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Existence results of Brezis-Browder type for systems of Fredholm integral equations

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Abstract

In this article, we consider the following systems of Fredholm integral equations:

$$u_i(t) = h_i(t) + \int_0^T g_i(t,s) f_i(s, u_1(s), u_2(s), \ldots, u_n(s))\,ds, \quad t \in [0,T],\ 1 \le i \le n,$$
$$u_i(t) = h_i(t) + \int_0^\infty g_i(t,s) f_i(s, u_1(s), u_2(s), \ldots, u_n(s))\,ds, \quad t \in [0,\infty),\ 1 \le i \le n.$$

Using an argument originating from Brezis and Browder [Bull. Am. Math. Soc. **81**, 73-78 (1975)] and a fixed point theorem, we establish the existence of solutions of the first system in $(C[0, T])^n$, whereas for the second system, the existence criteria are developed separately in $(C_l[0,\infty))^n$ as well as in $(BC[0,\infty))^n$. For both systems, we further seek the existence of *constant-sign* solutions, which include *positive* solutions (the usual consideration) as a special case. Several examples are also included to illustrate the results obtained.

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Keywords: system of Fredholm integral equations, Brezis-Browder arguments, constant-sign solutions

1 Introduction

In this article, we shall consider the system of Fredholm integral equations:

$$u_i(t) = h_i(t) + \int_0^T g_i(t,s) f_i(s, u_1(s), u_2(s), \ldots, u_n(s))\,ds, \quad t \in [0,T], \quad 1 \le i \le n \tag{1.1}$$

where $0 < T < \infty$, and also the following system on the half-line

$$u_i(t) = h_i(t) + \int_0^\infty g_i(t,s) f_i(s, u_1(s), u_2(s), \ldots, u_n(s))\,ds, \quad t \in [0,\infty), \quad 1 \le i \le n \tag{1.2}$$

Throughout, let $u = (u_1, u_2,..., u_n)$. We are interested in establishing the existence of solutions u of the system (1.1) in $(C[0, T])^n = C[0, T] \times C[0, T] \times \mathbb{P} \times C[0, T]$ (n times), whereas for the system (1.2), we shall seek a solution in $(C_l[0, \infty))^n$ as well as in $(BC[0, \infty))^n$. Here, $BC[0, \infty)$ denotes the space of functions that are bounded and continuous on $[0, \infty)$ and $C_l[0, \infty) = \{x \in BC[0, \infty) : \lim_{t\to\infty} x(t) \text{ exists}\}$.

We shall also tackle the existence of *constant-sign* solutions of (1.1) and (1.2). A solution u of (1.1) (or (1.2)) is said to be of *constant sign* if for each $1 \le i \le n$, we have

$\theta_i u_i(t) \geq 0$ for all $t \in [0, T]$ (or $t \in [0,\infty)$), where $\theta_i \in \{-1, 1\}$ is fixed. Note that when $\theta_i = 1$ for all $1 \leq i \leq n$, a constant-sign solution reduces to a *positive* solution, which is the usual consideration in the literature.

In the literature, there is a vast amount of research on the existence of positive solutions of the nonlinear Fredholm integral equations:

$$y(t) = h(t) + \int_0^T g(t,s)f(y(s))\mathrm{d}s, \quad t \in [0,T] \tag{1.3}$$

and

$$y(t) = h(t) + \int_0^\infty g(t,s)f(y(s))\mathrm{d}s, \quad t \in [0,\infty). \tag{1.4}$$

Particular cases of (1.3) are also considered in [1-3]. The reader is referred to the monographs [[4,5], and the references cited therein] for the related literature. Recently, a generalization of (1.3) and (1.4) to systems similar to (1.1) and (1.2) have been made, and the existence of single and multiple constant-sign solutions has been established for these systems in [6-10].

The technique used in these articles has relied heavily on various fixed point results such as Krasnosel'skii's fixed point theorem in a cone, Leray-Schauder alternative, Leggett-Williams' fixed point theorem, five-functional fixed point theorem, Schauder fixed point theorem, and Schauder-Tychonoff fixed point theorem. In the current study, we will make use of an argument that originates from Brezis and Browder [11]; therefore, the technique is different from those of [6-10] and the results subsequently obtained are also different. The present article also extends, improves, and complements the studies of [5,12-23]. Indeed, we have generalized the problems to (i) *systems*; (ii) more *general* form of nonlinearities f_i, $1 \leq i \leq n$,; and (iii) existence of *constant-sign* solutions.

The outline of the article is as follows. In Section 2, we shall state the necessary fixed point theorem and compactness criterion, which are used later. In Section 3, we tackle the existence of solutions of system (1.1) in $(C[0, T])^n$, while Sections 4 and 5 deal with the existence of solutions of system (1.2) in $(C_l[0, \infty))^n$ and $(BC[0, \infty))^n$, respectively. In Section 6, we seek the existence of *constant-sign* solutions of (1.1) and (1.2) in $(C[0, T])^n$, $(C_l[0, \infty))^n$ and $(BC[0, \infty))^n$. Finally, several examples are presented in Section 7 to illustrate the results obtained.

2 Preliminaries

In this section, we shall state the theorems that are used later to develop the existence criteria–Theorem 2.1 [24] is Schauder's nonlinear alternative for continuous and compact maps, whereas Theorem 2.2 is the criterion of compactness on $C_l[0, \infty)$ [[16], p. 62].

Theorem 2.1 [24]*Let B be a Banach space with* $E \subseteq B$ *closed and convex. Assume U is a relatively open subset of E with* $0 \in U$ *and* $S : \overline{U} \to E$ *is a continuous and compact map. Then either*

(a) *S has a fixed point in* $\overline{U}$, *or*

(b) *there exist* $u \in \partial U$ *and* $\lambda \in (0, 1)$ *such that* $u = \lambda Su$.

Theorem 2.2 [[16], p. 62] *Let $P \subset C_l[0, \infty)$. Then P is compact in $C_l[0, \infty)$ if the following hold:*

(a) *P is bounded in $C_l[0, \infty)$.*

(b) *Any $y \in P$ is equicontinuous on any compact interval of* $[0, \infty)$.

(c) *P is equiconvergent, i.e., given $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that $|y(t) - y(\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $y \in P$.*

3 Existence results for (1.1) in $(C[0, T])^n$

Let the Banach space $B = (C[0, T])^n$ be equipped with the norm:

$$||u|| = \max_{1 \le i \le n} \sup_{t \in [0,T]} |u_i(t)| = \max_{1 \le i \le n} |u_i|_0$$

where we let $|u_i|_0 = \sup_{t \in [0,T]} |u_i(t)|$, $1 \leq i \leq n$. Throughout, for $u \in B$ and $t \in [0, T]$, we shall denote

$$||u(t)|| = \max_{1 \le i \le n} |u_i(t)|.$$

Moreover, for each $1 \leq i \leq n$, let $1 \leq p_i \leq \infty$ be an integer and q_i be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$. For $x \in L^{p_i}[0, T]$, we shall define

$$\|x\|_{p_i} = \begin{cases} \left(\int_0^T |x(s)|^{p_i} \mathrm{d}s \right)^{\frac{1}{p_i}}, & 1 \le p_i < \infty \\ \operatorname*{ess\,sup}_{s \in [0,T]} |x(s)|, & p_i = \infty. \end{cases}$$

Our first existence result uses Theorem 2.1.

Theorem 3.1 *For each $1 \leq i \leq n$, assume (C1)- (C4) hold where*

(C1) $h_i \in C[0, T]$, *denote* $H_i \equiv \sup_{t \in [0, T]} |h_i(t)|$,

(C2) $f_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *is a L^{q_i}-Carathéodory function:*

(i) *the map $u \alpha f_i(t, u)$ is continuous for almost all $t \in$* $[0, T]$,;

(ii) *the map $t \alpha f_i(t, u)$ is measurable for all $u \in \mathbb{R}^n$;*

(iii) *for any* $r > 0$, *there exists $\mu_{r,i} \in L^{q_i}[0, T]$ such that $|u| \leq r$ implies $|f_i(t, u)| \leq \mu_{r,i}(t)$ for almost all $t \in$* $[0, T]$;

(C3) $g_i^t(s) = g_i(t, s) \in L^{p_i}[0, T]$ *for each $t \in$* $[0, T]$;

(C4) *the map $t \mapsto g_i^t$ is continuous from* $[0, T]$ *to* $L^{p_i}[0, T]$.

In addition, suppose there is a constant $M > 0$, independent of λ, with $||u|| \neq M$ for any solution $u \in (C[0, T])^n$ to

$$u_i(t) = \lambda \left(h_i(t) + \int_0^T g_i(t, s) f_i(s, u(s)) \mathrm{d}s \right), \quad t \in [0, T], \quad 1 \le i \le n \quad (3.1)_\lambda$$

for each $\lambda \in (0, 1)$. Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof Let the operator S be defined by

$$Su(t) = (S_1 u(t), S_2 u(t), \ldots, S_n u(t)), \quad t \in [0, T] \tag{3.2}$$

where

$$S_iu(t) = h_i(t) + \int_0^T g_i(t,s)f_i(s,u(s))\mathrm{d}s, \quad t \in [0,T], \quad 1 \le i \le n. \tag{3.3}$$

Clearly, the system (1.1) is equivalent to $u = Su$, and $(3.1)_\lambda$ is the same as $u = \lambda Su$.

