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On generalized Srivastava-Owa fractional operators in the unit disk

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Abstract

This article introduces a generalization for the Srivastava-Owa fractional operators in the unit disk. Conditions are given for the fractional integral operator to be bounded in Bergman space. Some properties for the above operator are also provided. Moreover, applications of these operators are posed in the geometric functions theory and fractional differential equations.

1 Introduction

Recently, the theory of fractional calculus has found interesting applications in the theory of analytic functions. The classical definitions of fractional operators and their generalizations have fruitfully been applied in obtaining, for example, the characterization properties, coefficient estimates [1], distortion inequalities [2] and convolution structures for various subclasses of analytic functions and the works in the research monographs. In [3], Srivastava and Owa gave definitions for fractional operators (derivative and integral) in the complex z-plane **C** as follows:

Definition 1.1. The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \le \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane **C** containing the origin, and the multiplicity of $(z - \zeta)^{-\alpha}$ is removed by requiring $\log(z - \zeta)$ to be real when $(z - \zeta) > 0$.

Definition 1.2. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane (**C**) containing the origin, and the multiplicity of $(z - \zeta)^{\alpha-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $(z - \zeta) > 0$.

Remark 1.1. From Definitions 1.1 and 1.2, we have $D_z^0 f(z) = f(0)$, $\lim_{\alpha\to 0} I_z^\alpha f(z) = f(z)$ and $\lim_{\alpha\to 0} D_z^{1-\alpha} f(z) = f'(z)$. Moreover,

$$D_z^\alpha \left\{z^\mu\right\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} \left\{z^{\mu-\alpha}\right\}, \quad \mu > -1; \ \ 0 \le \alpha < 1$$

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and

$$I_z^\alpha\left\{z^\mu\right\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\left\{z^{\mu-\alpha}\right\}, \quad \mu > -1; \ \ 0 \le \alpha < 0.$$

Further properties of these operators can be found in [4,5].

2 Generalized integral operator

For $0 < p < 1$, the Bergman space $\mathcal{A}^p$ is the set of functions f analytic in the unit disk $U := \{z : z \in \mathbf{C}; |z| < 1\}$ with $\|f\|^p_{\mathcal{A}^p} < \infty$, where the norm is defined by

$$\|f\|^p_{\mathcal{A}^p} = \frac{1}{\pi}\int_U |f(z)|^p dA < \infty, \quad z \in U,$$

and dA is denoted Lebesgue area measure.

To derive a formula for the generalized fractional integral, consider for natural $n \in \mathbf{N} = \{1, 2, ...\}$ and real μ, the n-fold integral of the form

$$I_z^{\alpha,\mu} f(z) = \int_0^z \zeta_1^\mu d\zeta_1 \int_0^{\zeta_1} \zeta_2^\mu d\zeta_2 \ldots \int_0^{\zeta_{n-1}} \zeta_n^\mu f(\zeta_n) d\zeta_n. \tag{1}$$

By employing the Dirichlet technique yields

$$\int_0^z \zeta_1^\mu d\zeta_1 \int_0^{\zeta_1} \zeta^\mu f(\zeta) d\zeta = \int_0^z \zeta^\mu f(\zeta) d\zeta \int_\zeta^z \zeta_1^\mu d\zeta_1 = \frac{1}{\mu+1}\int_0^z \left(z^{\mu+1} - \zeta^{\mu+1}\right)^{n-1} \zeta^\mu f(\zeta) d\zeta.$$

Repeating the above step n - 1 times we have

$$\int_0^z \zeta_1^\mu d\zeta_1 \int_0^{\zeta_1} \zeta_2^\mu d\zeta_2 \ldots \int_0^{\zeta_{n-1}} \zeta_n^\mu f(\zeta_n) d\zeta_n = \frac{(\mu+1)^{1-n}}{(n-1)!}\int_0^z \left(z^{\mu+1} - \zeta^{\mu+1}\right)^{n-1} \zeta^\mu f(\zeta) d\zeta.$$

which implies the fractional operator type

$$I_z^{\alpha,\mu} f(z) = \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^\mu f(\zeta) d\zeta, \tag{2}$$

where a and $\mu \neq -1$ are real numbers and the function $f(z)$ is analytic in simply-connected region of the complex z-plane **C** containing the origin, and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$. When $\mu = 0$, we arrive at the standard Srivastava-Owa fractional integral, which is used to define the Srivastava-Owa fractional derivatives.

