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Existence results for a coupled system of nonlinear fractional 2m-point boundary value problems at resonance

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Abstract

A 2m-point boundary value problem for a coupled system of nonlinear fractional differential equations is considered in this article. An existence result is obtained with the use of the coincidence degree theory.

MSC: 34B17; 34L09.

Keywords: coincidence degree, fractional nonlinear differential equation, 2m-point boundary conditions

1. Introduction

In this article, we will consider a 2m-point boundary value problem (BVP) at resonance for a coupled system of nonlinear fractional differential equations given by

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,\ v(t),\ D_{0+}^{\beta-1}v(t),\ D_{0+}^{\beta-2}v(t)),\ 0 < t < 1, \\ D_{0+}^{\beta}v(t) = g(t,\ u(t),\ D_{0+}^{\alpha-1}u(t),\ D_{0+}^{\alpha-2}u(t)),\ 0 < t < 1, \end{cases} \tag{1.1}$$

$$I_{0+}^{3-\alpha}u(t)|_{t=0} = 0, \quad D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m} a_i D_{0+}^{\alpha-2}u(\xi_i), \quad u(1) = \sum_{i=1}^{m} b_i u(\eta_i), \tag{1.2}$$

$$I_{0+}^{3-\beta}v(t)|_{t=0} = 0, \quad D_{0+}^{\beta-2}v(1) = \sum_{j=1}^{m} c_j D_{0+}^{\beta-2}v(\gamma_j), \quad v(1) = \sum_{j=1}^{m} d_j v(\delta_j), \tag{1.3}$$

where $2 < \alpha, \beta \le 3$, $0 < \xi_1 < \ldots < \xi_m < 1$, $0 < \eta_1 < \ldots < \eta_m < 1$, $0 < \gamma_1 < \ldots < \gamma_m < 1$, $0 < \delta_1 < \ldots < \delta_m < 1$, $a_i, b_i, c_j, d_j \in R$, $f, g : [0, 1] \times R^3 \to R$, f, g satisfies Carathéodory conditions, D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville fractional derivative and fractional integral, respectively.

Setting:

$$\Lambda_1 = \frac{1}{\alpha(\alpha-1)}\left(1-\sum_{i=1}^{m} a_i\xi_i^{\alpha+1}\right), \quad \Lambda_2 = \frac{1}{\alpha(\alpha-1)}\left(1-\sum_{i=1}^{m} a_i\xi_i^{\alpha}\right),$$
$$\Lambda_3 = \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\left[1-\sum_{i=1}^{m} b_i\eta_i^{2\alpha-1}\right], \quad \Lambda_4 = \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left[1-\sum_{i=1}^{m} b_i\eta_i^{2\alpha-2}\right],$$
$$\Delta_1 = \frac{1}{\beta(\beta-1)}\left(1-\sum_{j=1}^{n-2} c_j\gamma_j^{\beta+1}\right), \quad \Delta_2 = \frac{1}{\beta(\beta-1)}\left(1-\sum_{j=1}^{n-2} c_j\gamma_j^{\beta}\right),$$
$$\Delta_3 = \frac{(\Gamma(\beta))^2}{\Gamma(2\beta)}\left[1-\sum_{j=1}^{m} d_i\delta_j^{2\beta-1}\right], \quad \Delta_4 = \frac{\Gamma(\beta)\Gamma(\beta-1)}{\Gamma(2\beta-1)}\left[1-\sum_{j=1}^{m} d_j\delta_j^{2\beta-2}\right].$$

In this article, we will always suppose that the following conditions hold:
(C1):

$$\sum_{i=1}^{m} a_i\xi_i = \sum_{i=1}^{m} a_i = 1, \quad \sum_{i=1}^{m} b_i\eta_i^{\alpha-1} = \sum_{i=1}^{m} b_i\eta_i^{\alpha-2} = 1,$$
$$\sum_{j=1}^{m} c_j\gamma_j = \sum_{j=1}^{m} c_j = 1, \quad \sum_{j=1}^{m} d_j\delta_j^{\beta-1} = \sum_{j=1}^{m} d_j\delta_j^{\beta-2} = 1;$$

(C2):

$$\Lambda = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3 \neq 0, \quad \Delta = \Delta_1\Delta_4 - \Delta_2\Delta_3 \neq 0.$$

The subject of fractional calculus has gained considerable popularity and importance because of its frequent appearance in various fields such as physics, chemistry, and engineering. In consequence, the subject of fractional differential equations has attracted much attention. For details, refer to [1-4] and the references therein. Some basic theory for the initial value problems of fractional differential equations(FDE) involving Riemann-Liouville differential operator has been discussed by Lakshmikantham [5-7], El-Sayed et al. [8,9], Diethelm and Ford [10], Bai [11], and so on. Also, there are some articles which deal with the existence and multiplicity of solutions for nonlinear FDE BVPs using techniques of topological degree theory. For example, Su [12] considered the BVP of the coupled system

$$\begin{cases} D^{\alpha}u(t) = f(t,\ v(t),\ D^{\mu}v(t)), \\ D^{\beta}v(t) = g(t, u(t),\ D^{\nu}u(t)). \end{cases}$$

By using the Schauder fixed point theorem, one existence result was given.

However, there are few articles which consider the BVP at resonance for nonlinear ordinary differential equations of fractional order. In [13], Zhang and Bai investigated the nonlinear nonlocal problem

$$D_{0+}^{\alpha}u(t) = f(t,\ u(t)), \quad 0 < t < 1,$$
$$u(0) = 0, \quad \beta u(\eta) = u(1),$$

where $1 < \alpha \leq 2$, we consider the case $\beta\eta^{\alpha-1} = 1$, i.e., the resonance case.

In [14], Bai investigated the BVP at resonance

$$D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)) + e(t), \quad 0 < t < 1,$$

$$I_{0+}^{2-\alpha}u(t)\mid_{t=0} = 0, \quad D_{0+}^{\alpha-1}u(1) = \sum_{i=0}^{m-2}\beta_i D_{0+}^{\alpha-1}u(\eta_i)$$

is considered, where $1 < \alpha \le 2$ is a real number, $D_{0^+}^{\alpha}$ and $I_{0^+}^{\alpha}$ are the standard Riemann-Liouville fractional derivative and fractional integral, respectively, and $f : [0, 1] \times R^2 \to R$ is continuous, and $e(t) \in L^1[0, 1]$, $m \ge 2$, $0 < \xi_i < 1$, $\beta_i \in R$, $i = 1, 2, ..., m - 2$, are given constants such that $\sum_{i=1}^{m-2}\beta_i = 1$.

The coupled system (1.1)-(1.3) happens to be at resonance in the sense that the associated linear homogeneous coupled system

$$\begin{cases} D_{0+}^{\alpha}u(t) = 0, \ 0 < t < 1, \\ D_{0+}^{\beta}v(t) = 0, \ 0 < t < 1, \end{cases}$$

$$I_{0+}^{3-\alpha}u(t)|_{t=0} = 0, \quad D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m} a_i D_{0+}^{\alpha-2}u(\xi_i), \quad u(1) = \sum_{i=1}^{m} b_i u(\eta_i),$$

$$I_{0+}^{3-\beta}v(t)|_{t=0} = 0, \quad D_{0+}^{\beta-2}v(1) = \sum_{j=1}^{m} c_j D_{0+}^{\beta-2}v(\gamma_j), \quad v(1) = \sum_{i=1}^{m} d_j v(\delta_j)$$

has $(u(t), v(t)) = (at^{\alpha-1} + bt^{\alpha-2}, ct^{\beta-1} + dt^{\beta-2})$, $a, b, c, d \in R$ as a nontrivial solution.

The purpose of this article is to study the existence of solution for BVP (1.1)-(1.3) at resonance case, and establish an existence theorem under nonlinear growth restriction of f. Our method is based upon the coincidence degree theory of Mawhin.