Note that S maps $(C[0, T])^n$ into $(C[0, T])^n$, i.e., $S_i : (C[0, T])^n \to C[0, T]$, $1 \le i \le n$. To see this, note that for any $u \in (C[0, T])^n$, there exits $r > 0$ such that $||u|| < r$. Since f_i is a L^{q_i}-Carathéodory function, there exists $\mu_{r,i} \in L^{q_i}[0,T]$ such that $|f_i(s, u)| \le \mu_{r,i}(s)$ for almost all $s \in [0, T]$. Hence, for any $t_1, t_2 \in [0, T]$, we find for $1 \le i \le n$,

$$|S_iu(t_1) - S_iu(t_2)| \le |h_i(t_1) - h_i(t_2)| + \left[\int_0^T |g_i^{t_1}(s) - g_i^{t_2}(s)|^{p_i}\mathrm{d}s\right]^{\frac{1}{p_i}} \left\|\mu_{r,i}\right\|_{q_i} \to 0 \tag{3.4}$$

as $t_1 \to t_2$, where we have used (C1) and (C3). This shows that $S : (C[0, T])^n \to (C[0, T])^n$.

Next, we shall prove that $S : (C[0, T])^n \to (C[0, T])^n$ is continuous. Let $u^m = (u_1^m, u_2^m, \ldots, u_n^m) \to u$ in $(C[0, T])^n$, i.e., $u_i^m \to u_i$ in $C[0, T]$, $1 \le i \le n$. We need to show that $Su^m \to Su$ in $(C[0, T])^n$, or equivalently $S_iu^m \to S_iu$ in $C[0, T]$, $1 \le i \le n$. There exists $r > 0$ such that $||u^m||$, $||u|| < r$. Since f_i is a L^{q_i}-Carathéodory function, there exists $\mu_{r,i} \in L^{q_i}[0,T]$ such that $|f_i(s, u^m)|$, $|f_i(s, u)| \le \mu_{r,i}(s)$ for almost all $s \in [0, T]$. Using a similar argument as in (3.4), we get for any $t_1, t_2 \in [0, T]$ and $1 \le i \le n$:

$$|S_iu^m(t_1) - S_iu^m(t_2)| \to 0 \quad \text{and} \quad |S_iu(t_1) - S_iu(t_2)| \to 0 \tag{3.5}$$

as $t_1 \to t_2$. Furthermore, $S_iu^m(t) \to S_iu(t)$ pointwise on $[0, T]$, since, by the Lebesgue-dominated convergence theorem,

$$|S_iu^m(t) - S_iu(t)| \le \sup_{t\in[0,T]} ||g_i^t||_{p_i} \left[\int_0^T |f_i(s,u^m(s)) - f_i(s,u(s))|^{q_i}\mathrm{d}s\right]^{\frac{1}{q_i}} \to 0 \tag{3.6}$$

as $m \to \infty$. Combining (3.5) and (3.6) and using the fact that $[0, T]$ is compact, gives for all $t \in [0, T]$,

$$|S_iu^m(t) - S_iu(t)| \le |S_iu^m(t) - S_iu^m(t_1)| + |S_iu^m(t_1) - S_iu(t_1)| + |S_iu(t_1) - S_iu(t)| \to 0 \tag{3.7}$$

as $m \to \infty$. Hence, we have proved that $S : (C[0, T])^n \to (C[0, T])^n$ is continuous.

Finally, we shall show that $S : (C[0, T])^n \to (C[0, T])^n$ is completely continuous. Let Ω be a bounded set in $(C[0, T])^n$ with $||u|| \le r$ for all $u \in \Omega$. We need to show that $S_i\Omega$ is relatively compact for $1 \le i \le n$. Clearly, $S_i\Omega$ is uniformly bounded, since there exists $\mu_{r,i} \in L^{q_i}[0,T]$ such that $|f_i(s, u)| \le \mu_{r,i}(s)$ for all $u \in \Omega$ and *a.e.* $s \in [0, T]$, and hence

$$|S_iu|_0 \le H_i + \sup_{t\in[0,T]} \left\|g_i^t\right\|_{p_i} \cdot \left\|\mu_{r,i}\right\|_{q_i} \equiv K_i, \quad u \in \Omega. \tag{3.8}$$

Further, using a similar argument as in (3.4), we see that $S_i\Omega$ is equicontinuous. It follows from the Arzéla-Ascoli theorem [[5], Theorem 1.2.4] that $S_i\Omega$ is relatively compact.

We now apply Theorem 2.1 with $U = \{u \in (C[0, T])^n : ||u|| < M\}$ and $B = E = (C[0, T])^n$ to obtain the conclusion of the theorem. □

Our subsequent results will apply Theorem 3.1. To do so, we shall show that any solution u of $(3.1)_\lambda$ is bounded above. This is achieved by bounding the integral of $|f_i(t, u(t))|$ (or $|f_i(t,u(t))|^{\rho_i}$) on two complementary subsets of $[0, T]$, namely $\{t \in [0, T] : ||u(t)|| \le r\}$ and $\{t \in [0, T] : ||u(t)|| > r\}$, where ρ_i and r are some constants–this technique originates from the study of Brezis and Browder [11]. In the next four theorems (Theorems 3.2-3.5), we shall apply Theorem 3.1 to the case $p_i = \infty$ and $q_i = 1$, $1 \le i \le n$.

Theorem 3.2. *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4) with* $p_i = \infty$ *and* $q_i = 1$, *(C5) and (C6) where*

(C5) *there exist* $B_i > 0$ *such that for any* $u \in (C[0, T])^n$,

$$\int_0^T \left[f_i(t,u(t)) \int_0^T g_i(t,s) f_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t \le B_i,$$

(C6) *there exist* $r > 0$ *and* $\alpha_i > 0$ *with* $r\alpha_i > H_i$ *such that for any* $u \in (C[0, T])^n$,

$$u_i(t) f_i(t,u(t)) \ge r\alpha_i |f_i(t,u(t))| \textit{ for } ||u(t)|| > r \textit{ and a.e. } t \in [0,T].$$

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof We shall employ Theorem 3.1, and so let $u = (u_1, u_2, l...., u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$.

Define

$$I = \{t \in [0,T] : ||u(t)|| \le r\} \quad \text{and} \quad J = \{t \in [0,T] : ||u(t)|| > r\}. \tag{3.9}$$

Clearly, $[0, T] = I \cup J$, and hence $\int_0^T = \int_I + \int_J$.

Let $1 \le i \le n$. If $t \in I$, then by (C2), there exists $\mu_{r,i} \in L^1[0, T]$ such that $|f_i(t, u(t))| \le \mu_{r,i}(t)$. Thus, we get

$$\int_I |f_i(t,u(t))|\mathrm{d}t \le \int_I \mu_{r,i}(t)\mathrm{d}t \le \int_0^T \mu_{r,i}(t)\mathrm{d}t = ||\mu_{r,i}||_1. \tag{3.10}$$

On the other hand, if $t \in J$, then it is clear from (C6) that $u_i(t)f_i(t, u(t)) \ge 0$ for *a.e.* $t \in [0, T]$. It follows that

$$\int_J u_i(t) f_i(t,u(t))\mathrm{d}t \ge r\alpha_i \int_J |f_i(t,u(t))|\mathrm{d}t. \tag{3.11}$$

We now multiply $(3.1)_\lambda$ by $f_i(t, u(t))$, then integrate from 0 to T to get

$$\int_0^T u_i(t) f_i(t,u(t))\mathrm{d}t = \lambda \int_0^T h_i(t) f_i(t,u(t))\mathrm{d}t + \lambda \int_0^T \left[f_i(t,u(t)) \int_0^T g_i(t,s) f_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t. \tag{3.12}$$

Using (C5) in (3.12) yields

$$\int_0^T u_i(t) f_i(t,u(t))\mathrm{d}t \le H_i \int_0^T |f_i(t,u(t))|\mathrm{d}t + B_i. \tag{3.13}$$

Splitting the integrals in (3.13) and applying (3.11), we get

$$\int_I u_i(t)f_i(t,u(t))\mathrm{d}t + r\alpha_i \int_J |f_i(t,u(t))|\mathrm{d}t \le H_i \int_I |f_i(t,u(t))|\mathrm{d}t + H_i \int_J |f_i(t,u(t))|\mathrm{d}t + B_i$$

or

$$\begin{aligned}(r\alpha_i - H_i)\int_J |f_i(t,u(t))|\mathrm{d}t &\le H_i \int_I |f_i(t,u(t))|\mathrm{d}t + \int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + B_i \\ &\le (H_i + r)\|\mu_{r,i}\|_1 + B_i\end{aligned}$$

where we have used (3.10) in the last inequality. It follows that

$$\int_J |f_i(t,u(t))|\mathrm{d}t \le \frac{(H_i + r)\|\mu_{r,i}\|_1 + B_i}{r\alpha_i - H_i} \equiv k_i. \tag{3.14}$$

