x_i\|^p)$ for all $x_1, ..., x_n \in X$ with $x_2 \perp x_i$. Then, we can choose

$$\alpha = \begin{cases} |2-c-n|^{1-p} & \text{if } |2-c-n| < 1 \\ |2-c-n|^{p-1} & \text{if } |2-c-n| > 1 \end{cases}.$$

and we get the desired result. □

Theorem 2.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (3) for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\varphi(x_1, \ldots, x_n) \le \frac{\alpha\varphi((2-c-n)x_1, \ldots, (2-c-n)x_n)}{|2-c-n|^2}$$

for all $x_1, ..., x_n \in X$ with $x_2 \perp x_i$ and fixed positive real number c. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x)\|_Y \le \frac{\alpha\varphi(0, x, 0, \ldots, 0)}{|2-c-n|^2 - |2-c-n|^2\alpha} \tag{9}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := (2-c-n)^2 g\left(\frac{x}{2-c-n}\right)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then,

$$\| g(x) - h(x)\|_Y \le \varepsilon\varphi(0, x, 0, \ldots, 0)$$

for all $x \in X$. Hence,

$$\begin{aligned}\| Jg(x) - Jh(x)\|_Y &= \left\|(2-c-n)^2 g\left(\frac{x}{2-c-n}\right) - (2-c-n)^2 h\left(\frac{x}{2-c-n}\right)\right\|_Y \\ &\le |2-c-n|^2 \left\|g\left(\frac{x}{2-c-n}\right) - h\left(\frac{x}{2-c-n}\right)\right\|_Y \\ &\le |2-c-n|^2 \varphi\left(0, \frac{x}{2-c-n}, 0, \ldots, 0\right) \\ &\le |2-c-n|^2 \frac{\alpha\varepsilon}{|2-c-n|^2}\varphi(0, x, 0, \ldots, 0) \\ &= \alpha\varepsilon\varphi(0, x, 0, \ldots, 0)\end{aligned}$$

for all $x \in X$. So $\mathrm{d}(g, h) = \varepsilon$ implies that $\mathrm{d}(Jg, Jh) \le \alpha\varepsilon$. This means that

$$\mathrm{d}(Jg, Jh) \le \alpha \mathrm{d}(g, h)$$

for all $g, h \in S$.

It follows from (5) that

$$\begin{aligned}\left\|f(x) - (2-c-n)^2 f\left(\frac{x}{2-c-n}\right)\right\|_Y &\le \varphi\left(0, \frac{x}{2-c-n}, 0, \ldots, 0\right) \\ &\le \frac{\alpha}{|2-c-n|^2}\varphi(0, x, 0, \ldots, 0).\end{aligned}$$

So

$$\mathrm{d}(f, Jf) \le \frac{\alpha}{|2-c-n|^2}.$$

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2-c-n}\right) = \frac{1}{(2-c-n)^2} Q(x) \tag{10}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : \mathrm{d}(h, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\| f(x) - Q(x)\|_Y \le \mu\varphi(0, x, 0, \ldots, 0)$$

for all $x \in X$;

(2) $\mathrm{d}(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{m\to\infty} (2-c-n)^{2m} g\left(\frac{x}{(2-c-n)^m}\right) = Q(x)$$

for all $x \in X$;

(3) $\mathrm{d}(f, Q) \le \frac{1}{1-\alpha}\mathrm{d}(f, Jf)$, which implies the inequality

$$\mathrm{d}(f, Q) \le \frac{\alpha}{|2-c-n|^2 - |2-c-n|^2\alpha}.$$

This implies that the inequality (9) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.2. *Let θ be a positive real number and p a real number with p > 1. Let f : X → Y be a mapping with f(0) = 0 and satisfying (8). Then there exists a unique orthogonally quadratic mapping Q : X → Y such that*

$$\| f(x) - Q(x) \| = \begin{cases} \frac{|2-c-n|^p \theta ||x||^p}{|2-c-n|^3 - |2-c-n|^{2+p}} & if\ |2-c-n| < 1 \\ \frac{|2-c-n| \theta ||x||^p}{|2-c-n|^{p+2} - |2-c-n|^3} & if\ |2-c-n| > 1 \end{cases}.$$

for all x ∈ X.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x_1, \ldots, x_n) = \theta \left(\sum_{i=1}^n \| x_i \|^p \right)$ for all $x_1, \ldots, x_n \in X$ with $x_2 \perp x_i$. Then, we can choose

$$\alpha = \begin{cases} |2-c-n|^{p-1} \text{ if } |2-c-n| < 1 \\ |2-c-n|^{1-p} \text{ if } |2-c-n| > 1 \end{cases}.$$

and we get the desired result. □

3. Fuzzy stability of the quadratic functional equation (1)

In this section, using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in fuzzy Banach spaces.