Now, we will briefiy recall some notation and an abstract existence result.

Let Y, Z be real Banach spaces, $L : domL \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that

$$Y = KerL \oplus KerP, \quad Z = ImQ \oplus ImL, \ ImP = KerL, \ KerQ = ImL.$$

It follows that $L|_{domL \cap KerP} : domL \cap Ker\ P \to ImL$ is invertible. We denote the inverse of the map by K_p. If Ω is an open bounded subset of Y such that $domL \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ will be called L-compact on Ω if $QN(\overline{\Omega})$ is bounded, and $K_p(I - Q)N : \overline{\Omega} \to Y$ is compact.

The theorem that we used is Theorem 2.4 of [15].

Theorem 1.1. Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:

(i) $Lx \neq \lambda Nx \ \forall (x, \lambda) \in [domL \backslash KerL \cap \partial\Omega] \times [0, 1]$;

(ii)) $Nx \notin ImL, \forall x \in KerL \cap \partial\Omega$;

(iii) $deg(JQN|_{KerL}, KerL \cap \Omega, 0) \neq 0$;

where $Q : Z \to Z$ is a projection as above with $KerQ = ImL$, and $J : ImQ \to KerL$ is any isomorphism. Then, the equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

The rest of this article is organized as follows. In Section 2, we give some notation and lemmas. In Section 3, we establish a theorem of existence of a solution for the problem (1.1)-(1.3).

2. Background materials and preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory. These definitions can be found in the recent literature [1-14,16].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) ds,$$

provided the right side is pointwise defined on $(0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

Where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0, \infty)$.

Definition 2.3. We say that the map $f : [0, 1] \times R^n \to R$ satisfies Carathéodory conditions with respect to $L^1[0, 1]$ if the following conditions are satisfied:

(i) for each $z \in R^n$, the mapping $t \to f(t, z)$ is Lebesgue measurable;

(ii) for almost every $t \in [0, 1]$, the mapping $t \to f(t, z)$ is continuous on R^n;

(iii) for each $r > 0$, there exists $\rho_r \in L^1([0, 1], R)$ such that, for a.e. $t \in [0, 1]$ and every $|z| \le r$, we have $f(t, z) \le \rho_r(t)$.

Lemma 2.1. [13] Assume that $u \in C(0, 1) \cap L^1(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L^1(0, 1)$. Then,

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_1 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$$

for some $c_i \in R$, $i = 1, 2, ..., N$, where N is the smallest integer grater than or equal to α.

We use the classical Banach space $C[0, 1]$ with the norm

$$||u||_\infty = \max_{t\in[0,1]} |u(t)|,$$

$L[0, 1]$ with the norm $||u||_1 = \int_0^1 |u(t)|dt$. For $n \in N$, we denote by $AC^n[0, 1]$ the space of functions $u(t)$ which have continuous derivatives up to order $n - 1$ on $[0, 1]$ such that $u^{(n-1)}(t)$ is absolutely continuous:

$AC^n[0, 1] = \{u|[0, 1] \to R \text{ and } D^{n-1}u(t) \text{ is absolutely continuous in } [0, 1]\}$.

Definition 2.4. Given $\mu > 0$ and $N = [\mu] + 1$ we can define a linear space

$$C^{\mu}[0, 1] = \{u(t)\ |u(t) = I_{0^+}^{\mu} x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \cdots + c_N t^{\mu-(N-1)},\ t \in (0, 1)\},$$

where $x \in C[0, 1]$, $c_i \in R$, $i = 1, 2, ..., N - 1$.

Remark 2.1. By means of the linear functional analysis theory, we can prove that with the

$$||u||_{C^\mu} = ||D_{0^+}^{\mu} u||_\infty + \cdots + ||D_{0^+}^{\mu-(N-1)} u||_\infty + ||u||_\infty,$$

$C^{\mu}[0, 1]$ is a Banach space.

Remark 2.2. If μ is a natural number, then C^{μ} [0, 1] is in accordance with the classical Banach space C^{n} [0, 1].

Lemma 2.2. [13]$f \subset C^{\mu}$ [0, 1] is a sequentially compact set if and only if f is uniformly bounded and equicontinuous. Here, uniformly bounded means there exists $M >$ 0, such that for every $u \in f$

$$||u||_{C^{\mu}} = ||D_{0+}^{\mu}u||_{\infty} + \cdots + ||D_{0+}^{\mu-(N-1)}u||_{\infty} + ||u||_{\infty} < M,$$

and equicontinuous means that $\forall\varepsilon > 0$, $\exists\delta > 0$, such that

$$|u(t_1) - u(t_2)| < \varepsilon \quad (\forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta, \forall u \in f)$$

and

$$|D_{0+}^{\alpha-i}u(t_1) - D_{0+}^{\alpha-i}u(t_2)| < \varepsilon$$
$$(\forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta, \forall u \in f, \forall i = 1, 2, \ldots, N-1)$$

Lemma 2.3. [14] Let $\alpha > 0$, $n = [\alpha] + 1$. Assume that $u \in L^1$ (0, 1) with a fractional integration of order $n - \alpha$ that belongs to $AC^n[0, 1]$. Then, the equality

$$(I_{0+}^{\alpha}D_{0+}^{\alpha}u)(t) = u(t) - \sum_{i=1}^{n} \frac{((I_{0+}^{n-\alpha}u)(t))^{n-i}|_{t=0}}{\Gamma(\alpha - i + 1)} t^{\alpha-i}$$

holds almost everywhere on [0, 1].

Definition 2.5. [14] Let $I_{0+}^{\alpha}(L^1(0,1))$, $\alpha > 0$ denote the space of functions $u(t)$, represented by fractional integral of order α of a summable function: $u = I_{0+}^{\alpha}v,\ v \in L^1(0,1)$.

In the following lemma, we use the unified notation of both for fractional integrals and fractional derivatives assuming that $I_{0+}^{\alpha} = D_{0+}^{-\alpha}$ for $\alpha > 0$.

Let $Z_1 = L^1[0, 1]$, with the norm $||y|| = \int_0^1 |y(s)|ds$, $Y_1 = C^{\alpha-1}[0, 1]$, $Y_2 = C^{\beta-1}[0, 1]$, defined by Remark 2.1, with the norm

$$||u||_{Y_1} = ||D_{0^+}^{\alpha-1}u||_{\infty} + ||D_{0^+}^{\alpha-2}u||_{\infty} + ||u||_{\infty},$$
$$||v||_{Y_2} = ||D_{0^+}^{\beta-1}v||_{\infty} + ||D_{0^+}^{\beta-2}v||_{\infty} + ||v||_{\infty},$$

where $Y = Y_1 \times Y_2$ is a Banach space, with the norm

$$||(u, v)||_Y = \max\{||u||_{Y_1}, ||v||_{Y_2}\},$$

and $Z = Z_1 \times Z_1$ is a Banach space, with the norm

$$||(x, y)||_Z = \max\{||x||_1, ||y||_1\}.$$

Define L_1 to be the linear operator from $domL_1 \cap Y_1$ to Z_1 with

$$domL_1 = \{u \in C^{\alpha-1}[0,1] | D_{0+}^{\alpha}u \in L^1[0,1],\ u \text{ satisfies } (1.2)\},$$

and

$$L_1 u = D_{0^+}^{\alpha}u,\ u \in domL_1.$$

Define L_2 to be the linear operator from $domL_2 \cap Y_2$ to Z_1 with

$$domL_2 = \{v \in C^{\beta-1}[0,1] | D_{0+}^{\beta}v \in L^1[0,1],\ v \text{ satisfies } (1.3)\},$$

and

$$L_2v = D^{\beta}_{0^+}v,\ v \in domL_2.$$

Define L to be the linear operator from $domL \cap Y$ to Z with

$$domL = \{(u,\ v) \in Y | u \in domL_1,\ v \in domL_2\},$$

and

$$L(u,\ v) = (L_1u,\ L_2v),$$

we define $N : Y \to Z$ by setting

$$N(u,\ v) = (N_1v,\ N_2u),$$

where $N_1 : Y_2 \to Z_1$ is defined by

$$N_1v(t) = f(t,\ v(t),\ D^{\beta-1}_{0+}v(t)),\ D^{\beta-2}_{0+}v(t)),$$

and $N_2 : Y_1 \to Z_2$ is defined by

$$N_2u(t) = f(t,\ u(t),\ D^{\alpha-1}_{0+}u(t)),\ D^{\alpha-2}_{0+}u(t)).$$

Then, the coupled system of BVPs (1.1) can be written as

$$L(u,\ v) = N(u,\ v).$$

3. Main results

Lemma 3.1. The mapping $L : domL \subset Y \to Z$ is a Fredholm operator of index zero.