Finally, it is clear from $(3.1)_\lambda$ that for $t \in [0, T]$ and $1 \le i \le n$,

$$\begin{aligned}|u_i(t)| &\le H_i + \int_0^T |g_i(t,s)f_i(s,u(s))|\mathrm{d}s \\ &= H_i + \left(\int_I + \int_J\right)|g_i(t,s)f_i(s,u(s))|\mathrm{d}s \\ &\le H_i + \left(\sup_{t\in[0,T]} \|g_i^t\|_\infty\right)(\|\mu_{r,i}\|_1 + k_i) \equiv l_i\end{aligned} \tag{3.15}$$

where we have applied (3.10) and (3.14) in the last inequality. Thus, $|u_i|_0 \le l_i$ for $1 \le i \le n$ and $\|u\| \le \max_{1\le i\le n} l_i \equiv L$. It follows from Theorem 3.1 (with $M = L + 1$) that (1.1) has a solution $u^* \in (C[0, T])^n$. □

Theorem 3.3 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4) with* $p_i = \infty$ *and* $q_i = 1$, *(C7) and (C8) where*

(C7) *there exist constants* $a_i \ge 0$ *and* b_i *such that for any* $u \in (C[0, T])^n$,

$$\int_0^T \left[f_i(t,u(t))\int_0^T g_i(t,s)f_i(s,u(s))\mathrm{d}s\right]\mathrm{d}t \le a_i \int_0^T |f_i(t,u(t))|\mathrm{d}t + b_i,$$

(C8) *there exist* $r > 0$ *and* $\alpha_i > 0$ *with* $r\alpha_i > H_i + a_i$ *such that for any* $u \in (C[0, T])^n$,

$$u_i(t)f_i(t,u(t)) \ge r\alpha_i|f_i(t,u(t))| \text{ for } \|u(t)\| > r \text{ and a.e. } t \in [0,T].$$

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof The proof follows that of Theorem 3.2 until (3.12). Let $1 \le i \le n$. We use (C7) in (3.12) to get

$$\begin{aligned}\int_0^T u_i(t)f_i(t,u(t))\mathrm{d}t &\le \int_0^T |h_i(t)f_i(t,u(t))|\mathrm{d}t + \lambda\int_0^T \left[f_i(t,u(t))\int_0^T g_i(t,s)f_i(s,u(s))\mathrm{d}s\right]\mathrm{d}t \\ &\le (H_i + a_i)\int_0^T |f_i(t,u(t))|\mathrm{d}t + |b_i|.\end{aligned} \tag{3.16}$$

Splitting the integrals in (3.16) and applying (3.11) gives

$$(r\alpha_i - H_i - a_i)\int_J |f_i(t,u(t))|\mathrm{d}t \le (H_i + a_i)\int_I |f_i(t,u(t))|\mathrm{d}t + \int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + |b_i|$$
$$\le (H_i + a_i + r)||\mu_{r,i}||_1 + |b_i|$$

where we have also used (3.10) in the last inequality. It follows that

$$\int_J |f_i(t,u(t))|\mathrm{d}t \le \frac{(H_i + a_i + r)||\mu_{r,i}||_1 + |b_i|}{r\alpha_i - H_i - a_i} \equiv k_i. \tag{3.17}$$

The rest of the proof follows that of Theorem 3.2. □

Theorem 3.4 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4) with* $p_i = \infty$ *and* $q_i = 1$*, (C9) and (C10) where*

(C9) *there exist constants* $a_i \ge 0$, $0 < \tau_i \le 1$ *and* b_i *such that for any* $u \in (C[0, T])^n$,

$$\int_0^T \left[f_i(t,u(t)) \int_0^T g_i(t,s) f_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t \le a_i \left[\int_0^T |f_i(t,u(t))|\mathrm{d}t \right]^{\tau_i} + b_i,$$

(C10) *there exist* $r > 0$ *and* $\beta_i > 0$ *such that for any* $u \in (C[0, T])^n$,

$$u_i(t)f_i(t,u(t)) \ge \beta_i||u(t)|| \cdot |f_i(t,u(t))| \textit{ for } ||u(t)|| > r \textit{ and a.e. } t \in [0,T].$$

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof Let $u = (u_1, u_2,..., u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$. Define

$$r_0 = \max\left\{ r, \max_{1\le i\le n} \frac{H_i + a_i 2^{\tau_i} + 1}{\beta_i} \right\},$$
$$I_0 = \{t \in [0,T] : ||u(t)|| \le r_0\} \quad \text{and} \quad J_0 = \{t \in [0,T] : ||u(t)|| > r_0\}. \tag{3.18}$$

Clearly, $[0, T] = I_0 \cup J_0$ and hence $\int_0^T = \int_{I_0} + \int_{J_0}$.

Let $1 \le i \le n$. If $t \in I_0$, then by (C2) there exists $\mu_{r_0,i} \in L^1[0,T]$ such that $|f_i(t, u(t))| \le \mu_{r_0,i}(t)$ and

$$\int_{I_0} |f_i(t,u(t))|\mathrm{d}t \le \int_{I_0} \mu_{r_0,i}(t)\mathrm{d}t \le \int_0^T \mu_{r_0,i}(t)\mathrm{d}t = ||\mu_{r_0,i}||_1. \tag{3.19}$$

Further, if $t \in J_0$, then by (C10) we have

$$\int_{J_0} u_i(t)f_i(t,u(t))\mathrm{d}t \ge \beta_i \int_{J_0} ||u(t)|| \cdot |f_i(t,u(t))|\mathrm{d}t \ge \beta_i r_0 \int_{J_0} |f_i(t,u(t))|\mathrm{d}t. \tag{3.20}$$

Now, using (3.20) and (C9) in (3.12) gives

$$\begin{aligned} \beta_i r_0 \int_{J_0} |f_i(t,u(t))|\mathrm{d}t &\le \int_{I_0} |u_i(t)f_i(t,u(t))|\mathrm{d}t + \int_0^T |h_i(t)f_i(t,u(t))|\mathrm{d}t \\ &\quad + a_i \left[\int_0^T |f_i(t,u(t))|\mathrm{d}t \right]^{\tau_i} + |b_i| \\ &\le \int_{I_0} u_i(t)f_i(t,u(t))\mathrm{d}t + \int_0^T |h_i(t)f_i(t,u(t))|\mathrm{d}t \\ &\quad + a_i 2^{\tau_i} \left\{ \left[\int_{I_0} |f_i(t,u(t))|\mathrm{d}t \right]^{\tau_i} + \left[\int_{J_0} |f_i(t,u(t))|\mathrm{d}t \right]^{\tau_i} \right\} + |b_i| \end{aligned} \tag{3.21}$$

where in the last inequality, we have made use of the inequality:

$$(x+y)^{\alpha} \le 2^{\alpha}(x^{\alpha}+y^{\alpha}), \quad x, y \ge 0, \ \ \alpha \ge 0.$$

Now, noting (3.19) we find that

$$\begin{aligned}
&\int_{I_0} |u_i(t)f_i(t,u(t))|\mathrm{d}t + \int_{I_0} |h_i(t)f_i(t,u(t))|\mathrm{d}t + a_i 2^{\tau_i}\left[\int_{I_0} |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + |b_i| \\
&\quad \le (r_0 + H_i)||\mu_{r_0,i}||_1 + a_i 2^{\tau_i}(||\mu_{r_0,i}||_1)^{\tau_i} + |b_i| \equiv k'_i
\end{aligned} \tag{3.22}$$

Substituting (3.22) in (3.21) then yields

$$\begin{aligned}
\beta_i r_0 \int_{J_0} |f_i(t,u(t))|\mathrm{d}t &\le \int_{J_0} |h_i(t)f_i(t,u(t))|\mathrm{d}t + a_i 2^{\tau_i}\left[\int_{J_0} |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + k'_i \\
&\le H_i \int_{J_0} |f_i(t,u(t))|\mathrm{d}t + a_i 2^{\tau_i}\left[\int_{J_0} |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + k'_i.
\end{aligned}$$

Since $\tau_i \le 1$, there exists a constant k''_i such that

$$(\beta_i r_0 - H_i - a_i 2^{\tau_i}) \int_{J_0} |f_i(t,u(t))|\mathrm{d}t \le k''_i$$

which leads to

$$\int_{J_0} |f_i(t,u(t))|\mathrm{d}t \le \frac{k''_i}{\beta_i r_0 - H_i - a_i 2^{\tau_i}} \equiv k_i. \tag{3.23}$$

Finally, it is clear from $(3.1)_\lambda$ that for $t \in [0, T]$ and $1 \le i \le n$,

$$\begin{aligned}
|u_i(t)| &\le H_i + \int_0^T |g_i(t,s)f_i(s,u(s))|\mathrm{d}s \\
&= H_i + \left(\int_{I_0} + \int_{J_0}\right) |g_i(t,s)f_i(s,u(s))|\mathrm{d}s \\
&\le H_i + \left(\sup_{t\in[0,T]} ||g_i^t||_\infty\right)(||\mu_{r_0,i}||_1 + k_i) \equiv l_i
\end{aligned} \tag{3.24}$$

where we have applied (3.19) and (3.23) in the last inequality. The conclusion now follows from Theorem 3.1. □