Throughout this section, assume that X is a vector space and that (Y, N) is a fuzzy Banach space. In the rest of the paper, let 2 - n - c > 1.

Theorem 3.1. *Let $\phi : X^n \to [0, \infty)$ be a function such that there exists an α < 1 with*

$$\varphi\left(\frac{x_1}{2-c-n}, \ldots, \frac{x_n}{2-c-n}\right) \leq \frac{\alpha}{(2-c-n)^2} \varphi(x_1, \ldots, x_n) \tag{11}$$

for all $x_1, \ldots, x_n \in X$. Let f : X → Y be a mapping with f(0) = 0 and satisfying

$$\begin{aligned} & N\left(cf\left(\sum_{i=1}^n x_i\right) + \sum_{j=2}^n f\left(\sum_{i=1}^n x_i - (n+c-1)x_j\right)\right. \\ & \left. -(n+c-1)\left(f(x_1) + c\sum_{i=2}^n f(x_i) + \sum_{i<j,j=3}^n \left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right),\ t\right) \\ & \geq \frac{t}{t + \varphi(x_1, \ldots, x_n)} \end{aligned} \tag{12}$$

for all $x_1, \ldots, x_n \in X$ and all t > 0. Then the limit

$$Q(x) := N - \lim_{m\to\infty} (2-c-n)^{2m} f\left(\frac{x}{(2-c-n)^m}\right)$$

exists for each x ∈ X and defines a unique quadratic mapping Q : X → Y such that

$$N(f(x) - Q(x),\ t) \geq \frac{((2-c-n)^2 - (2-c-n)^2\alpha)t}{((2-c-n)^2 - (2-c-n)^2\alpha)t + \alpha\varphi(0, x, 0, \ldots, 0)}. \tag{13}$$

Proof. Putting $x_2 = x$ and $x_1 = x_3 = \ldots = x_n = 0$ in (12), we have

$$N\left(f((2-c-n)x) - (2-c-n)^2 f(x), t\right) \geq \frac{t}{t + \varphi(0, x, 0, \ldots, 0)} \tag{14}$$

for all $x \in X$ and $t > 0$.

Replacing x by $\frac{x}{2-c-n}$ in (14), we obtain

$$N\left(f(x) - (2-c-n)^2 f\left(\frac{x}{2-c-n}\right),\ t\right) \geq \frac{t}{t + \varphi(0, \frac{x}{2-c-n}, 0, \ldots, 0)}. \qquad (15)$$

for all $y \in X$ and $t > 0$.

By (15), we have

$$N\left(f(x) - (2-c-n)^2 f\left(\frac{x}{2-c-n}\right), \frac{\alpha t}{(2-c-n)^2}\right) \geq \frac{t}{t + \varphi(0, x, 0, \ldots, 0)}. \quad (16)$$

Consider the set

$$S := \{g : X \to Y; g(0) = 0\}$$

and the generalized metric d in S defined by

$$\mathrm{d}(f,\ g) = \inf_{\mu \in \mathbb{R}^+} \left\{N(g(x) - h(x),\ \mu t) \geq \frac{t}{t + \varphi(0, x, 0, \ldots, 0)},\ \forall x \in X,\ t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [[61], Lemma 2.1]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := (2-c-n)^2 g\left(\frac{x}{2-c-n}\right)$$

for all $x \in X$. Let $g, h \in S$ satisfy $\mathrm{d}(g, h) = \epsilon$. Then,

$$N(g(x) - h(x),\ \varepsilon t) \geq \frac{t}{t + \varphi(0, x, 0, \ldots, 0)}$$

for all $x \in X$ and $t > 0$. Hence,

$$\begin{aligned}
&N(Jg(x) - Jh(x),\ \alpha\varepsilon t) \\
&= N\left((2-c-n)^2 g\left(\frac{x}{2-c-n}\right) - (2-c-n)^2 h\left(\frac{x}{2-c-n}\right),\ \alpha\varepsilon t\right) \\
&= N\left(g\left(\frac{x}{2-c-n}\right) - h\left(\frac{x}{2-c-n}\right),\ \frac{\alpha\varepsilon t}{(2-c-n)^2}\right) \\
&\geq \frac{\frac{\alpha t}{(2-c-n)^2}}{\frac{\alpha t}{(2-c-n)^2} + \varphi(0, \frac{x}{2-c-n}, 0, \ldots, 0)} \\
&\geq \frac{\frac{\alpha t}{(2-c-n)^2}}{\frac{\alpha t}{(2-c-n)^2} + \frac{\alpha}{(2-c-n)^2}\varphi(0, x, 0, \ldots, 0)} \\
&= \frac{t}{t + \varphi(0, x, 0, \ldots, 0)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus, $\mathrm{d}(g, h) = \epsilon$ implies that $\mathrm{d}(Jg, Jh) \leq \alpha\epsilon$. This means that

$$\mathrm{d}(Jg,\ Jh) \leq \alpha \mathrm{d}(g,\ h)$$

for all $g, h \in S$. It follows from (16) that

$$\mathrm{d}(f,\ Jf) \leq \frac{\alpha}{(2-c-n)^2}$$

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q\left(\frac{x}{2-c-n}\right) = \frac{1}{(2-c-n)^2}Q(x) \tag{17}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$\Omega = \{h \in S : \mathrm{d}(g,\ h) < \infty\}.$$

This implies that Q is a unique mapping satisfying (17) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x),\ \mu t) \geq \frac{t}{t + \varphi(0, x, 0, \dots, 0)}$$

for all $x \in X$ and $t > 0$.