Proof. Let $L_1u = D^{\alpha}_{0^+}u$, by Lemma 2.3, $D^{\alpha}_{0+}u(t) = 0$ has solution

$$\begin{aligned} u(t) &= \sum_{i=1}^{3} \frac{((I^{3-\alpha}_{0+}u)(t))^{3-i}|_{t=0}}{\Gamma(\alpha - i + 1)} t^{\alpha-i} \\ &= \frac{((I^{3-\alpha}_{0+}u)(t))''|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{((I^{3-\alpha}_{0+}u)(t))'|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{((I^{3-\alpha}_{0+}u)(t))|_{t=0}}{\Gamma(\alpha-2)} t^{\alpha-3} \\ &= \frac{D^{\alpha-1}_{0+}u(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{D^{\alpha-2}_{0+}u(t)|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{((I^{3-\alpha}_{0+}u)(t))|_{t=0}}{\Gamma(\alpha-2)} t^{\alpha-3}. \end{aligned}$$

Combine with (1.2), so

$$KerL_1 = \{at^{\alpha-1} + bt^{\alpha-2} | a,\ b \in R\} \cong R^2.$$

Similarly, let $L_2v = D^{\beta}_{0^+}v$, by Lemmas 2.3, 2.4, $D^{\beta}_{0+}v(t) = 0$, combine with (1.3), so

$$KerL_2 = \{ct^{\beta-1} + dt^{\beta-2} | c,\ d \in R\} \cong R^2.$$

It is clear that

$$KerL = \{(at^{\alpha-1} + bt^{\alpha-2},\ ct^{\beta-1} + dt^{\beta-2}) | a,\ b,\ c,\ d \in R\} \cong R^2 \times R^2.$$

Let $(x, y) \in ImL$, then there exists $(u, v) \in domL$, such that $(x, y) = L(u, v)$, that is $u \in Y_1$, $x = D^{\alpha}_{0+}u$ and $v \in Y_2$, $y = D^{\beta}_{0+}v$. By Lemma 2.3, we have

$$I_{0+}^{\alpha}x(t) = u(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - c_3 t^{\alpha-3},$$
$$I_{0+}^{\beta}y(t) = v(t) - d_1 t^{\beta-1} - d_2 t^{\beta-2} - d_3 t^{\beta-3},$$

where

$$c_1 = \frac{D_{0+}^{\alpha-1}u(t)|_{t=0}}{\Gamma(\alpha)}, \quad c_2 = \frac{D_{0+}^{\alpha-2}u(t)|_{t=0}}{\Gamma(\alpha-1)}, \quad c_3 = \frac{I_{0+}^{3-\alpha}u(t)|_{t=0}}{\Gamma(\alpha-2)},$$
$$d_1 = \frac{D_{0+}^{\beta-1}v(t)|_{t=0}}{\Gamma(\beta)}, \quad d_2 = \frac{D_{0+}^{\beta-2}v(t)|_{t=0}}{\Gamma(\beta-1)}, \quad d_3 = \frac{I_{0+}^{3-\beta}v(t)|_{t=0}}{\Gamma(\beta-2)},$$

and by the boundary condition (1.2), we obtain $c_3 = 0$, c_1, c_2 can be any constant, and x satisfies

$$\begin{cases} \int_0^1 (1-s)x(s)ds - \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)x(s)ds = 0, \\ \int_0^1 (1-s)^{\alpha-1}x(s)ds - \sum_{i=1}^m b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1}x(s)ds = 0. \end{cases} \tag{3.1}$$

Similarly, by the boundary condition (1.3), we obtain $d_3 = 0$, d_1, d_2 can be any constant, and y satisfies

$$\begin{cases} \int_0^1 (1-s)y(s)ds - \sum_{j=1}^m c_j \int_0^{\gamma_j} (\gamma_j - s)y(s)ds = 0, \\ \int_0^1 (1-s)^{\beta-1}y(s)ds - \sum_{j=1}^m d_j \int_0^{\delta_j} (\delta_j - s)^{\beta-1}y(s)ds = 0. \end{cases} \tag{3.2}$$

On the other hand, suppose x, $y \in Z_1$ satisfy (3.1), (3.2), respectively, let $v(t) = I_{0+}^{\beta}y(t)$, $v(t) = I_{0+}^{\beta}y(t)$, then $u \in domL_1$, $D_{0+}^{\alpha}u(t) = x(t)$ and $v \in domL_2$, $D_{0+}^{\beta}v(t) = y(t)$. That is to say, $(x, y) \in ImL$. From the above argument, we obtain

$$ImL = \{(x, y) \in Z | x \text{ satisfies } (3.1), y\text{satisfies } (3.2)\}.$$

Consider the continuous linear mapping A_i, B_i, T_i, R_i, $Q_i : Z_1 \to Z_1$, $i = 1, 2$ and $Q : Z \to Z$ defined by

$$A_1 x = \int_0^1 (1-s)x(s)ds - \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)x(s)ds,$$
$$A_2 x = \int_0^1 (1-s)^{\alpha-1}x(s)ds - \sum_{i=1}^m b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1}x(s)ds$$

and

$$B_1 y = \int_0^1 (1-s)y(s)ds - \sum_{j=1}^m c_j \int_0^{\gamma_j} (\gamma_j - s)y(s)ds,$$
$$B_2 y = \int_0^1 (1-s)^{\beta-1}y(s)ds - \sum_{j=1}^m d_j \int_0^{\delta_j} (\delta_j - s)^{\beta-1}y(s)ds. \tag{3.3}$$
$$T_1 x = \frac{1}{\Lambda}(\Lambda_4 A_1 x - \Lambda_2 A_2 x), \quad T_2 x = \frac{1}{\Lambda}(\Lambda_3 A_1 x - \Lambda_1 A_2 x)$$

and

$$R_1 y = \frac{1}{\Delta}(\Delta_4 B_1 y - \Delta_2 B_2 y), \quad R_2 y = \frac{1}{\Delta}(\Delta_3 B_1 y - \Delta_1 B_2 y). \tag{3.4}$$

Since the conditions (C1) and (C2) hold, the mapping defined by

$$\begin{cases} Q_1x(t) &= (T_1x(t))t^{\alpha-1} + (T_2x(t))t^{\alpha-2}, \\ Q_2y(t) &= (R_1y(t))t^{\beta-1} + (R_2y(t))t^{\beta-2} \end{cases} \tag{3.5}$$

is well-defined. It is clear that $dimImQ_1 = dimImQ_2 = 2$.