Theorem 3.5 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1), (C2)-(C4) with* $p_i = \infty$ *and* $q_i = 1$, *(C10), (C11) and (C12) where*

(C11) *there exist* $r > 0$, $\eta_i > 0$, $\gamma_i > 0$ *and* $\phi_i \in L^{\frac{\gamma_i+1}{\gamma_i}}[0,T]$ *such that for any* $u \in (C[0, T])^n$,

$$||u(t)|| \ge \eta_i |f_i(t,u(t)|^{\gamma_i} + \phi_i(t) \textit{ for } ||u(t)|| > r \textit{ and a.e. } t \in [0,T],$$

(C12) *there exist* $a_i \ge 0$, $0 < \tau_i < \gamma_i + 1$, b_i, *and* $\psi_i \in L^{\frac{\gamma_i+1}{\gamma_i}}[0,T]$ *with* $\psi_i \ge 0$ *almost everywhere on* $[0, T]$, *such that for any* $u \in (C[0, T])^n$,

$$\int_0^T \left[f_i(t,u(t)) \int_0^T g_i(t,s)f_i(s,u(s))\mathrm{d}s\right]\mathrm{d}t \le a_i\left[\int_0^T \psi_i(t)|f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + b_i.$$

Also, $\varphi_i \in C[0, T]$, $h_i \in L^{\frac{\gamma_i+1}{\gamma_i}}[0, T]$, $\psi_i \in C[0, T]$ *and* $\int\limits_0^T |g_i(t,s)|^{\frac{\gamma_i+1}{\gamma_i}} ds \in C[0, T]$.

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof Let $u = (u_1, u_2,..., u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$. Define the sets I and J as in (3.9). Let $1 \leq i \leq n$. Applying (C10) and (C11), we get

$$\begin{aligned}\int\limits_J u_i(t)f_i(t,u(t))dt &\geq \beta_i \int\limits_J ||u(t)|| \cdot |f_i(t,u(t))|dt \\ &\geq \beta_i\eta_i \int\limits_J |f_i(t,u(t))|^{\gamma_i+1}dt + \beta_i \int\limits_J \phi_i(t)|f_i(t,u(t))|dt.\end{aligned} \tag{3.25}$$

Using (3.25) and (C12) in (3.12), we obtain

$$\begin{aligned}&\beta_i\eta_i \int\limits_J |f_i(t,u(t))|^{\gamma_i+1}dt \\ &\leq \int\limits_I |u_i(t)f_i(t,u(t))|dt + \beta_i \int\limits_J |\phi_i(t)f_i(t,u(t))|dt + \int\limits_0^T |h_i(t)f_i(t,u(t))|dt \\ &\quad + a_i\left[\int\limits_0^T \psi_i(t)|f_i(t,u(t))|dt\right]^{\tau_i} + |b_i| \\ &\leq \int\limits_I |u_i(t)f_i(t,u(t))|dt + \beta_i \int\limits_J |\phi_i(t)f_i(t,u(t))|dt + \int\limits_0^T |h_i(t)f_i(t,u(t))|dt \\ &\quad + a_i 2^{\tau_i}\left\{\left[\int\limits_I \psi_i(t)|f_i(t,u(t))|dt\right]^{\tau_i} + \left[\int\limits_J \psi_i(t)|f_i(t,u(t))|dt\right]^{\tau_i}\right\} + |b_i|.\end{aligned} \tag{3.26}$$

Now, in view of (3.10) and (C12), we have

$$\begin{aligned}&\int\limits_I |u_i(t)f_i(t,u(t))|dt + \int\limits_I |h_i(t)f_i(t,u(t))|dt + a_i 2^{\tau_i}\left[\int\limits_I \psi_i(t)|f_i(t,u(t))|dt\right]^{\tau_i} + |b_i| \\ &\leq (r + H_i)||\mu_{r,i}||_1 + a_i 2^{\tau_i}\left[\int\limits_I \psi_i(t)\mu_{r,i}(t)dt\right]^{\tau_i} + |b_i| \equiv \overline{k}_i.\end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.26) and using Hölder's inequality, we find

$$\begin{aligned}&\beta_i\eta_i \int\limits_J |f_i(t,u(t))|^{\gamma_i+1}dt \\ &\leq \beta_i \int\limits_J |\phi_i(t)f_i(t,u(t))|dt + \int\limits_J |h_i(t)f_i(t,u(t))|dt + a_i 2^{\tau_i}\left[\int\limits_J \psi_i(t)|f_i(t,u(t))|dt\right]^{\tau_i} + \overline{k}_i \\ &\leq \beta_i\left[\int\limits_0^T |\phi_i(t)|^{\frac{\gamma_i+1}{\gamma_i}}dt\right]^{\frac{\gamma_i+1}{\gamma_i}}\left[\int\limits_J |f_i(t,u(t))|^{\gamma_i+1}dt\right]^{\frac{1}{\gamma_i+1}} \\ &\quad + \left[\int\limits_0^T |h_i(t)|^{\frac{\gamma_i+1}{\gamma_i}}dt\right]^{\frac{\gamma_i}{\gamma_i+1}}\left[\int\limits_J |f_i(t,u(t))|^{\gamma_i+1}dt\right]^{\frac{1}{\gamma_i+1}} \\ &\quad + a_i 2^{\tau_i}\left[\int\limits_0^T |\psi_i(t)|^{\frac{\gamma_i+1}{\gamma_i}}dt\right]^{\frac{\tau_i\gamma_i}{\gamma_i+1}}\left[\int\limits_J |f_i(t,u(t))|^{\gamma_i+1}dt\right]^{\frac{\tau_i}{\gamma_i+1}} + \overline{k}_i.\end{aligned}$$

Since $\frac{1}{\gamma_i+1} < 1$ and $\frac{\tau_i}{\gamma_i+1} < 1$, there exists a constant k_i such that

$$\int_J |f_i(t, u(t))|^{\gamma_i+1} \mathrm{d}t \le k_i. \tag{3.28}$$

Finally, it is clear from $(3.1)_\lambda$ that for $t \in [0, T]$ and $1 \le i \le n$,

$$\begin{aligned} |u_i(t)| &\le H_i + \left(\int_I + \int_J \right) |g_i(t, s) f_i(s, u(s))| \mathrm{d}s \\ &\le H_i + \left(\sup_{t \in [0,T]} ||g_i^t||_\infty \right) ||\mu_{r,i}||_1 \\ &\quad + \left\{ \sup_{t \in [0,T]} \left[\int_0^T |g_i(t,s)|^{\frac{\gamma_i+1}{\gamma_i}} \mathrm{d}s \right]^{\frac{\gamma_i}{\gamma_i+1}} \right\} \left[\int_J |f_i(t, u(t))|^{\gamma_i+1} \mathrm{d}t \right]^{\frac{1}{\gamma_i+1}} \\ &\le l_i \end{aligned} \tag{3.29}$$

where we have used (3.28) and (C12) in the last inequality, and l_i is some constant. The conclusion is now immediate by Theorem 3.1. □

In the next six results (Theorem 3.6-3.11), we shall apply Theorem 3.1 for general p_i and q_i.

Theorem 3.6 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C5), (C10) and (C13) where*

(C13) *there exist* $r > 0$, $\eta_i > 0$, $\gamma_i > 0$ *and* $\phi_i \in L^{p_i}[0, T]$*such that for any* $u \in (C[0, T])^n$,

$$||u(t)|| \ge \eta_i |f_i(t, u(t)|^{\gamma_i} + \phi_i(t) \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, T].$$

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof Let $u = (u_1, u_2,..., u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$. Define the sets I and J as in (3.9). Let $1 \le i \le n$. If $t \in I$, then by (C2), there exists $\mu_{r,i} \in L^{q_i}[0, T]$ such that $|f_i(t, u(t))| \le \mu_{r,i}(t)$. Consequently, we have

$$\int_I |f_i(t, u(t))| \mathrm{d}t \le \int_I \mu_{r,i}(t) \mathrm{d}t \le \int_0^T \mu_{r,i}(t) \mathrm{d}t \le T^{\frac{1}{p_i}} ||\mu_{r,i}||_{q_i}. \tag{3.30}$$

On the other hand, using (C10) and (C13), we derive at (3.25).

Next, applying (C5) in (3.12) leads to (3.13). Splitting the integrals in (3.13) and using (3.25), we find that

$$\begin{aligned} &\beta_i \eta_i \int_J |f_i(t, u(t))|^{\gamma_i+1} \mathrm{d}t \\ &\le \beta_i \int_J |\phi_i(t) f_i(t, u(t))| \mathrm{d}t + H_i \int_J |f_i(t, u(t))| \mathrm{d}t + B_i + \int_I (|u_i(t)| + H_i) |f_i(t, u(t))| \mathrm{d}t \\ &\le \beta_i \int_J |\phi_i(t) f_i(t, u(t))| \mathrm{d}t + H_i \int_J |f_i(t, u(t))| \mathrm{d}t + B_i + (r + H_i) T^{\frac{1}{p_i}} ||\mu_{r,i}||_{q_i} \\ &= \beta_i \int_J |\phi_i(t) f_i(t, u(t))| \mathrm{d}t + H_i \int_J |f_i(t, u(t))| \mathrm{d}t + B'_i \end{aligned} \tag{3.31}$$

where (3.30) has been used in the last inequality and $B'_i \equiv B_i + (r + H_i) T^{\frac{1}{p_i}} ||\mu_{r,i}||_{q_i}$.