Theorem 2.1. Let $\alpha > 0, 0 < p < \infty$ and $\mu \in \mathbf{R}$. Then, the operator $I_z^{\alpha,\mu}$ is bounded in $\mathcal{A}^p$ and

$$\left\|I_z^{\alpha,\mu} f(z)\right\|^p_{\mathcal{A}^p} \le C \left\|f(z)\right\|^p_{\mathcal{A}^p},$$

where

$$C := \int_0^1 \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\left(1 - w^{\mu+1}\right)^{\alpha-1} w^\mu dw\right|^p.$$

Proof. Assume that $f(z) \in \mathcal{A}^p$. Then, we have

$$\begin{aligned}\|I_z^{\alpha,\mu} f(z)\|_{\mathcal{A}^p}^p &= \frac{1}{\pi}\int_U |I_z^{\alpha,\mu} f(z)|^p dA \\ &= \frac{1}{\pi}\int_0^1 \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha-1}\zeta^\mu f(\zeta)d\zeta\right|^p dA \\ &= \frac{1}{\pi}\int_0^1 \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(1-\frac{\zeta^{\mu+1}}{z^{\mu+1}}\right)^{\alpha-1} z^{(\mu+1)(\alpha-1)}\zeta^\mu f(\zeta)d\zeta\right|^p dA.\end{aligned}$$

Let $w := \dfrac{\zeta}{z}$, then we obtain

$$\begin{aligned}\|I_z^{\alpha,\mu} f(z)\|_{\mathcal{A}^p}^p &= \frac{1}{\pi}\int_0^1 \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\frac{\zeta}{w}} \left(1-w^{\mu+1}\right)^{\alpha-1} z^{(\mu+1)(\alpha)} w^\mu f(wz)dw\right|^p dA \\ &\leq \frac{1}{\pi}\int_0^1 \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_U \left(1-w^{\mu+1}\right)^{\alpha-1} w^\mu f(wz)dw\right|^p dA \\ &\leq \int_0^1 \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_U \left(1-w^{\mu+1}\right)^{\alpha-1} w^\mu dw\right|^p \left(\frac{1}{\pi}\int_U |f(\zeta)|^p dA\right) \\ &:= C\,\|f\|_{\mathcal{A}^p}^p.\end{aligned}$$

This completes the proof.

Next, we give semigroup properties of the integral operator.

Theorem 2.2. Let f be analytic in the unit disk. Then, operator (2) satisfies

$$I_z^{\alpha+\beta,\mu} f = I_z^{\alpha,\mu} I_z^{\beta,\mu} f, \quad \alpha > 0,\ \beta > 0. \tag{3}$$

Proof. For function f by using Dirichlet technique yields

$$\begin{aligned}I_z^{\alpha,\mu} I_z^{\beta,\mu} f(z) &= \frac{(\mu+1)^{(1-\alpha)}}{\Gamma(\alpha)}\int_0^z \left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha-1}\zeta^\mu I_\zeta^{\beta,\mu} f(\zeta)d\zeta \\ &= \frac{(\mu+1)^{(1-\alpha)+(1-\beta)}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^z \left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha-1}\zeta^\mu \left(\int_\xi^\zeta \left(\zeta^{\mu+1}-\xi^{\mu+1}\right)^{\beta-1}\xi^\mu f(\xi)\right) d\xi d\zeta \\ &= \frac{(\mu+1)^{(1-\alpha)+(1-\beta)}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^z \xi^\mu f(\xi)\left(\int_\xi^\zeta \left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha-1}\left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\beta-1}\zeta^\mu\right) d\zeta\, d\xi.\end{aligned} \tag{4}$$