Recall (C1) and (C2) and note that

$$\begin{aligned} T_1(T_1xt^{\alpha-1}) &= \frac{1}{\Lambda}(\Lambda_4A_1(T_1xt^{\alpha-1}) - \Lambda_2A_2(T_1xt^{\alpha-1})) \\ &= \frac{1}{\Lambda}\left[\Lambda_4\left(\frac{\Lambda_4\Lambda_1}{\Lambda}A_1x - \frac{\Lambda_1\Lambda_2}{\Lambda}A_2x\right) - \Lambda_2\left(\frac{\Lambda_4\Lambda_3}{\Lambda}A_1x - \frac{\Lambda_2\Lambda_3}{\Lambda}A_2x\right)\right] \\ &= T_1x, \end{aligned}$$

and similarly we can derive that

$$T_1(T_2xt^{\alpha-2}) = 0, \quad T_2(T_1xt^{\alpha-1}) = 0, \quad T_2(T_2xt^{\alpha-2}) = T_2x.$$

Hence, for $x \in Z_1$, it follows from the four relations above that

$$\begin{aligned} Q_1^2x &= Q_1((T_1x)t^{\alpha-1} + (T_2x)t^{\alpha-2}) \\ &= T_1((T_1x)t^{\alpha-1} + (T_2x)t^{\alpha-2})t^{\alpha-1} + T_2((T_1x)t^{\alpha-1} + (T_2x)t^{\alpha-2})t^{\alpha-2} \\ &= (T_1x)t^{\alpha-1} + (T_2x)t^{\alpha-2} \\ &= Q_1x, \end{aligned}$$

that is, the map Q_1 is idempotent. In fact, Q_1 is a continuous linear projector.

Similarly, the map Q_2 is a continuous linear projector.

Therefore,

$$Q(x, y) = (Q_1x, Q_2y).$$

It is clear that Q is a continuous linear projector.

Note $(x, y) \in ImL$ implies $Q(x, y) = (Q_1x, Q_2y) = (0, 0)$. Conversely, if $Q(x, y) = (0, 0)$, so

$$\begin{cases} \Lambda_4A_1x - \Lambda_2A_2x = 0, \\ \Lambda_1A_2x - \Lambda_3A_1x = 0, \end{cases}$$

$$\begin{cases} \Delta_4B_1y - \Delta_2B_2y = 0, \\ \Delta_1B_2y - \Delta_3B_1y = 0, \end{cases}$$

but

$$\begin{vmatrix} \Lambda_4 & -\Lambda_2 \\ -\Lambda_3 & \Lambda_1 \end{vmatrix} = \Lambda_4\Lambda_1 - \Lambda_2\Lambda_3 \neq 0,$$

$$\begin{vmatrix} \Delta_4 & -\Delta_2 \\ -\Delta_3 & \Delta_1 \end{vmatrix} = \Delta_4\Delta_1 - \Delta_2\Delta_3 \neq 0,$$

then we must have $A_ix = B_iy = 0$, $i = 1, 2$, that is, $(x, y) \in ImL$. In fact, $KerQ = ImL$.

Take $(x, y) \in Z$ in the form $(x, y) = ((x, y) - Q(x, y)) + Q(x, y)$ so that $((x, y) - Q(x, y)) \in KerQ = ImL$, $Q(x, y) \in ImQ$. Thus, $Z = ImL + ImQ$. Let $(x, y) \in ImL \cap ImQ$ and assume that $(x, y) = (at^{\alpha-1} + bt^{\alpha-2}, ct^{\beta-1} + dt^{\beta-2})$ is not identically zero on [0, 1]. Then, since $(x, y) \in ImL$, from (3.1) and (3.2) and the condition (C2), we have

$$A_1x = \int_0^1 (1-s)(as^{\alpha-1} + bs^{\alpha-2})ds - \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)(as^{\alpha-1} + bs^{\alpha-2})ds = 0,$$
$$A_2x = \int_0^1 (1-s)^{\alpha-1}(as^{\alpha-1} + bs^{\alpha-2})ds - \sum_{i=1}^m b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1}(as^{\alpha-1} + bs^{\alpha-2})ds = 0,$$
$$B_1y = \int_0^1 (1-s)(cs^{\beta-1} + ds^{\beta-2})ds - \sum_{j=1}^m c_j \int_0^{\gamma j} (\gamma_j - s)(cs^{\beta-1} + ds^{\beta-2})ds = 0,$$
$$B_2y = \int_0^1 (1-s)^{\beta-1}(cs^{\beta-1} + ds^{\beta-2})ds - \sum_{j=1}^m d_j \int_0^{\delta_j} (\delta_j - s)^{\beta-1}(cs^{\beta-1} + ds^{\beta-2})ds = 0.$$

So,

$$\begin{cases} a\Lambda_1 + b\Lambda_2 = 0, \\ a\Lambda_3 + b\Lambda_4 = 0, \end{cases}$$
$$\begin{cases} c\Delta_1 + d\Delta_2 = 0, \\ c\Delta_3 + d\Delta_4 = 0, \end{cases}$$

but

$$\begin{vmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{vmatrix} = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3 \neq 0,$$
$$\begin{vmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & \Delta_4 \end{vmatrix} = \Delta_1\Delta_4 - \Delta_2\Delta_3 \neq 0,$$

we derive $a = b = c = d = 0$, which is a contradiction. Hence, $ImL \cap ImQ = \{0, 0\}$; thus, $Z = ImL \oplus ImQ$.

Now, $IndL = dimKerL - codimImL = 0$, and so L is a Fredholm operator of index zero.

Let $P_1 : Y_1 \to Y_1$, $P_2 : Y_2 \to Y_2$, $P : Y \to Y$ be defined by

$$P_1u(t) = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} u(t)|_{t=0} t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2} u(t)|_{t=0} t^{\alpha-2}, \quad t \in [0, 1],$$
$$P_2v(t) = \frac{1}{\Gamma(\beta)} D_{0+}^{\beta-1} v(t)|_{t=0} t^{\beta-1} + \frac{1}{\Gamma(\beta-1)} D_{0+}^{\beta-1} v(t)|_{t=0} t^{\beta-2}, \quad t \in [0, 1],$$

and

$$P(u, v) = (P_1u, P_2v).$$

Note that P_1, P_2, P are continuous linear projectors and

$$\begin{aligned} KerP &= (KerP_l, KerP_2) \\ &= \{(u, v) \in Y | D_{0+}^{\alpha-1}u(0) = D_{0+}^{\alpha-2}u(0) = 0, D_{0+}^{\beta-1}v(0) = D_{0+}^{\beta-2}v(0) = 0\}. \end{aligned}$$

It is clear that $Y = KerL \oplus KerP$.

Note that the projectors P and Q are exact. Define by $K_p : ImL \to domL \cap KerP$ by

$$K_P(x, y) = (I_{0+}^{\alpha}x, I_{0+}^{\beta}y),$$
$$K_px(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}x(s)ds = I_{0+}^{\alpha}x(t), \quad x \in ImL.$$

Hence, we have

$$D_{0+}^{\alpha-1}(K_p x)t = \int_0^t x(s)ds, \; D_{0+}^{\alpha-2}(K_p x)t = \int_0^t (t-s)x(s)ds.$$

Then,

$$||K_p x||_\infty \le \frac{1}{\Gamma(\alpha)}||x||_1, \quad ||D_{0+}^{\alpha-1}(K_p x)||_\infty \le ||x||_1, \quad ||D_{0+}^{\alpha-2}(K_p x)||_\infty \le ||x||_1,$$

and thus

$$||K_p x||_{Y_1} \le \left(\frac{1}{\Gamma(\alpha)} + 2\right)||x||_1$$

and

$$K_p y(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}y(s)ds = I_{0+}^{\beta}y(t), \quad y \in ImL.$$

Hence, we have

$$D_{0+}^{\beta-1}(K_p y)t = \int_0^t y(s)ds, \quad D_{0+}^{\beta-2}(K_p y)t = \int_0^t (t-s)y(s)ds.$$