Now, an application of Hölder's inequality gives

$$\int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t \le \left[\int_0^T |\phi_i(t)|^{\frac{\gamma_i+1}{\gamma_i}}\mathrm{d}t\right]^{\frac{\gamma_i}{\gamma_i+1}} \cdot \left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}}. \tag{3.32}$$

Another application of Hölder's inequality yields

$$\int_0^T |\phi_i(t)|^{\frac{\gamma_i+1}{\gamma_i}}\mathrm{d}t \le T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i\gamma_i}}\left[\int_0^T |\phi_i(t)|^{p_i}\mathrm{d}t\right]^{\frac{\gamma_i+1}{\gamma_i p_i}}. \tag{3.33}$$

Substituting (3.33) into (3.32) then leads to

$$\int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t \le T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}||\phi_i||_{p_i}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}}. \tag{3.34}$$

Further, using Hölder's inequality again, we get

$$\int_J |f_i(t,u(t))|\mathrm{d}t \le T^{\frac{\gamma_i}{\gamma_i+1}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}}. \tag{3.35}$$

Substituting (3.34) and (3.35) into (3.31), we obtain

$$\beta_i\eta_i\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \le A_i\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} + B_i' \tag{3.36}$$

where $A_i \equiv T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}\beta_i||\phi_i||_{p_i} + H_iT^{\frac{\gamma_i}{\gamma_i+1}}$. Since $\frac{1}{\gamma_i+1} < 1$, from (3.36), there exists a constant k_i such that

$$\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \le k_i. \tag{3.37}$$

Finally, it is clear from $(3.1)_\lambda$ that for $t \in [0, T]$ and $1 \le i \le n$,

$$\begin{aligned} |u_i(t)| &\le H_i + \left(\int_I + \int_J\right)\left|g_i(t,s)f_i(s\ ,u(s))\right|\mathrm{d}s \\ &\le H_i + \left(\sup_{t\in[0,T]}||g_i^t||_{p_i}\right)||\mu_{r,i}||_{q_i} + T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}\left(\sup_{t\in[0,T]}||g_i^t||_{p_i}\right)\left[\int_J |f_i(s,u(s))|^{\gamma_i+1}\mathrm{d}s\right]^{\frac{1}{\gamma_i+1}} \\ &\le l_i(\text{a constant}), \end{aligned} \tag{3.38}$$

where in the second last inequality a similar argument as in (3.34) is used, and in the last inequality we have used (3.37). An application of Theorem 3.1 completes the proof. □

Theorem 3.7 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C7), (C10) and (C13). Then, (1.1) has at least one solution in* $(C[0, T])^n$.

Proof Let $u = (u_1, u_2,..., u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$. Define the sets I and J as in (3.9). Let $1 \le i \le n$. As in the proof of Theorems 3.3 and 3.6, respectively, (C7) leads to (3.16), whereas (C10) and (C13) yield (3.25).

Splitting the integrals in (3.16) and applying (3.25), we find that

$$\begin{aligned}
&\beta_i\eta_i \int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&\le \beta_i \int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + (H_i + a_i)\int_J |f_i(t,u(t))|\mathrm{d}t + |b_i| + \int_I (|u_i(t)| + H_i + a_i)|f_i(t,u(t))|\mathrm{d}t \\
&\le \beta_i \int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + (H_i + a_i)\int_J |f_i(t,u(t))|\mathrm{d}t + |b_i| + (r + H_i + a_i)T^{\frac{1}{p_i}}||\mu_{r,i}||_{q_i} \\
&= \beta_i \int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + (H_i + a_i)\int_J |f_i(t,u(t))|\mathrm{d}t + B''_i
\end{aligned} \tag{3.39}$$

where $B''_i \equiv |b_i| + (r + H_i + a_i)T^{\frac{1}{p_i}}||\mu_{r,i}||_{q_i}$. Substituting (3.34) and (3.35) into (3.39) then leads to

$$\beta_i\eta_i \int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \le A'_i\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} + B''_i \tag{3.40}$$

where $A'_i \equiv T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}\beta_i||\phi_i||_{p_i} + (H_i + a_i)T^{\frac{\gamma_i}{\gamma_i+1}}$. Since $\frac{1}{\gamma_i+1} < 1$, from (3.40), we can obtain (3.37) where k_i is some constant. The rest of the proof proceeds as that of Theorem 3.6. □

Theorem 3.8 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C10), (C13), and (C14) where*

(C14) *there exist constants* $a_i \ge 0$, $0 < \tau_i < \gamma_i + 1$ *and* b_i *such that for any* $u \in (C[0, T])^n$,

$$\int_0^T \left[f_i(t,u(t))\int_0^T g_i(t,s)f_i(s,u(s))\mathrm{d}s\right]\mathrm{d}t \le a_i\left[\int_0^T |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + b_i.$$

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof Let $u = (u_1, u_2,..., u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$. Define the sets I and J as in (3.9). Let $1 \le i \le n$. From the proof of Theorem 3.6, we see that (C10) and (C13) lead to (3.25).

Using (3.25) and (C14) in (3.12), we obtain

$$\begin{aligned}
&\beta_i\eta_i \int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&\le \int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + \beta_i \int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + \int_0^T |h_i(t)f_i(t,u(t))|\mathrm{d}t \\
&\quad + a_i\left[\int_0^T |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + |b_i| \\
&\le \int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + \beta_i \int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + \int_0^T |h_i(t)f_i(t,u(t))|\mathrm{d}t \\
&\quad + a_i 2^{\tau_i}\left\{\left[\int_I |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + \left[\int_J |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i}\right\} + |b_i|.
\end{aligned} \tag{3.41}$$

Note that

$$\begin{aligned}
&\int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + \int_I |h_i(t)f_i(t,u(t))|\mathrm{d}t + a_i 2^{\tau_i}\left[\int_I |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + |b_i| \\
&\le (r+H_i)\int_I |f_i(t,u(t))|\mathrm{d}t + a_i 2^{\tau_i}\left[\int_I |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + |b_i| \\
&\le (r+H_i)T^{\frac{1}{p_i}}\|\mu_{r,i}\|_{q_i} + a_i 2^{\tau_i}T^{\frac{\tau_i}{p_i}}(\|\mu_{r,i}\|_{q_i})^{\tau_i} + |b_i| \equiv k'_i
\end{aligned} \tag{3.42}$$

where we have used (3.30) in the last inequality. Substituting (3.42) into (3.41) and using (3.34) and (3.35) then provides

$$\begin{aligned}
&\beta_i\eta_i\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&\le \beta_i\int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + \int_J |h_i(t)f_i(t,u(t))|\mathrm{d}t + a_i 2^{\tau_i}\left[\int_J |f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + k'_i \\
&\le \beta_i T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}\|\phi_i\|_{p_i}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} + H_i T^{\frac{\gamma_i}{\gamma_i+1}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} \\
&\quad + a_i 2^{\tau_i}T^{\frac{\tau_i\gamma_i}{\gamma_i+1}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{\tau_i}{\gamma_i+1}} + k'_i.
\end{aligned} \tag{3.43}$$

Since $\frac{1}{\gamma_i+1} < 1$ and $\frac{\tau_i}{\gamma_i+1} < 1$, there exists a constant k_i such that (3.37) holds. The rest of the proof is similar to that of Theorem 3.6. □

Theorem 3.9 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C10), (C13), and (C15) where*

(C15) *there exist constants* $d_i \ge 0$, $0 < \tau_i < \gamma_i + 1$ *and* e_i *such that for any* $u \in (C[0, T])^n$,

$$\int_0^T \left[f_i(t,u(t))\int_0^T g_i(t,s)f_i(s,u(s))\mathrm{d}s\right]\mathrm{d}t \le d_i\left[\int_0^T |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + e_i.$$

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof Let $u = (u_1, u_2, \ldots, u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$. Define the sets I and J as in (3.9). Let $1 \le i \le n$. As before, we see that (C10) and (C13) lead to (3.25).