Let $w := \dfrac{\zeta^{\mu+1}-\xi^{\mu+1}}{z^{\mu+1}-\xi^{\mu+1}}$, we pose

$$\begin{aligned}\int_\xi^\zeta \left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha-1}\left(\zeta^{\mu+1}-\xi^{\mu+1}\right)^{\beta-1}\zeta^\mu d\zeta &= \frac{\left(z^{\mu+1}-\xi^{\mu+1}\right)^{\alpha+\beta-1}}{\mu+1}\int_0^1 (1-\omega)^{\alpha-1}\omega^{\beta-1}d\omega \\ &= \frac{\left(z^{\mu+1}-\xi^{\mu+1}\right)^{\alpha+\beta-1}}{\mu+1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.\end{aligned} \tag{5}$$

By (4) and (5), we obtain

$$\begin{aligned}I_z^{\alpha,\mu} I_z^{\beta,\mu} f(z) &= \frac{(\mu+1)^{(2-\alpha-\beta)}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^z \frac{\left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha+\beta-1}}{\mu+1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\xi^\mu f(\xi)d\xi \\ &= \frac{(\mu+1)^{(2-\alpha-\beta)}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^z \frac{\left(z^{\mu+1}-\xi^{\mu+1}\right)^{\alpha+\beta-1}}{\mu+1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\xi^\mu f(\xi)d\xi \\ &= \frac{(\mu+1)^{(1-\alpha-\beta)}}{\Gamma(\alpha+\beta)}\int_0^z \left(z^{\mu+1}-\xi^{\mu+1}\right)^{\alpha+\beta-1}\xi^\mu f(\xi)d\xi \\ &= I_z^{\alpha+\beta,\mu}.\end{aligned} \tag{6}$$

Example 2.1. We find the generalized integral of the function $f(z) = z^{\nu}, \nu \in \mathbb{R}$. Let $\eta := \left(\frac{\zeta}{z}\right)^{\mu+1}$ then

$$\begin{aligned}
I_z^{\alpha,\mu} z^{\nu} &= \frac{(\mu+1)}{\Gamma(\alpha)} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu+\nu} d\zeta \\
&= \frac{z^{\alpha(\mu+1)+\nu}}{(\mu+1)^{\alpha}\Gamma(\alpha)} \int_0^1 \eta^{\frac{\nu}{\mu+1}} (1-\eta)^{\alpha-1} d\eta \\
&= \frac{z^{\alpha(\mu+1)+\nu}}{(\mu+1)^{\alpha}\Gamma(\alpha)} \int_0^1 \eta^{\frac{\nu+\mu+1}{\mu+1}} (1-\eta)^{\alpha-1} d\eta \\
&= \frac{z^{\alpha(\mu+1)+\nu}}{(\mu+1)^{\alpha}\Gamma(\alpha)} B\left(\frac{\nu+\mu+1}{\mu+1}, \alpha\right) \\
&= \frac{z^{\alpha(\mu+1)+\nu}}{(\mu+1)^{\alpha}\Gamma(\alpha)} \frac{\Gamma\left(\frac{\nu+\mu+1}{\mu+1}\right)\Gamma(\alpha)}{\Gamma\left(\alpha + \frac{\nu+\mu+1}{\mu+1}\right)} \\
&= \frac{z^{\alpha(\mu+1)+\nu}}{(\mu+1)^{\alpha}} \frac{\Gamma\left(\frac{\nu+\mu+1}{\mu+1}\right)}{\Gamma\left(\alpha + \frac{\nu+\mu+1}{\mu+1}\right)},
\end{aligned}$$

where B is the Beta function. When $\mu = 0$, we obtain $I_z^{\alpha} z^{\nu} = z^{\alpha+\nu} \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)}$ (see Remark 1.1).

In the next section, we will define generalized fractional derivatives for an arbitrary order. Some of its properties are discussed. Furthermore, applications involving this operator are illustrated.

3 Generalized differential operator

Corresponding to the generalized fractional integrals (2), we define the generalized differential operator.