Then,

$$||K_p y||_\infty \le \frac{1}{\Gamma(\beta)}||y||_1, \quad ||D_{0+}^{\beta-1}(K_p y)||_\infty \le ||y||_1, \quad ||D_{0+}^{\beta-2}(K_p y)||_\infty \le ||y||_1,$$

and thus

$$||K_p y||_{Y_2} \le \left(\frac{1}{\Gamma(\beta)} + 2\right)||y||_1,$$

so

$$\begin{aligned} \| K_p(x, y)\|_Y &= \| (I_{0+}^{\alpha}x, I_{0+}^{\beta}y)\|_Y \\ &= \max\{\| I_{0+}^{\alpha}x\|_{Y_1}, \| I_{0+}^{\beta}y\|_{Y_2}\} \\ &\le \max\left\{\left(\frac{1}{\Gamma(\alpha)+2}\right)\| x\|_1, \left(\frac{1}{\Gamma(\beta)+2}\right)\| y\|_1\right\}. \end{aligned}$$

For $(x, y) \in ImL$, we have

$$LK_P(x, y) = L(I_{0+}^{\alpha}x, I_{0+}^{\beta}y) = (D_{0+}^{\alpha}I_{0+}^{\alpha}x, D_{0+}^{\beta}I_{0+}^{\beta}y) = (x, y).$$

Also, if $(u, v) \in domL \cap KerP$, we have $u \in domL_1$, $D_{0+}^{\alpha-1}u(0) = D_{0+}^{\alpha-2}u(0) = 0$, $v \in domL_2$, $D_{0+}^{\beta-1}v(0) = D_{0+}^{\beta-2}v(0) = 0$, so the coefficients c_i, d_i, $i = 1, 2, 3$ in the expressions then

$$(K_p L_1)u(t) = I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},$$

where

$$c_1 = \frac{D_{0+}^{\alpha-1}u(t)|_{t=0}}{\Gamma(\alpha)}, \quad c_2 = \frac{D_{0+}^{\alpha-2}u(t)|_{t=0}}{\Gamma(\alpha-1)}, \quad c_3 = \frac{I_{0+}^{3-\alpha}u(t)|_{t=0}}{\Gamma(\alpha-2)},$$

$$(K_pL_2)v(t) = I_{0+}^{\beta}D_{0+}^{\beta}v(t) = v(t) \ + d_1t^{\beta-1} + d_2t^{\beta-2} + d_3t^{\beta-3},$$

where

$$d_1 = \frac{D_{0+}^{\beta-1}v(t)|_{t=0}}{\Gamma(\beta)}, \quad d_2 = \frac{D_{0+}^{\beta-2}v(t)|_{t=0}}{\Gamma(\beta-1)}, \quad d_3 = \frac{I_{0+}^{3-\beta}v(t)|_{t=0}}{\Gamma(\beta-2)},$$

and from the boundary value conditions (1.2), (1.3) and the fact that $(u, v) \in domL \cap KerP$, $P(u, v) = 0$, we have $c_i = d_i = 0$, thus

$$(K_pL)(u,\ v) = K_p(L_1u,\ L_2v) = (u,\ v).$$

This shows that $K_p = [L|_{domL\cap KerP}]^{-1}$.

Using (3.3)-(3.5), we write

$$K_p(I-Q)N(x,y) = \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t\ -s)^{\alpha-1}[N_1x(s)-Q_1N_1x(s)]ds,\right.$$
$$\left.\frac{1}{\Gamma(\beta)}\int_0^t (t\ -s)^{\beta-1}[N_2y(s)-Q_2N_{2y2}(s)]ds\right).$$

By Lemma 2.2 and a standard method, we obtain the following lemma.

Lemma 3.2. [16]$K_p(I - Q)N : Y \to Y$ is completely continuous.

In this section, we shall prove existence results for (1.1)-(1.3).

First, let us set the following notations for convenience:

$$m = 3 + \frac{1}{\Gamma(\alpha)} + +\frac{1}{\Gamma(\alpha-1)}, \quad n = 3 + \frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-1)},$$
$$j = 2 + \frac{1}{\Gamma(\alpha)}, \quad k = 2 + \frac{1}{\Gamma(\beta)},$$
$$q = 5 + \frac{2}{\Gamma(\alpha)} + +\frac{1}{\Gamma(\alpha-1)}, \quad w = 5 + \frac{2}{\Gamma(\beta)} + +\frac{1}{\Gamma(\beta-1)}.$$

Assume that the following conditions on the function $f(t, x, y, z)$, $g(t, x, y, z)$ are satisfied:

(H1) There exist functions $a_i(t)$, $b_i(t)$, $c_i(t)$, $d_i(t)$, $r_i(t) \in L^1[0, 1]$, $i = 1, 2$ and a constant $\theta_i \in [0, 1)$, $i = 1, 2$ such that for all $(x, y, z) \in R^3$, $t \in [0, 1]$, one of the following inequalities is satisfied:

$$|f(t,\ x,\ y,\ z)|\ \le a_1(t)|x| + b_1(t)|y| + c_1(t)|z| + d_1(t)|x|^{\theta_1} + r_1(t), \tag{3.6}$$

$$|g(t,\ x,\ y,\ z)|\ \le a_2(t)|x| + b_2(t)|y| + c_2(t)|z| + d_2(t)|x|^{\theta_2} + r_2(t). \tag{3.7}$$

(H2) There exists a constant $A > 0$, such that for $(u, v) \in domL\backslash KerL$ satisfying

$$\min\{|D_{0+}^{\alpha-1}u(t)| + |D_{0+}^{\alpha-2}u(t)|,\ |D_{0+}^{\beta-1}v(t)| + |D_{0+}^{\beta-2}v(t)|\} > A$$

or for all $t \in [0, 1]$, we have

$$A_1N_1v(t) \neq 0, \quad B_1N_2u(t) \neq 0 \quad \text{or} \quad A_2N_1v(t) \neq 0,\ B_2N_2u(t) \neq 0.$$

(H3) There exists a constant $B > 0$ such that for every $a, b, c, d \in R$ satisfying $\min\{a^2 + b^2, c^2 + d^2\} > B$ then either

$$aT_1N_1(at^{\alpha-1} + bt^{\alpha-2}) + bT_2N_1(at^{\alpha-1} + bt^{\alpha-2}) > 0, \tag{3.8}$$

$$cR_1N_2(ct^{\beta-1} + dt^{\beta-2}) + dR_2N_2(ct^{\beta-1} + dt^{\beta-2}) > 0, \tag{3.9}$$

or

$$aT_1N_1(at^{\alpha-1} + bt^{\alpha-2}) + bT_2N_1(at^{\alpha-1} + bt^{\alpha-2}) < 0, \tag{3.10}$$

$$cR_1N_2(ct^{\beta-1} + dt^{\beta-2}) + dR_2N_2(ct^{\beta-1} + dt^{\beta-2}) < 0, \tag{3.11}$$

Theorem 3.1 If (C1)-(C2) and (H1)-(H3) hold, then the BVP (1.1)-1.3) has at least one solution provided that

$$\begin{aligned}&\max\{q(\| a_1\|_1+ \| b_1\|_1+ \| c_1\|_1),\ w(\| a_2\|_1+ \| b_2\|_1+ \| c_2\|_1),\\ &j \| a_1\|_1 + n \| a_2\|_1 + j \| b_1\|_1 + n \| b_2\|_1 + j \| c_1\|_1 + n \| c_2\|_1,\\ &k \| a_1\|_1 + m \| a_2\|_1 + k \| b_1\|_1 + m \| b_2\|_1 + k \| c_1\|_1 + m \| c_2\|_1\} < 1.\end{aligned}$$

Proof. Set

$$\Omega_1 = \{(u,\ v) \in domL\backslash KerL\ : L(u,\ v) = \lambda N(u,\ v),\ \lambda \in [0, 1]\}.$$

Then, for $(u, v) \in \Omega_1$, $L(u, v) = \lambda N(u, v)$, thus $\lambda \neq 0$, $N(u, v) \in ImL = KerQ$, and hence $QN(u, v) = (0, 0)$ for all $t \in [0, 1]$. By the definition of Q, we have $Q_1N_1v(t) = Q_2N_2u(t) = 0$. It follows from (H2) that there exists $t_0, t_1 \in [0, 1]$, such that

$$\min\{|D_{0+}^{\alpha-1}u(t_0)| + |D_{0+}^{\alpha-2}u(t_0)|,\ |D_{0+}^{\beta-1}v(t_1)| + |D_{0+}^{\beta-2}v(t_1)|\} \leq A.$$