(2) $\mathrm{d}(J^m f, Q) \to 0$ as $m \to \infty$. This implies the equality

$$N\text{ - }\lim_{m\to\infty} (2-n-c)^{2m} f\left(\frac{x}{(2-n-c)^m}\right) = Q(x)$$

for all $x \in X$.

(3) $\mathrm{d}(f,\ Q) \leq \frac{\mathrm{d}(f,Jf)}{1-\alpha}$ with $f \in \Omega$, which implies the inequality

$$\mathrm{d}(f,\ Q) \leq \frac{\alpha}{(2-c-n)^2(1-\alpha)}.$$

This implies that the inequality (13) holds.

Using (11) and (12), we obtain

$$\begin{aligned} &N(2-c-n)^{2m}\left[cf\left(\sum_{i=1}^{n}\frac{x_i}{(2-c-n)^m}\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n}\frac{x_i}{(2-c-n)^m} - \frac{(n+c-1)x_j}{(2-c-n)^m}\right)\right. \\ &-(n+c-1)\left(f\left(\frac{x_1}{(2-c-n)^m}\right) + c\sum_{i=2}^{n} f\left(\frac{x_i}{(2-c-n)^m}\right)\right. \\ &\left.\left.+\sum_{i<j,j=3}^{n}\left(\sum_{i<j,j=3}^{n-1} f\left(\frac{x_i-x_j}{(2-c-n)^m}\right)\right)\right)\right], (2-c-n)^{2m}t\Bigg) \\ &\geq \frac{t}{t+\varphi\left(\frac{x_1}{(2-c-n)^m}, \dots, \frac{x_n}{(2-c-n)^m}\right)} \end{aligned} \tag{18}$$

for all $x_1, \dots, x_n \in X$, $t > 0$ and all $n \in \mathbb{N}$.

So by (11) and (18), we have

$$\begin{aligned} &N\Bigg((2-c-n)^{2m}\left[cf\left(\sum_{i=1}^{n}\frac{x_i}{(2-c-n)^m}\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n}\frac{x_i}{(2-c-n)^m} - \frac{(n+c-1)x_j}{(2-c-n)^m}\right)\right. \\ &-(n+c-1)\left(f\left(\frac{x_1}{(2-c-n)^m}\right) + c\sum_{i=2}^{n} f\left(\frac{x_i}{(2-c-n)^m}\right)\right. \\ &\left.\left.+\sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f\left(\frac{x_i-x_j}{(2-c-n)^m}\right)\right)\right)\right], t\Bigg) \\ &\geq \frac{\frac{t}{(2-c-n)^{2m}}}{\frac{t}{(2-c-n)^{2m}} + \frac{\alpha^m}{(2-c-n)^{2m}}\varphi(x_1, \dots, x_n)} \end{aligned}$$

for all $x_1, ..., x_n \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{\frac{t}{(2-c-n)^{2m}}}{\frac{t}{(2-c-n)^{2m}} + \frac{\alpha^m}{(2-c-n)^{2m}}\varphi(x_1,\ldots,x_n)} = 1$$

for all $x_1, ..., x_n \in X$ and all $t > 0$, we deduce that

$$N\left(cQ\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} Q\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right)\right.$$
$$\left. -(n+c-1)\left(Q(x_1) + c\sum_{i=2}^{n} Q(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} Q(x_i - x_j)\right)\right),\ t\right) = 1$$

for all $x_1, ..., x_n \in X$ and all $t > 0$. Thus the mapping $Q : X \to Y$ satisfying (1), as desired. This completes the proof. □