Definition 3.1. The generalized fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^{\alpha,\mu} f(z) := \frac{(\mu+1)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{\zeta^{\mu} f(\zeta)}{\left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha}} d\zeta; \quad 0 \le \alpha < 1, \tag{7}$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane **C** containing the origin, and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{-\alpha}$ is removed by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$.

Example 3.1. We find the generalized derivative of the function $f(z) = z^{\nu}$, $\nu \in$ **R**. In the same manner of Example 2.1, we let $\eta := \left(\frac{\zeta}{z}\right)^{\mu+1}$ then we have

$$
\begin{aligned}
D_z^{\alpha,\mu} z^{\nu} &= \frac{(\mu+1)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{\zeta^{\mu+\nu}}{\left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha}} d\zeta \\
&= \frac{(\mu+1)^{\alpha-1}}{\Gamma(1-\alpha)} \frac{d}{dz} z^{(1-\alpha)(\mu+1)+\nu} \int_0^1 \eta^{\frac{\nu+\mu+1}{\mu+1}-1} (1-\eta)^{(1-\alpha)-1} d\eta \\
&= \frac{(\mu+1)^{\alpha-1}\Gamma\left(\frac{\nu}{\mu+1}+1\right)}{\Gamma\left(\frac{\nu}{\mu+1}+1-\alpha\right)} z^{(1-\alpha)(\mu+1)+\nu-1}.
\end{aligned}
$$

When $\mu = 0$, we obtain $D_z^{\alpha} z^{\nu} = z^{\nu-\alpha} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)}$ (see Remark 1.1).

Next, we proceed to prove some relations of the generalized operators $I_z^{\alpha,\mu}$ and $D_z^{\alpha,\mu}$ for analytic functions of the form

$$
f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in U. \tag{8}
$$

By employing Theorem 2.2 and Example 2.1, we have the following proposition:

Proposition 3.1. Let f be analytic in U of the form (8). Then,

$$
D_z^{\alpha,\mu} I_z^{\alpha,\mu} f(z) = I_z^{\alpha,\mu} D_z^{\alpha,\mu} f(z) = f(z), \quad z \in U.
$$

4 Applications

In this section, we discuss some applications of the generalized operators (2) and (7) in geometric function theory and fractional differential equations.

4.1 Distortion inequalities involving fractional derivatives

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{9}
$$

Also, let $\mathcal{S}$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, univalent and convex in U. It is well known that if the function $f(z)$ given by (9) is in the class $\mathcal{S}$, then

$$
|a_n| \le n, \quad n \in \mathbf{N}\backslash\{1\}. \tag{10}
$$

Equality holds for the Koebe function

$$
f(z) = \frac{z}{(1-z)^2}, \quad z \in U.
$$

Moreover, if the function $f(z)$ given by (9) is in the class $\mathcal{K}$, then

$$
|a_n| \le 1, \quad n \in \mathbf{N}. \tag{11}
$$

Equality holds for the function

$$
f(z) = \frac{z}{1-z}, \quad z \in U.
$$

In our present investigation, we shall also make use of the Fox-Wright generalization ${}_q\Psi_p[z]$ of the hyperge-ometric ${}_qF_p$ function defined by [6]

$$\begin{aligned}{}_q\psi_p\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_q, A_q); \\ (\beta_1, B_1), \ldots, (\beta_p, B_p);\end{matrix}\; z\right] &= {}_q\psi_p\left[(\alpha_j, A_j)_{1,q;}(\beta_j, B_j)_{1,p;z}\right] \\ &:= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1 + nA_1)\ldots\Gamma(\alpha_q + nA_q)\, z^n}{\Gamma(\beta_1 + nB_1)\ldots\Gamma(\beta_q + nB_q)\, n!} \\ &= \sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + nA_j)\, z^n}{\Pi_{j=1}^{p}\Gamma(\beta_j + nB_j)\, n!},\end{aligned}$$

where $A_j > 0$ for all $j = 1,..., q$, $B_j > 0$ for all $j = 1,..., p$ and $1 + \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j \geq 0$ for suitable values $|z| < 1$.