Now

$$\begin{aligned}D_{0+}^{\alpha-1}u(t) &= D_{0+}^{\alpha-1}u(t_0) + \int_{t_0}^{t} D_{0+}^{\alpha}u(s)ds,\\ D_{0+}^{\alpha-2}u(t) &= D_{0+}^{\alpha-2}u(t_0) + \int_{t_0}^{t} D_{0+}^{\alpha-1}u(s)ds,\\ D_{0+}^{\beta-1}v(t) &= D_{0+}^{\beta-1}v(t_1) + \int_{t_1}^{t} D_{0+}^{\beta}v(s)ds,\\ D_{0+}^{\beta-2}v(t) &= D_{0+}^{\beta-2}v(t_1) + \int_{t_1}^{t} D_{0+}^{\beta-1}v(s)ds,\end{aligned}$$

and so

$$\begin{aligned}|D_{0+}^{\alpha-1}u(0)| &\leq \| D_{0+}^{\alpha-1}u(t) \|_\infty\\ &\leq |D_{0+}^{\alpha-1}u(t_0)|+ \| D_{0+}^{\alpha-1}u\|_1\\ &\leq A+ \| L_1u\|_1 \leq A+ \| N_1v\|_1,\end{aligned}$$

and

$$\begin{aligned}|D_{0+}^{\alpha-2}u(0)| &\le \| D_{0+}^{\alpha-2}u(t) \|_\infty \\ &\le |D_{0+}^{\alpha-2}u(t_0)| + \| D_{0+}^{\alpha-1}u\|_\infty \\ &\le |D_{0+}^{\alpha-2}u(t_0)| + |D_{0+}^{\alpha-1}u(t_0)| + \| D_{0+}^{\alpha}u\|_1 \\ &\le A + \| L_1u\|_1 \le A + \| N_1v\|_1.\end{aligned}$$

Similarly,

$$|D_{0+}^{\beta-1}v(0)| \le A + \| N_2u\|_1, \ |D_{0+}^{\beta-2}v(0)| \le A + \| N_2u\|_1.$$

Therefore, we have noted that $(I - P)(u, v) \in domL \cap KerP$ for $\forall(u, v) \in \Omega_1$.
Then,

$$\begin{aligned}\| P(u,v)\|_Y &= \| (P_1u, P_2v)\|_Y = \max\{\| P_1u\|_{Y_1}, \| P_2v\|_{Y_2}\} \\ &= \max\left\{ \left\| \frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}u(0)t^{\alpha-2} \right\|_{Y_1}, \right. \\ &\quad \left. \left\| \frac{1}{\Gamma(\beta)}D_{0+}^{\beta-1}v(0)t^{\beta-1} + \frac{1}{\Gamma(\beta-1)}D_{0+}^{\beta-2}v(0)t^{\beta-2} \right\|_{Y_2} \right\} \\ &= \max\left\{ \left\| \frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}u(0)t^{\alpha-2} \right\|_\infty \right. \\ &\quad + \| D_{0+}^{\alpha-1}u(0)\|_\infty + \| D_{0+}^{\alpha-1}u(0)t + D_{0+}^{\alpha-2}u(0)\|_\infty, \\ &\quad \left\| \frac{1}{\Gamma(\beta)}D_{0+}^{\beta-1}v(0)t^{\beta-1} + \frac{1}{\Gamma(\beta-1)}D_{0+}^{\beta-2}v(0)t^{\beta-2} \right\|_\infty + \| D_{0+}^{\beta-1}v(0)\|_\infty \\ &\quad \left. + \| D_{0+}^{\beta-1}v(0)t + D_{0+}^{\beta-2}v(0)\|_\infty \right\} \\ &= \max\left\{ \left(3 + \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right) (A + \| N_1v\|_1), \right. \\ &\quad \left. \left(3 + \frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta-1)}\right) (A + \| N_2u\|_1) \right\} \\ &= \max\{m(A + \| N_1v\|_1), \ n(A + \| N_2u\|_1)\}.\end{aligned}$$

$$\begin{aligned}\| (I-P)(u,v)\|_Y &= \| K_pL(I-P)(u,v)\|_Y = \| K_p(L_1u, L_2v)\|_Y \\ &\le \max\{(2 + \frac{1}{\Gamma(\alpha)}) \| L_1u\|_1, \ (2 + \frac{1}{\Gamma(\beta)}) \| L_2v\|_1\} \\ &\le \max\{j \| N_1v\|_1, \ k \| N_2u\|_1\},\end{aligned}$$

so, we have

$$\begin{aligned}\| (u,v)\|_Y &\le \| (I-P)(u,v)\|_Y + \| P(u,v)\|_Y \\ &= \max\{m(A + \| N_1v\|_1), \ n(A + \| N_2u\|_1)\} \\ &\quad + \max\{j \| N_1v\|_1, \ k \| N_2u\|_1\} \\ &= \max\{q \| N_1v\|_1 + mA, \ m(A + \| N_1v\|_1) + k \| N_2u\|_1, \\ &\quad n(A + \| N_2u\|_1) + j \| N_1v\|_1, \ w \| N_2u\|_1 + nA\}.\end{aligned} \tag{3.12}$$

If the first condition of (H1) is satisfied, then from (3.12), the proof can be divided into four cases:

Case 1. $||(u, v)||_Y \le q||N_1v||_1 + mA$.

From (3.6), we have

$$\| (u,v)\|_Y \le q[\| a_1\|_1 \| v\|_\infty + \| b_1\|_1 \| D_{0+}^{\beta-1}v\|_\infty + \| c_1\|_1 \| D_{0+}^{\beta-2}v\|_\infty + \| d_1\|_1 \| D_{0+}^{\beta-2}v \|_\infty^{\theta_1} + D],$$

where $D = \| r_1\|_1 + \frac{mA}{q}$, and consequently, for

$$\| v\|_\infty,\ \| D_{0+}^{\beta-1}v\|_\infty,\ \| D_{0+}^{\beta-2}v\|_\infty \le \| (u,\ v)\|_Y,$$

so

$$\begin{aligned}
\| v\|_\infty &\le \frac{q}{1-q\| a_1\|_1}[\| b_1\|_1 \| D_{0+}^{\beta-1}v\|_\infty + \| c_1\|_1 \| D_{0+}^{\beta-2}v\|_\infty + \| d_1\|_1 \| D_{0+}^{\beta-2}v \|_\infty^{\theta_1} + D],\\
\| D_{0+}^{\beta-1}v\|_\infty &\le \frac{q}{1-q\| a_1\|_1 - q\| b_1\|_1}[\| c_1\|_1 \| D_{0+}^{\beta-2}x\|_\infty + \| d_1\|_1 \| D_{0+}^{\beta-2}v \|_\infty^{\theta_1} + D],\\
\| D_{0+}^{\beta-2}v\|_\infty &\le \frac{q}{1-q\| a_1\|_1 - q\| b_1\|_1 - q\| c_1\|_1}(\| d_1\|_1 \| D_{0+}^{\beta-2}v \|_\infty^{\theta_1} + D).
\end{aligned}$$

But $\theta_1 \in [0, 1)$ and $\| a_1\|_1 + \| b_1\|_1 + \| c_1\|_1 \le \frac{1}{q}$, so there exists $A_1, A_2, A_3 > 0$ such that

$$\| D_{0+}^{\beta-2}v\|_\infty \le A_1, \quad \| D_{0+}^{\beta-1}v\|_\infty \le A_2, \quad \| v\|_\infty \le A_3.$$