Using (3.25) and (C15) in (3.12), we obtain

$$\begin{aligned}
&\beta_i\eta_i\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&\le \beta_i\int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + \int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + \int_0^T |h_i(t)f_i(t,u(t))|\mathrm{d}t \\
&\quad + d_i\left[\int_0^T |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + |e_i| \\
&\le \beta_i\int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + \int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + H_i\int_0^T |f_i(t,u(t))|\mathrm{d}t \\
&\quad + d_i 2^{\frac{\tau_i}{q_i}}\left\{\left[\int_J |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + \left[\int_I |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}}\right\} + |e_i|.
\end{aligned} \tag{3.44}$$

Now, it is clear that

$$\begin{aligned}
&\int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + H_i\int_I |f_i(t,u(t))|\mathrm{d}t + d_i 2^{\frac{\tau_i}{q_i}}\left[\int_I |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + |e_i| \\
&\quad\le (r+H_i)\int_I \mu_{r,i}(t)\mathrm{d}t + d_i 2^{\frac{\tau_i}{q_i}}\left[\int_I (\mu_{r,i}(t))^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + |e_i| \\
&\quad\le (r+H_i)\int_0^T \mu_{r,i}(t)\mathrm{d}t + d_i 2^{\frac{\tau_i}{q_i}}\left[\int_0^T (\mu_{r,i}(t))^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + |e_i| \equiv \hat{k}_i.
\end{aligned} \tag{3.45}$$

Moreover, an application of Hölder's inequality gives

$$\int_J |f_i(t,u(t))|^{q_i}\mathrm{d}t \le T^{\frac{\gamma_i+1-q_i}{\gamma_i+1}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{q_i}{\gamma_i+1}}. \tag{3.46}$$

Substituting (3.45) into (3.44) and using (3.34), (3.35) and (3.46) then leads to

$$\begin{aligned}
&\beta_i\eta_i\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&\quad\le \beta_i\int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + H_i\int_J |f_i(t,u(t))|\mathrm{d}t + d_i 2^{\frac{\tau_i}{q_i}}\left[\int_J |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + \hat{k}_i \\
&\quad\le \beta_i T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}||\phi_i||_{p_i}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} + H_i T^{\frac{\gamma_i}{\gamma_i+1}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} \\
&\quad\quad + d_i 2^{\frac{\tau_i}{q_i}} T^{\frac{\tau_i(\gamma_i+1-q_i)}{q_i(\gamma_i+1)}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{\tau_i}{\gamma_i+1}} + \hat{k}_i.
\end{aligned} \tag{3.47}$$

Noting $\frac{1}{\gamma_i+1} < 1$ and $\frac{\tau_i}{\gamma_i+1} < 1$, there exists a constant k_i such that (3.37) holds. The rest of the proof follows that of Theorem 3.6. □

Theorem 3.10 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C10), (C13) and (C16) where*

(C16) *there exist constants* $c_i \ge 0$, $d_i \ge 0$, $0 < \tau_i < \gamma_i + 1$ *and* e_i *with* $\beta_i\eta_i > 2c_i(2T)^{\frac{\gamma_i+1-q_i}{q_i}}$ *such that for any* $u \in (C[0, T])^n$,

$$\begin{aligned}
&\int_0^T \left[f_i(t,u(t))\int_0^T g_i(t,s)f_i(s,u(s))\mathrm{d}s\right]\mathrm{d}t \\
&\quad\le c_i\left[\int_0^T |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\gamma_i+1}{q_i}} + d_i\left[\int_0^T |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + e_i.
\end{aligned}$$

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof Let $u = (u_1, u_2,..., u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$. Define the sets I and J as in (3.9). Let $1 \le i \le n$. As before, we see that (C10) and (C13) lead to (3.25).

Using (3.25) and (C16) in (3.12) gives

$$\begin{aligned}
&\beta_i\eta_i \int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&\le \beta_i \int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + \int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + \int_0^T |h_i(t)f_i(t,u(t))|\mathrm{d}t \\
&\quad + c_i\left[\int_0^T |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\gamma_i+1}{q_i}} + d_i\left[\int_0^T |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + |e_i| \\
&\le \beta_i \int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + \int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + H_i\int_0^T |f_i(t,u(t))|\mathrm{d}t \\
&\quad + c_i 2^{\frac{\gamma_i+1}{q_i}}\left\{\left[\int_I |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\gamma_i+1}{q_i}} + \left[\int_J |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\gamma_i+1}{q_i}}\right\} \\
&\quad + d_i 2^{\frac{\tau_i}{q_i}}\left\{\left[\int_I |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + \left[\int_J |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}}\right\} + |e_i|.
\end{aligned} \tag{3.48}$$

Now, it is clear that

$$\begin{aligned}
&\int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + H_i\int_I |f_i(t,u(t))|\mathrm{d}t + c_i 2^{\frac{\gamma_i+1}{q_i}}\left[\int_I |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\gamma_i+1}{q_i}} \\
&\quad + d_i 2^{\frac{\tau_i}{q_i}}\left[\int_I |f_i(t,u(t))|^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + |e_i| \\
&\le (r+H_i)\int_I \mu_{r,i}(t)\mathrm{d}t + c_i 2^{\frac{\gamma_i+1}{q_i}}\left[\int_I (\mu_{r,i}(t))^{q_i}\mathrm{d}t\right]^{\frac{\gamma_i+1}{q_i}} \\
&\quad + d_i 2^{\frac{\tau_i}{q_i}}\left[\int_I (\mu_{r,i}(t))^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + |e_i| \\
&\le (r+H_i)\int_0^T \mu_{r,i}(t)\mathrm{d}t + c_i 2^{\frac{\gamma_i+1}{q_i}}\left[\int_0^T (\mu_{r,i}(t))^{q_i}\mathrm{d}t\right]^{\frac{\gamma_i+1}{q_i}} \\
&\quad + d_i 2^{\frac{\tau_i}{q_i}}\left[\int_0^T (\mu_{r,i}(t))^{q_i}\mathrm{d}t\right]^{\frac{\tau_i}{q_i}} + |e_i| \equiv k'_i.
\end{aligned} \tag{3.49}$$

Substituting (3.49) into (3.48) and then using (3.34), (3.35) and (3.46) leads to

$$\begin{aligned}
&\beta_i\eta_i\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&\le \beta_i T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}||\phi_i||_{p_i}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} + H_i T^{\frac{\gamma_i}{\gamma_i+1}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} \\
&+c_i 2^{\frac{\gamma_i+1}{q_i}}T^{\frac{\gamma_i+1-q_i}{q_i}}\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&+d_i 2^{\frac{\tau_i}{q_i}}T^{\frac{\tau_i(\gamma_i+1-q_i)}{q_i(\gamma_i+1)}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{\tau_i}{\gamma_i+1}} + k'_i.
\end{aligned}\tag{3.50}$$

Noting $\frac{1}{\gamma_i+1} < 1, \frac{\tau_i}{\gamma_i+1} < 1$ as well as $\beta_i\eta_i > 2c_i(2T)^{\frac{\gamma_i+1-q_i}{q_i}}$, from (3.50) there exists a constant k_i such that (3.37) holds. The rest of the proof proceeds as that of Theorem 3.6. □

Theorem 3.11 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C10), (C13) and (C17) where*

(C17) *there exist* $a_i \ge 0$, $0 < \tau_i < \gamma_i + 1$, b_i, *and* $\psi_i \in L^{p_i}[0,T]$ *with* $\psi_i \ge 0$ *almost everywhere on* $[0, T]$, *such that for any* $u \in (C[0, T])^n$,

$$\int_0^T \left[f_i(t,u(t))\int_0^T g_i(t,s)f_i(s,u(s))\mathrm{d}s\right]\mathrm{d}t \le a_i\left[\int_0^T \psi_i(t)\,|f_i(t,u(t))|\,\mathrm{d}t\right]^{\tau_i} + b_i.$$

Then, (1.1) has at least one solution in $(C[0, T])^n$.

Proof Let $u = (u_1, u_2,\ldots, u_n) \in (C[0, T])^n$ be any solution of $(3.1)_\lambda$ where $\lambda \in (0, 1)$. Define the sets I and J as in (3.9). Let $1 \le i \le n$. Once again, conditions (C10) and (C13) give rise to (3.25).

Similar to the proof of Theorem 3.5, we apply (3.25) and (C17) in (3.12) to get (3.26). Next, using (3.30) and Hölder's inequality, we find that

$$\begin{aligned}
&\int_I |u_i(t)f_i(t,u(t))|\mathrm{d}t + \int_I |h_i(t)f_i(t,u(t))|\mathrm{d}t + a_i 2^{\tau_i}\left[\int_I \psi_i(t)|f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + |b_i| \\
&\le (r+H_i)T^{\frac{1}{p_i}}||\mu_{r,i}||_{q_i} + a_i 2^{\tau_i}\left[\int_I \psi_i(t)\mu_{r,i}(t)\mathrm{d}t\right]^{\tau_i} + |b_i| \\
&\le (r+H_i)T^{\frac{1}{p_i}}||\mu_{r,i}||_{q_i} + a_i 2^{\tau_i}(||\psi_i||_{p_i}||\mu_{r,i}||_{q_i})^{\tau_i} + |b_i| \equiv k'_i.
\end{aligned}\tag{3.51}$$

Substituting (3.51) into (3.26) and applying (3.34) and (3.35), we find that

$$\begin{aligned}
&\beta_i\eta_i\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t \\
&\le \beta_i\int_J |\phi_i(t)f_i(t,u(t))|\mathrm{d}t + \int_J |h_i(t)f_i(t,u(t))|\mathrm{d}t + a_i 2^{\tau_i}\left[\int_J \psi_i(t)|f_i(t,u(t))|\mathrm{d}t\right]^{\tau_i} + k'_i \\
&\le \beta_i T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}||\phi_i||_{p_i}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} + H_i T^{\frac{\gamma_i}{\gamma_i+1}}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{1}{\gamma_i+1}} \\
&+a_i 2^{\tau_i}T^{\frac{\tau_i(\gamma_i p_i-\gamma_i-1)}{p_i(\gamma_i+1)}}(||\psi_i||_{p_i})^{\tau_i}\left[\int_J |f_i(t,u(t))|^{\gamma_i+1}\mathrm{d}t\right]^{\frac{\tau_i}{\gamma_i+1}} + k'_i.
\end{aligned}\tag{3.52}$$

Since $\frac{1}{\gamma_i+1} < 1$ and $\frac{\tau_i}{\gamma_i+1} < 1$, from (3.52), there exists a constant k_i such that (3.37) holds. The rest of the proof proceeds as that of Theorem 3.6. □

Remark 3.1 In Theorem 3.5, the conditions (C10) and (C11) can be replaced by the following, which is evident from the proof.