Theorem 4.1. Let $f \in \mathcal{S}$. Then,

$$\left|D_z^{\alpha,\mu} f(z)\right| \leq r^{(1-\alpha)(\mu+1)}(\mu + 1)^{\alpha-1}\left[\begin{matrix}(2, 1), \left(1 + \frac{1}{\mu+1}, \frac{1}{\mu+1}\right); \\ \left(1 - \alpha + \frac{1}{\mu+1}, \frac{1}{\mu+1}\right);\end{matrix}\; r\right], \tag{12}$$
$$(r = |z|;\; z \in U;\; 0 < \alpha < 1),$$

where the equality holds true for the Koebe function.

Proof. Suppose that the function $f(z) \in \mathcal{S}$ is given by (9). Then, by using Example 3.1, we obtain

$$\begin{aligned}D_z^{\alpha,\mu} f(z) &= \sum_{n=1}^{\infty}\frac{(\mu+1)^{\alpha-1}\Gamma\left(\frac{n}{\mu+1} + 1\right)}{\Gamma\left(\frac{n}{\mu+1} + 1 - \alpha\right)} a_n z^{(1-\alpha)(\mu+1)+n-1}, \quad a_1 = 1 \\ &= z^{(1-\alpha)(\mu+1)}(\mu + 1)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\mu+1} + 1\right)}{\Gamma\left(\frac{n+1}{\mu+1} + 1 - \alpha\right)} a_{n+1} z^n.\end{aligned}$$

Thus,

$$\begin{aligned}\left|D_z^{\alpha,\mu} f(z)\right| &\leq r^{(1-\alpha)(\mu+1)}(\mu + 1)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\mu+1} + 1\right)}{\Gamma\left(\frac{n+1}{\mu+1} + 1 - \alpha\right)}(n + 1)r^n \\ &= r^{(1-\alpha)(\mu+1)}(\mu + 1)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma(2 + n)\Gamma\left(1 + \frac{1}{\mu+1} + \frac{n}{\mu+1}\right) r^n}{\Gamma\left(1 - \alpha + \frac{1}{\mu+1} + \frac{n}{\mu+1}\right)}\frac{r^n}{n!} \\ &= r^{(1-\alpha)(\mu+1)}(\mu + 1)^{\alpha-1}\left[\begin{matrix}(2, 1), \left(1 + \frac{1}{\mu+1}, \frac{1}{\mu+1}\right); \\ \left(1 - \alpha + \frac{1}{\mu+1}, \frac{1}{\mu+1}\right);\end{matrix}\; r\right] \\ &= r^{(1-\alpha)(\mu+1)}(\mu + 1)^{\alpha-1}\, {}_2\Phi_1[r].\end{aligned}$$

In the same manner of Theorem 4.1, we have a distortion inequality involving the Fox-Wright function, which is given by the following:

Theorem 4.2. Let $f \in \mathcal{S}$. Then,

$$\left|D_z^{\alpha+1,\mu} f(z)\right| \le \frac{(\mu+1)^\alpha}{r^{\alpha(\mu+1)}} \left[\begin{matrix} (2,1), \left(1+\frac{1}{\mu+1}, \frac{1}{\mu+1}\right); \\ \\ \left(\frac{1}{\mu+1} - \alpha \frac{1}{\mu+1}\right); \end{matrix} \; r\right], \tag{13}$$
$$(r = |z|;\ z \in U;\ 0 < \alpha < 1),$$

where the equality holds true for the Koebe function.

Theorem 4.3. Let $f \in \mathcal{K}$. Then,

$$\left|D_z^{\alpha+1,\mu} f(z)\right| \le r^{(1-\alpha)(\mu+1)}(\mu+1)^{\alpha-1} \left[\begin{matrix} (1,1), \left(1+\frac{1}{\mu+1}, \frac{1}{\mu+1}\right); \\ \\ \left(1-\alpha\frac{1}{\mu+1} - \alpha \frac{1}{\mu+1}\right); \end{matrix} \; r\right], \tag{14}$$
$$(r = |z|;\ z \in U;\ 0 < \alpha < 1),$$

where the equality holds true for the Koebe function.