Therefore, for all $(u, v) \in \Omega_1$,

$$\| (u,\ v)\|_Y = \max\{\| v\|_\infty,\ \| D_{0+}^{\beta-1}v\|_\infty,\ \| D_{0+}^{\beta-2}v\|_\infty\} \le \max\{A_1,\ A_2,\ A_3\},$$

we can prove that Ω_1 is also bounded.
Case 2. $||(u, v)||_Y \le w||N_2u||_1 + nA$.
The proof is similar to that of case 1. Here, we omit it, where

$$\| a_2\|_1 + \| b_2\|_1 + \| c_2\|_1 \le \frac{1}{w}.$$

Case 3. $||(u, v)||_Y \le n(A + ||N_2u||_1) + j||N_1v||_1$.
From (3.6) and (3.7), we have

$$\begin{aligned}
\| (u,v)\|_Y \le\ & j[\| a_1\|_1 \| v\|_\infty + \| b_1\|_1 \| D_{0+}^{\beta-1}v\|_\infty + \| c_1\|_1 \| D_{0+}^{\beta-2}v\|_\infty\\
&+ \| d_1\|_1 \| D_{0+}^{\beta-2}v \|_\infty^{\theta_1} + \| r_1\|_1] + n[\| a_2\|_1 \| u\|_\infty\\
&+ \| b_2\|_1 \| D_{0+}^{\alpha-1}u\|_\infty + \| c_2\|_1 \| D_{0+}^{\alpha-2}u\|_\infty\\
&+ \| d_2\|_1 \| D_{0+}^{\alpha-2}u \|_\infty^{\theta_2} + A + \| r_2\|_1],
\end{aligned}$$

$$\begin{aligned}
\| v\|_\infty \le\ & \frac{1}{1-j\| a_1\|_1}[j\| b_1\|_1 \| D_{0+}^{\beta-1}v\|_\infty + j\| c_1\|_1 \| D_{0+}^{\beta-2}v\|_\infty\\
&+ j\| d_1\|_1 \| D_{0+}^{\beta-2}v \|_\infty^{\theta_1} + j\| r_1\|_1 + n\| a_2\|_1 \| u\|_\infty\\
&+ n\| b_2\|_1 \| D_{0+}^{\alpha-1}u\|_\infty + n\| c_2\|_1 \| D_{0+}^{\alpha-2}u\|_\infty\\
&+ n\| d_2\|_1 \| D_{0+}^{\alpha-2}u \|_\infty^{\theta_2} + nA + n\| r_2\|_1],
\end{aligned}$$

$$\begin{aligned}
\| u\|_\infty \le\ & \frac{1}{1-j\| a_1\|_1 - n\| a_2\|_1}[j\| b_1\|_1 \| D_{0+}^{\beta-1}v\|_\infty + j\| c_1\|_1 \| D_{0+}^{\beta-2}v\|_\infty\\
&+ j\| d_1\|_1 \| D_{0+}^{\beta-2}v \|_\infty^{\theta_1} + j\| r_1\|_1 + n\| b_2\|_1 \| D_{0+}^{\alpha-1}u\|_\infty + n\| c_2\|_1 \| D_{0+}^{\alpha-2}u\|_\infty\\
&+ n\| d_2\|_1 \| D_{0+}^{\alpha-2}u \|_\infty^{\theta_2} + nA + n\| r_2\|_1],
\end{aligned}$$

$$\begin{aligned}
\| D_{0+}^{\beta-1}v\|_\infty \le\ & \frac{1}{1-j\| a_1\|_1 - n\| a_2\|_1 - j\| b_1\|_1}[j\| c_1\|_1 \| D_{0+}^{\beta-2}v\|_\infty\\
&+ j\| d_1\|_1 \| D_{0+}^{\beta-2}v \|_\infty^{\theta_1} + j\| r\|_1 + n\| b_2\|_1 \| D_{0+}^{\alpha-1}u\|_\infty + n\| c_2\|_1 \| D_{0+}^{\alpha-2}u\|_\infty\\
&+ n\| d_2\|_1 \| D_{0+}^{\alpha-2}u \|_\infty^{\theta_2} + nA + n\| r\|_2],
\end{aligned}$$

$$\begin{aligned}\| D_{0+}^{\alpha-1}u\|_\infty &\leq \frac{1}{1-j\,\|\,a_1\|_1-n\,\|\,a_2\|_1-j\,\|\,b_1\|_1-n\,\|\,b_2\|_1}[j\,\|\,c_1\|_1\,\|\,D_{0+}^{\beta-2}v\|_\infty \\ &\quad +j\,\|\,d_1\|_1\,\|\,D_{0+}^{\beta-2}v\,\|_\infty^{\theta_1}+j\,\|\,r_1\|_1+n\,\|\,c_2\|_1\,\|\,D_{0+}^{\alpha-2}u\|_\infty \\ &\quad +n\,\|\,d_2\|_1\,\|\,D_{0+}^{\alpha-2}u\,\|_\infty^{\theta_2}+nA\;+n\,\|\,r_2\|_1],\end{aligned}$$

$$\| D_{0+}^{\beta-2}v\|_\infty \leq \frac{j\,\|\,d_1\|_1\,\|\,D_{0+}^{\beta-2}v\,\|_\infty^{\theta_1}+j\,\|\,r_1\|_1+n\,\|\,d_2\|_1\|D_{0+}^{\alpha-2}u\,\|_\infty^{\theta_2}+nA+n\|r_2\|_1}{1-j\,\|\,a_1\|_1-n\,\|\,a_2\|_1-j\,\|\,b_1\|_1-n\,\|\,b_2\|_1-j\|c_1\|_1-n\,\|\,c_2\|_1\,\|},$$

$$\| D_{0+}^{\alpha-2}u\|_\infty \leq \frac{j\,\|\,d_1\|_1\,\|\,D_{0+}^{\beta-2}v\,\|_\infty^{\theta_1}+j\,\|\,r_1\|_1+n\,\|\,d_2\|_1\|D_{0+}^{\alpha-2}u\,\|_\infty^{\theta_2}+nA+n\|r_2\|_1}{1-j\,\|\,a_1\|_1-n\,\|\,a_2\|_1-j\,\|\,b_1\|_1-n\,\|\,b_2\|_1-j\|c_1\|_1-n\,\|\,c_2\|_1\,\|}.$$

If $n\,\|\,d_2\|_1\,\|\,D_{0+}^{\alpha-2}u\,\|_\infty^{\theta_2}\geq j\,\|\,d_1\|_1\,\|\,D_{0+}^{\beta-2}v\,\|_\infty^{\theta_1}$, then we have

$$\| D_{0+}^{\alpha-2}u\|_\infty \leq \frac{j\|r_1\|_1+2n\,\|\,d_2\|_1\|D_{0+}^{\alpha-2}u\,\|_\infty^{\theta_2}+nA+n\|r_2\|_1}{1-j\,\|\,a_1\|_1-n\,\|\,a_2\|_1-j\,\|\,b_1\|_1-n\|b_2\|_1-j\,\|\,c_1\|_1-n\,\|\,c_2\|_1\,\|}.$$

But $\theta_2 \in [0, 1)$ and $j||a_1||1 + n||a_2||1 + j||b_1||1 + n||b_2||1 + j||c_1||_1 + n||c_2||_1 <1$, so there exists $A_i >0$, $i = 1, ..., 6$ such that

$$\begin{aligned}&\| D_{0+}^{\beta-2}v\|_\infty \leq A_1,\ \| D_{0+}^{\beta-1}v\|_\infty \leq A_2,\ \| v\|_\infty \leq A_3,\\ &\| D_{0+}^{\alpha-2}u\|_\infty \leq A_4,\ \| D_{0+}^{\alpha-1}u\|_\infty \leq A_5,\ \| u\|_\infty \leq A_6.\end{aligned}$$

Therefore, for all $(u, v) \in \Omega_1$,

$$\begin{aligned}\| (u,\ v)\|_Y &= \max\{\| v\|_\infty,\ \| D_{0+}^{\beta-1}v\|_\infty,\ \| D_{0+}^{\beta-2}v\|_\infty\ \| u\|_\infty,\ \| D_{0+}^{\alpha-1}u\|_\infty,\ \| D_{0+}^{\alpha-2}u\|_\infty\}\\ &\leq \max\{A_i\},\ i=1,\ \ldots,6.\end{aligned}$$

If $n\,\|\,d_2\|_1\,\|\,D_{0+}^{\alpha-2}u\,\|_\infty^{\theta_2}\leq j\,\|\,d_1\|_1\,\|\,D_{0+}^{\beta-2}v\,\|_\infty^{\theta_1}$, similarly to the above argument, we can also prove that Ω_1 is bounded.