Corollary 3.1. *Let $\theta \geq 0$ and let r be a real number with $r > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying*

$$N\left(cf\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - (n+c-1)x_j\right)\right.$$
$$\left. -(n+c-1)\left(f(x_1) + c\sum_{i=2}^{n} f(x_i) + \sum_{i<j,j=3}^{n}\left(\sum_{i=2}^{n-1} f(x_i - x_j)\right)\right),\ t\right) \tag{19}$$
$$\geq \frac{t}{t + \theta\left(\sum_{i=1}^{n} \|x_i\|^r\right)}$$

for all $x_1, ..., x_n \in X$ and all $t > 0$. Then

$$Q(x) := N - \lim_{m\to\infty} (2-n-c)^{2m} f\left(\frac{x}{(2-n-c)^m}\right)$$

exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{((2-c-n)^{2r} - (2-c-n)^2)t}{((2-c-n)^{2r} - (2-c-n)^2)t + \theta\|x\|^r}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x_1,\ldots,x_n) := \theta\left(\sum_{i=1}^{n} \|x_i\|^r\right)$$

for all $x_1, ..., x_n \in X$. Then, we can choose $\alpha = (2 - c - n)^{2-2r}$ and we get the desired result. □

Theorem 3.2. *Let $\phi : X^n \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi((2-c-n)x_1,\ldots,(2-c-n)x_n) \leq (2-c-n)^2\alpha\varphi(x_1,\ldots,x_n) \tag{20}$$

for all $x_1, ..., x_n \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (12). Then the limit

$$Q(x) := N - \lim_{m\to\infty} \frac{f((2-c-n)^m x)}{(2-c-n)^{2m}}$$

exists for each $x \in X$ *and defines a unique quadratic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x), t) \geq \frac{(2-c-n)^2(1-\alpha)t}{(2-c-n)^2(1-\alpha)t + \varphi(0, x, 0, \dots, 0)} \tag{21}$$

Proof. Let (S, d) be the generalized metric space defined as in the proof of Theorem 3.1. Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{(2-c-n)^2} g((2-c-n)x)$$

for all $x \in X$. Let $g, h \in S$ be such that $\mathrm{d}(g, h) = \epsilon$. Then,

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(0, x, 0, \dots, 0)}$$

for all $x \in X$ and $t > 0$. Hence,

$$\begin{aligned}
N(Jg(x) - Jh(x), \alpha\varepsilon t) &= N\left(\frac{g((2-c-n)x)}{(2-c-n)^2} - \frac{h((2-c-n)x)}{(2-c-n)^2}, \alpha\varepsilon t\right) \\
&= N(g((2-c-n)x) - h((2-c-n)x), (2-c-n)^2\alpha\varepsilon t) \\
&\geq \frac{(2-c-n)^2\alpha t}{(2-c-n)^2\alpha t + (2-c-n)^2\alpha\varphi(0, x, 0, \dots, 0)} \\
&= \frac{t}{t + \varphi(0, x, 0, \dots, 0)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus, $\mathrm{d}(g, h) = \epsilon$ implies that $\mathrm{d}(Jg, Jh) \leq \alpha\epsilon$. This means that

$$\mathrm{d}(Jg, Jh) \leq \alpha \mathrm{d}(g, h)$$

for all $g, h \in S$.

It follows from (14) that

$$N\left(\frac{f((2-c-n)x)}{(2-c-n)^2} - f(x), \frac{t}{(2-c-n)^2}\right) \geq \frac{t}{t + \varphi(0, x, 0, \dots, 0)}$$

for all $x \in X$ and $t > 0$. So $\mathrm{d}(f, Jf) \leq \frac{1}{(2-c-n)^2}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$(2-c-n)^2 Q(x) = Q((2-c-n)x) \tag{22}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$\Omega = \{h \in S : \mathrm{d}(g, h) < \infty\}.$$

This implies that Q is a unique mapping satisfying (22) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(0, x, 0, \dots, 0)}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^m f, Q) \to 0$ as $m \to \infty$. This implies the equality

$$\lim_{m\to\infty} N - \frac{f((2-c-n)^m x)}{(2-c-n)^{2m}} = Q(x)$$

for all $x \in X$.

(3) $d(f, Q) \le \dfrac{d(f, Jf)}{1-\alpha}$ with $f \in \Omega$, which implies the inequality

$$d(f, Q) \le \frac{1}{(2-c-n)^2(1-\alpha)}.$$

This implies that the inequality (21) holds.

The rest of the proof is similar to that of the proof of Theorem 3.1. □

Corollary 3.2. *Let $\theta \ge 0$ and let r be a real number with $0 < r < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (19). Then the limit*

$$Q(x) := N - \lim_{m\to\infty} \frac{f((2-c-n)^m x)}{(2-c-n)^{2m}}$$

exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{((2-c-n)^2 - (2-c-n)^{2r})t}{((2-c-n)^2 - (2-c-n)^{2r})t + \theta \|x\|^r}.$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x_1, \ldots, x_n) := \theta\left(\sum_{i=1}^{n} \|x_i\|^r\right)$$

for all $x_1, \ldots, x_n \in X$. Then, we can choose $\alpha = (2 - c - n)^{2r-2}$ and we get the desired result. □

Author details
[1]Department of Mathematics, College of Sciences, Yasouj University, Yasouj 75914-353, Iran [2]Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea [3]Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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