Proof. Suppose that the function $f(z) \in \mathcal{K}$. is given by (9). Then, we pose

$$\begin{aligned}\left|D_z^{\alpha,\mu} f(z)\right| &\le r^{(1-\alpha)(\mu+1)}(\mu+1)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{\mu+1}+1\right)}{\Gamma\left(\frac{n+1}{\mu+1}+1-\alpha\right)} r^n \\ &= r^{(1-\alpha)(\mu+1)}(\mu+1)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\Gamma(1+n)\Gamma\left(1+\frac{1}{\mu+1}+\frac{n}{\mu+1}\right) r^n}{\Gamma\left(1-\alpha+\frac{1}{\mu+1}+\frac{n}{\mu+1}\right)} \frac{r^n}{n!} \\ &= r^{(1-\alpha)(\mu+1)}(\mu+1)^{\alpha-1} \left[\begin{matrix} (1,1), \left(1+\frac{1}{\mu+1}, \frac{1}{\mu+1}\right); \\ \\ \left(1-\alpha+\frac{1}{\mu+1}, \frac{1}{\mu+1}\right); \end{matrix} \; r\right].\end{aligned}$$

4.2 Fractional differential equations

In this section, we focus our attention on the fractional differential equation of the form

$$D_z^{\alpha,\mu} u(z) = f\left(z, u(z)\right), \tag{15}$$

subject to the initial condition $u(0) = 0$, where $u : U \to \mathbf{C}$ is an analytic function for all $z \in U$, and $f : U \times \mathbf{C} \to \mathbf{C}$ is an analytic function in $z \in U$. Let $\mathcal{B}$ represent complex Banach space of analytic functions in the unit disk.

Theorem 4.4. (Existence) Let the function $f : U \times \mathbf{C} \to \mathbf{C}$ be analytic such that $||f|| \le M$; $M \ge 0$. Then, there exists a function $u : U \to \mathbf{C}$ solving the problem (15).

Proof. Define the set $S : \{u \in \mathcal{B} : \|u\| \le r, r > 0\}$, and the operator $P : S \to S$ by

$$(Pu)(z) := \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu} f(\zeta, u(\zeta)) d\zeta; \quad \alpha \in (0,1). \tag{16}$$

First, we show that P is bounded operator:

$$\begin{aligned} \left|(Pu)(z)\right| &= \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu} f(\zeta, u(\zeta)) d\zeta\right| \\ &\le \frac{M(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left|\left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu}\right| d\zeta \\ &\le \frac{M(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} B(1,\alpha) \\ &: = r, \end{aligned}$$

that is $\|P_u\|_{\mathcal{B}} = \sup_{z\in U} \left|(Pu)(z)\right|$. We proceed to prove that $P : S \to S$ is continuous operator. Since f is continuous function on $U \times S$, then it is uniformly continuous on a compact set $\tilde{U} \times S$, where

$$\tilde{U} := \{z \in U : |z| \le \ell,\ 0 < \ell < 1\}.$$

Hence, given $\epsilon > 0$, $\exists\delta > 0$ such that for all $u,v \in S$ we have

$$\|f(z,u) - f(z,v)\| < \frac{\varepsilon\Gamma(\alpha)}{(\mu+1)^{1-\alpha} B(1,\alpha) \ell^{\alpha(\mu+1)}} \quad \text{for } \|u - v\| < \delta, \quad \text{then}$$

$$\begin{aligned} \left|(Pu)(z) - (Pv)(z)\right| &= \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu} f(\zeta, u(\zeta)) d\zeta \right. \\ &\quad \left. - \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu} f(\zeta, v(\zeta)) d\zeta\right| \\ &\le \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left|\left(z^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu}\right| \times \left|f(\zeta, u(\zeta)) - f(\zeta, v(\zeta))\right| d\zeta \\ &\le \frac{\ell^{\alpha(\mu+1)}(\mu+1)^{1-\alpha} B(1,\alpha)}{\Gamma(\alpha)} \times \frac{\varepsilon\Gamma(\alpha)}{(\mu+1)^{1-\alpha} B(1,\alpha) \ell^{\alpha(\mu+1)}} \\ &= \varepsilon. \end{aligned}$$