(C10)' There exist $r > 0$ and $\beta_i > 0$ such that for any $u \in (C[0, T])^n$,

$$u_i(t)f_i(t, u(t)) \geq \beta_i|u_i|_0 \cdot |f_i(t, u(t))| \text{ for } ||u(t)|| > r \text{ and } a.e.\ t \in [0, T],$$

where we denote $|u_i|_0 = \sup_{t\in[0,T]} |u_i(t)|$.

(C11)' There exist $r > 0$, $\eta_i > 0$, $\gamma_i > 0$ and $\phi_i \in L^{\frac{\gamma_i+1}{\gamma_i}}[0, T]$ such that for any $u \in (C[0, T])^n$,

$$|u_i|_0 \geq \eta_i|f_i(t, u(t)|^{\gamma_i} + \phi_i(t) \text{ for } ||u(t)|| > r \text{ and } a.e.\ t \in [0, T].$$

Remark 3.2 In Theorems 3.6-3.11, the conditions (C10) and (C13) can be replaced by (C10)' and (C13)' below, and the proof will be similar.

(C13)' There exist $r > 0$, $\eta_i > 0$, $\gamma_i > 0$, and $\phi_i \in L^{p_i}[0, T]$ such that for any $u \in (C[0, T])^n$,

$$|u_i|_0 \geq \eta_i|f_i(t, u(t))|^{\gamma_i} + \phi_i(t) \text{ for } ||u(t)|| > r \text{ and } a.e.\ t \in [0, T].$$

4 Existence results for (1.2) in $(C_l[0, \infty))^n$

Let the Banach space $B = (C_l[0, \infty))^n$ be equipped with the norm:

$$||u|| = \max_{1\leq i\leq n} \sup_{t\in[0,\infty)} |u_i(t)| = \max_{1\leq i\leq n} |u_i|_0$$

where we let $|u_i|_0 = \sup_{t\in[0,\infty)} |u_i(t)|$, $1 \leq i \leq n$. Throughout, for $u \in B$ and $t \in [0, \infty)$, we shall denote that

$$||u(t)|| = \max_{1\leq i\leq n} |u_i(t)|.$$

Moreover, for each $1 \leq i \leq n$, let $1 \leq p_i \leq \infty$ be an integer and q_i be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$. For $x \in L^{p_i}[0, \infty)$, we shall define that

$$||x||_{p_i} = \begin{cases} \left(\int_0^\infty |x(s)|^{p_i} ds\right)^{\frac{1}{p_i}}, & 1 \leq p_i < \infty \\ \operatorname{ess} \sup_{s\in[0,\infty)} |x(s)|, & p_i = \infty. \end{cases}$$

We shall apply Theorem 2.1 to obtain the first existence result for (1.2) in $(C_l[0, \infty))^n$.

Theorem 4.1 *For each* $1 \leq i \leq n$, *assume (D1)-(D5) hold where*

(D1) $h_i \in C_l[0, \infty)$, *denote* $H_i \equiv \sup_{t\in[0,\infty)} |h_i(t)|$,

(D2) $f_i : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ *is a* L^1*-Carathéodory function, i.e.,*

(i) *the map* $u \alpha f_i(t, u)$ *is continuous for almost all* $t \in [0, \infty)$,

(ii) *the map* $t \alpha f_i(t, u)$ *is measurable for all* $u \in \mathbb{R}^n$,

(iii) *for any* $r > 0$, *there exists* $\mu_{r,i} \in L^1[0, \infty)$ *such that* $|u| \leq r$ *implies* $|f_i(t, u)| \leq \mu_{r,i}(t)$ *for almost all* $t \in [0, \infty)$.

(D3) $g_i^t(s) = g_i(t,s) \in L^\infty[0,\infty)$*for each* $t \in [0, \infty)$,

(D4) *the map* $t \mapsto g_i^t$*is continuous from* $[0, \infty)$ *to* $L^\infty [0, \infty)$,

(D5) *there exists* $\tilde{g}_i \in L^\infty[0,\infty)$*such that* $g_i^t \to \tilde{g}_i$*in* $L^\infty[0, \infty)$ *as* $t \to \infty$, *i.e.*,

$$\lim_{t\to\infty} ||g_i^t - \tilde{g}_i||_\infty = \lim_{t\to\infty} ess \sup_{s\in[0,\infty)} |g_i(t,s) - \tilde{g}_i(s)| = 0.$$

In addition, suppose there is a constant $M > 0$, independent of λ, with $||u|| \neq M$ for any solution $u \in (C_l[0, \infty))^n$ to

$$u_i(t) = \lambda \left(h_i(t) + \int_0^\infty g_i(t,s) f_i(s,u(s)) \mathrm{d}s \right), \quad t \in [0,\infty),\ 1 \le i \le n \quad (4.1)_\lambda$$

for each $\lambda \in (0, 1)$. Then, (1.2) has at least one solution in $(C_l[0, \infty))^n$.

Proof To begin, let the operator S be defined by

$$Su(t) = (S_1u(t), S_2u(t), \ldots, S_nu(t)), \quad t \in [0,\infty) \tag{4.2}$$

where

$$S_iu(t) = h_i(t) + \int_0^\infty g_i(t,s) f_i(s,u(s)) \mathrm{d}s, t \in [0,\infty), \quad 1 \le i \le n. \tag{4.3}$$

Clearly, the system (1.2) is equivalent to $u = Su$, and $(4.1)_\lambda$ is the same as $u = \lambda Su$.

First, we shall show that $S : (C_l[0, \infty))^n \to (C_l[0, \infty))^n$, or equivalently $S_i : (C_l[0, \infty))^n \to C_l[0, \infty)$, $1 \le i \le n$. Let $u \in (C_l[0, \infty))^n$. Then, there exists $r > 0$ such that $||u|| \le r$, and from (D2) there exists $\mu_{r,i} \in L^1[0, \infty)$ such that $|f_i(s, u)| \le \mu_{r,i}(s)$ for almost all $s \in [0, \infty)$. Let $t_1, t_2 \in [0, \infty)$. Together with (D1) and (D4), we find that

$$\begin{aligned} |S_iu(t_1) - S_iu(t_2)| &\le |h_i(t_1) - h_i(t_2)| + \int_0^\infty |g_i^{t_1}(s) - g_i^{t_2}(s)|\mu_{r,i}(s)\mathrm{d}s \\ &\le |h_i(t_1) - h_i(t_2)| + ||g_i^{t_1} - g_i^{t_2}||_\infty ||\mu_{r,i}||_1 \to 0 \end{aligned} \tag{4.4}$$

as $t_1 \to t_2$. Hence, $S_iu \in C[0, \infty)$.

To see that S_iu is bounded, we have for $t \in [0, \infty)$,

$$|S_iu(t)| \le H_i + \int_0^\infty |g_i(t,s)|\mu_{r,i}(s)\mathrm{d}s \le H_i + ||g_i^t||_\infty ||\mu_{r,i}||_1. \tag{4.5}$$

By (D5), there exists $T_1 > 0$ such that for $t > T_1$,

$$||g_i^t||_\infty \le ||\tilde{g}_i||_\infty + 1.$$

On the other hand, for $t \in [0, T_1]$, we have

$$||g_i^t||_\infty \le \sup_{t\in[0,T_1]} ||g_i^t||_\infty.$$

Hence,

$$\sup_{t\in[0,\infty)} ||g_i^t||_\infty \le \max\left\{ \sup_{t\in[0,T_1]} ||g_i^t||_\infty,\ ||\tilde{g}_i||_\infty + 1 \right\} \equiv K_i. \tag{4.6}$$

It follows from (4.5) that for $t \in [0, \infty)$,

$$|S_i u(t)| \le H_i + K_i ||\mu_{r,i}||_1 \equiv M_i. \tag{4.7}$$

Hence, $S_i u$ is bounded.

It remains to check the existence of the limit $\lim_{t\to\infty} S_i u(t)$. We claim that

$$\lim_{t\to\infty} S_i u(t) = h_i(\infty) + \int_0^\infty \tilde{g}_i(s) f_i(s, u(s))\mathrm{d}s \tag{4.8}$$

where $h_i(\infty) \equiv \lim_{t\to\infty} h_i(t)$. In fact, it follows from (D5) that

$$\int_0^\infty \left|[g_i^t(s) - \tilde{g}_i(s)] f_i(s, u(s))\right| \mathrm{d}s \le ||g_i^t - \tilde{g}_i||_\infty ||\mu_{r,i}||_1 \to 0$$

as $t \to \infty$. This implies

$$\lim_{t\to\infty} \int_0^\infty g_i^t(s) f_i(s, u(s))\mathrm{d}s = \int_0^\infty \tilde{g}_i(s) f_i(s, u(s))\mathrm{d}s$$

and so (4.8) is proved. We have hence shown that $S : (C_l[0, \infty))^n \to (C_l[0, \infty))^n$.