Case 4. $||(u, v)||_Y \leq m(A + ||N_1v||_1) + k||N_2u||_1$.

The proof is similar to that of case 3. Here, we omit it, where

$$k\,\|\,a_1\|_1+m\,\|\,a_2\|_1+k\,\|\,b_1\|_1+m\,\|\,b_2\|_1+k\,\|\,c_1\|_1+m\,\|\,c_2\|_1 < 1.$$

Let

$$\Omega_2 = \{(u,\ v)\in KerL\ : N(u,\ v)\in ImL\}$$

for $(u, v) \in \Omega_2$, $(u, v) \in KerL = \{(u, v) \in domL|(at^{\alpha-1}+bt^{\alpha-2}, ct^{\beta-1}+dt^{\beta-2}), a, b, c, d \in R, t \in [0, 1]\}$ and $QN(u, v) = (0, 0)$; thus

$$\begin{aligned}&T_1N_1(at^{\alpha-1}+bt^{\alpha-2}) = T_2N_1(at^{\alpha-1}+bt^{\alpha-2}) = 0,\\ &R_1N_2(ct^{\beta-1}+dt^{\beta-2}) = R_2N_2(ct^{\beta-1}+dt^{\beta-2}) = 0.\end{aligned}$$

By (H3), $\min\{a^2 + b^2, c^2 + d^2\} \leq B$ that is, Ω_2 is bounded.

We define the isomorphism $J : KerL \to ImQ$ by

$$J(at^{\alpha-1}+bt^{\alpha-2},\ ct^{\beta-1}+dt^{\beta-2}) = (at^{\alpha-1}+bt^{\alpha-2},\ ct^{\beta-1}+dt^{\beta-2}).$$

If the first part of (H3) is satisfied, and then let

$$\Omega_3 = \{(u,\ v)\in KerL : -\lambda J(u,\ v)+(1-\lambda)QN(u,\ v) = (0,0),\ \lambda\in[0,1]\},$$

For every $(u, v) = (at^{\alpha-1} + bt^{\alpha-2}, ct^{\beta-1} + dt^{\beta-2}) \in \Omega_3$,

$$\begin{aligned}&\lambda(at^{\alpha-1} + bt^{\alpha-2},\ ct^{\beta-1} + dt^{\beta-2})\\ &\quad = (1-\lambda)((T_1N_1(at^{\alpha-1} + bt^{\alpha-2})t^{\alpha-1} + T_2N_1(at^{\alpha-1} + bt^{\alpha-2})t^{\alpha-1},\\ &\qquad R_1N_2(ct^{\beta-1} + dt^{\beta-2})t^{\beta-1} + T_2N_2(ct^{\beta-1} + dt^{\beta-2})t^{\beta-1}).\end{aligned}$$

If $\lambda = 1$, then $a = b = c = d = 0$, and if $\min\{a^2 + b^2, c^2 + d^2\} > B$, then by (H3)

$$\begin{aligned}\lambda(a^2+b^2, c^2+d^2) &= (1-\lambda)(aT_1N_1(at^{\alpha-1} + bt^{\alpha-2}) + bT_2N_1(at^{\alpha-1} + bt^{\alpha-2}),\\ &\qquad cR_1N_2(ct^{\beta-1} + dt^{\beta-2}) + dT_2N_2(ct^{\beta-1} + dt^{\beta-2})) < (0,0).\end{aligned}$$

which, in either case, is a contradiction. If the other part of (H3) is satisfied, then we take

$$\Omega_3 = \{(u,\ v) \in KerL : \lambda J(u,\ v) + (1-\lambda)QN(u,\ v) = (O,\ 0),\ \lambda \in [0,1]\},$$

and, again, obtain a contradiction. Thus, in either case

$$\begin{aligned}\|(u,\ v)\|_Y &= \|(at^{\alpha-1} + bt^{\alpha-2},\ ct^{\beta-1} + dt^{\beta-2})\|_Y\\ &= \max\{\|at^{\alpha-1} + bt^{\alpha-2}\|_{Y_1},\ \|ct^{\beta-1} + dt^{\beta-2}\|_{Y_2}\ \|\}\\ &= \max\{(1+2\Gamma(\alpha))|a| + (1+\Gamma(\alpha-1))|b|,\\ &\quad (1+2\Gamma(\beta))|c| + (1+\Gamma(\beta-1))|d|\}\\ &\leq \max\{[(1+2\Gamma(\alpha)) + (1+\Gamma(\alpha-1))]B,\\ &\quad [(1+2\Gamma(\beta)) + (1+\Gamma(\beta-1))]B\}\\ &\leq (4 + 2\Gamma(\alpha) + \Gamma(\alpha-1) + 2\Gamma(\beta) + \Gamma(\beta-1))B,\end{aligned}$$

for all $x \in \Omega_3$, that is, Ω_3 is bounded.

In the following, we shall prove that all the conditions of Theorem 1.1 are satisfied. Set Ω to be a bounded open set of Y such that $\cup_{i=1}^{3}\overline{\Omega} \subset \Omega$. By Lemma 3.2, the operator $K_p(I - Q)N : \overline{\Omega} \rightarrow Y$ is compact N thus is L-compact on $\overline{\Omega}$.

Then by the above argument, we have

(i) $L(u, v) \neq \lambda N(u, v)$ for every $((u, v), \lambda) \in [domL\backslash KerL \cap \partial\Omega] \times [0, 1]$;

(ii) $N(u, v) \notin ImL$, for every $(u, v) \in KerL \cap \partial\Omega$.

Finally, we will prove that (iii) of Theorem 1.1 is satisfied.

Let $H((u, v), \lambda) = \pm\lambda J(u, v) + (1 - \lambda)QN(u, v)$, where I is the identity operator in the Banach space Y. According to the above argument, we know that $H((u, v), \lambda) \neq 0$, for all $(u, v) \in \partial\Omega \cap KerL$, and thus, by the homotopy property of degree,

$$\begin{aligned}g(QN|_{KerL},\ KerL \cap \Omega,\ (0,0)) &= deg(H(\cdot,\ 0),\ KerL \cap \Omega,\ (0,0))\\ &= deg(H(\cdot,\ 1),\ KerL \cap \Omega,\ (0,0))\\ &= deg(\pm I,\ KerL \cap \Omega,\ (0,0))\\ &= sgn\left(\left(\pm\begin{vmatrix}\frac{\Delta_4}{\Delta} & \frac{-\Delta_2}{\Lambda}\\ \frac{-\Lambda 3}{\Lambda} & \frac{\Delta_1}{\Lambda}\end{vmatrix}, \pm\begin{vmatrix}\frac{\Delta_4}{\Delta} & \frac{-\Delta_2}{\Delta}\\ \frac{-\Delta_3}{\Delta} & \frac{\Delta_1}{\Delta}\end{vmatrix}\right)\right)\\ &\neq 0,\end{aligned}$$

Then, by Theorem 1.1, $L(u, v) = N(u, v)$has at least one solution in $domL \cap \overline{\Omega}$, and so, the BVP (1.1)-(1.3) has at least one solution in the space Y.

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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