Thus, P is a continuous mapping on S. Now, we show that P is an equicontinuous mapping on S. For $z_1, z_2 \in \tilde{U}$ such that $z_1 \neq z_2$, then for all $u \in S$ we obtain

$$\begin{aligned} \left|(Pu)(z_1) - (Pu)(z_2)\right| &= \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^{z_1} \left(z_1^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu} f(\zeta, u(\zeta)) d\zeta \right. \\ &\quad \left. - \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^{z_2} \left(z_2^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu} f(\zeta, u(\zeta)) d\zeta\right| \\ &\le \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^{z_1} \left|\left(z_1^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu} f(\zeta, u(\zeta))\right| d\zeta \\ &\quad + \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^{z_2} \left|\left(z_2^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu} f(\zeta, u(\zeta))\right| d\zeta \\ &\le \frac{M(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \left(\int_0^{z_1} \left|\left(z_1^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu}\right| d\zeta + \int_0^{z_2} \left|\left(z_2^{\mu+1} - \zeta^{\mu+1}\right)^{\alpha-1} \zeta^{\mu}\right| d\zeta\right) \\ &\le \frac{2M\ell^{\alpha(\mu+1)}(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} B(1,\alpha), \end{aligned}$$

which is independent on u. Hence, P is an equicontinuous mapping on S. The Arzela-Ascoli theorem yields that every sequence of functions from $P(S)$ has got a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. Schauder's fixed point theorem asserts that P has a fixed point. By construction, a fixed point of P is a solution of the initial value problem (15).

Theorem 4.5. (Uniqueness) Let the function f be bounded and fulfill a Lipschitz condition with respect to the second variable: i.e.,

$$\left\|f(z,u)-f(z,v)\right\| \le L\left\|u-v\right\|$$

for some $L > 0$ independent of u,v and z. If $\frac{L(\mu+1)^{1-\alpha}B(1,\alpha)}{\Gamma(\alpha)} < 1$, then there exists a unique function $u : U \to \mathbf{C}$ solving the initial value problem (15).

Proof. We need only to prove that the operator P in Equation 3 has a unique fixed point.

$$\begin{aligned}\left|(Pu)(z)-(Pv)(z)\right| &= \left|\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha-1}\zeta^{\mu} f(\zeta,u(\zeta))d\zeta \right.\\ &\left. -\frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha-1}\zeta^{\mu} f(\zeta,v(\zeta))d\zeta\right|\\ &\le \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left|\left(z^{\mu+1}-\zeta^{\mu+1}\right)^{\alpha-1}\zeta^{\mu}\right| \times \left|f(\zeta,u(\zeta))-f(\zeta,v(\zeta))\right| d\zeta\\ &\le \frac{L(\mu+1)^{1-\alpha}B(1,\alpha)}{\Gamma(\alpha)}\left\|u-v\right\|.\end{aligned}$$

Then, for all u,v, we obtain

$$\left\|Pu-Pv\right\| \le \frac{L(\mu+1)^{1-\alpha}B(1,\alpha)}{\Gamma(\alpha)}\left\|u-v\right\|.$$

Thus, the operator P is a contraction mapping then in view of Banach fixed point theorem, P has a unique fixed point which corresponds to the solution of the initial value problem (15).

5 Conclusion

From above, we made a generalization to one of the most important differential and integral operators (Srivastava-Owa operators) of arbitrary order in the unit disk. We found that the generalize integral operator satisfying the semi-group property. Furthermore, their applications appeared in the theory of geometric functions and fractional differential equations by establishing the sufficient conditions for the existence and uniqueness of Cauchy problem in the unit disk.

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Competing interests
The authors declare that they have no competing interests.

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