Next, we shall prove that $S : (C_l[0, \infty))^n \to (C_l[0, \infty))^n$ is continuous. Let $\{u^m\}$ be a sequence in $(C_l[0, \infty))^n$ and $u^m = (u_1^m, u_2^m, \ldots, u_n^m) \to u$. In $(C_l[0, \infty))^n$, i.e., $u_i^m \to u_i$, in $C_l[0, \infty)$, $1 \le i \le n$. We need to show that $Su^m \to Su$ in $(C_l[0, \infty))^n$, or equivalently $S_i u^m \to S_i u$ in $C_l[0, \infty)$, $1 \le i \le n$. There exists $r > 0$ such that $||u^m||$, $||u|| < r$, Noting (D2), there exists $\mu_{r,i} \in L^1[0, \infty)$ such that $|f_i(s, u^m)|$, $|f_i(s, u)| \le \mu_{r,i}(s)$ for almost all $s \in [0, \infty)$. Denote $S_i u(\infty) \equiv \lim_{t\to\infty} S_i u(t)$ and $S_i u^m(\infty) \equiv \lim_{t\to\infty} S_i u^m(t)$. In view of (4.8), we get that

$$|S_i u^m(\infty) - S_i u(\infty)| \le \int_0^\infty |\tilde{g}_i(s)[f_i(s, u^m(s)) - f_i(s, u(s))]|\mathrm{d}s. \tag{4.9}$$

Since

$$|\tilde{g}_i(s)[f_i(s, u^m(s)) - f_i(s, u(s))]| \to 0 \text{ as } m \to \infty \text{ for almost every } s \in [0, \infty)$$

and

$$|\tilde{g}_i(s)[f_i(s, u^m(s)) - f_i(s, u(s))]| \le 2\mu_{r,i}(s)\, |\tilde{g}_i(s)| \in L^1[0, \infty),$$

by the Lebesgue-dominated convergence theorem, it is clear from (4.9) that

$$|S_i u^m(\infty) - S_i u(\infty)| \to 0 \text{ as } m \to \infty. \tag{4.10}$$

Further, using (4.8) again we find that

$$\begin{aligned} |S_i u(t) - S_i u(\infty)| &\le |h_i(t) - h_i(\infty)| + \int_0^\infty \left|g_i^t(s) - \tilde{g}_i(s)\right| \mu_{r,i}(s)\mathrm{d}s \\ &\le |h_i(t) - h_i(\infty)| + ||g_i^t - \tilde{g}_i||_\infty ||\mu_{r,i}||_1 \to 0 \end{aligned} \tag{4.11}$$

as $t \to \infty$. Similarly, we also have that

$$|S_i u^m(t) - S_i u^m(\infty)| \to 0 \text{ as } t \to \infty. \tag{4.12}$$

Combining (4.10)-(4.12), we have

$$|S_i u^m(t) - S_i u(t)| \to 0 \text{ as } t \to \infty \text{ and } m \to \infty$$

or equivalently, there exist $\hat{T} > 0$ such that

$$|S_i u^m(t) - S_i u(t)| \to 0 \text{ as } m \to \infty, \text{ for all } t > \hat{T}. \tag{4.13}$$

It remains to check the convergence in $[0, \hat{T}]$. As in (4.4), we find for any $|S_i u^m(t_1) - S_i u^m(t_2)| \to 0 \quad \text{and} \quad |S_i u(t_1) - S_i u(t_2)| \to 0,$

$$|S_i u^m(t_1) - S_i u^m(t_2)| \to 0 \quad \text{and} \quad |S_i u(t_1) - S_i u(t_2)| \to 0 \tag{4.14}$$

as $t_1 \to t_2$. Furthermore, $S_iu^m(t) \to S_iu(t)$ pointwise on $[0, \hat{T}]$, since, by the Lebesgue-dominated convergence theorem,

$$|S_i u^m(t) - S_i u(t)| \le \sup_{t\in[0,\hat{T}]} ||g_i^t||_\infty \int_0^\infty |f_i(s, u^m(s)) - f_i(s, u(s))| ds \to 0 \tag{4.15}$$

as $m \to \infty$. Combining (4.14) and (4.15) and the fact that $[0, \hat{T}]$ is compact yields

$$|S_i u^m(t) - S_i u(t)| \to 0 \text{ as } m \to \infty, \text{ for all } t \in [0, \hat{T}] \tag{4.16}$$

Coupling (4.13) and (4.16), we see that $S_iu^m \to S_iu$ in $C_l[0, \infty)$.

Finally, we shall show that $S : (C_l[0, \infty))^n \to (C_l[0, \infty))^n$ is completely continuous. Let Ω be a bounded set in $(C_l[0, \infty))^n$ with $||u|| \le r$ for all $u \in \Omega$ We need to show that $S_i\Omega$ is relatively compact for $1 \le i \le n$. First, we see that $S_i\Omega$ is bounded; in fact, this follows from an earlier argument in (4.7). Next, using a similar argument as in (4.4), we see that $S_i\Omega$ is equicontinuous. Moreover, $S_i\Omega$ is equiconvergent follows as in (4.11). By Theorem 2.2, we conclude that $S_i\Omega$ is relatively compact. Hence, $S : (C_l[0, \infty))^n \to (C_l[0, \infty))^n$ is completely continuous.

We now apply Theorem 2.1 with $U = \{u \in (C_l[0, \infty))^n : ||u|| < M\}$ and $B = E = (C_l[0, \infty))^n$ to obtain the conclusion of the theorem. □

Remark 4.1 In Theorem 4.1, the conditions (D2)-(D5) can be stated in terms of general p_i and q_i as follows, and the proof will be similar:

(D2)' $f_i : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a L^{q_i}-Carathéodory function, i.e.,

(i) the map $u \alpha f_i(t, u)$ is continuous for almost all $t \in [0, \infty)$,

(ii) the map $t \alpha f_i(t, u)$ is measurable for all $u \in \mathbb{R}^n$,

(iii) for any $r > 0$, there exists $\mu_{r,i} \in L^{q_i}[0, \infty)$ such that $|u| \le r$ implies $|f_i(t, u)| \le \mu_{r,i}(t)$ for almost all $t \in [0, \infty)$,

(D3)' $g_i^t(s) = g_i(t, s) \in L^{p_i}[0, \infty)$, for each $t \in [0, \infty)$,

(D4)' the map $t \mapsto g_i^t$ is continuous from $[0, \infty)$ to $L^{p_i}[0, \infty)$,

(D5)' there exists $\tilde{g}_i \in L^{p_i}[0, \infty)$ such that $g_i^t \to \tilde{g}_i$, in $L^{p_i}[0, \infty)$ as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} ||g_i^t - \tilde{g}_i||_{p_i} = \lim_{t\to\infty} \left(\int_0^\infty |g_i(t, s) - \tilde{g}_i(s)|^{p_i} ds \right)^{\frac{1}{p_i}} = 0.$$

Our subsequent Theorems 4.2-4.5 use an argument originating from Brezis and Browder [11]. These results are parallel to Theorems 3.2-3.5 for system (1.1).

Theorem 4.2 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(D1)-(D5), (C5)*$_\infty$, *and (C6)*$_\infty$ *where*

(C5)$_\infty$ *there exist* $B_i > 0$ *such that for any* $u \in (C_l[0, \infty))^n$,

$$\int_0^\infty \left[f_i(t, u(t)) \int_0^\infty g_i(t, s) f_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \le B_i,$$

(C6)$_\infty$ *there exist* $r > 0$ *and* $\alpha_i > 0$ *with* $r\alpha_i > H_i$ *such that for any* $u \in (C_l[0, \infty))^n$,

$$u_i(t) f_i(t, u(t)) \ge r\alpha_i |f_i(t, u(t))| \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, \infty).$$

Then, (1.2) has at least one solution in $(C_l[0, \infty))^n$.

Proof We shall employ Theorem 4.1, so let $u = (u_1, u_2, \dots, u_n) \in (C_l[0, \infty))^n$ be any solution of $(4.1)_\lambda$ where $\lambda \in (0, 1)$. The rest of the proof is similar to that of Theorem 3.2 with the obvious modification that $[0, T]$ be replaced by $[0, \infty)$. Also, noting (4.6) we see that the analog of (3.15) holds. □

In view of the proof of Theorem 4.2, we see that the proof of subsequent Theorems 4.3-4.5 will also be similar to that of Theorems 3.3-3.5 with the appropriate modification. As such, we shall present the results and omit the proof.

Theorem 4.3 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(D1)-(D5), (C7)*$_\infty$ *and (C8)*$_\infty$ *where*

(C7)$_\infty$ *there exist constants* $a_i \ge 0$ *and* b_i *such that for any* $u \in (C_l[0, \infty))^n$,

$$\int_0^\infty \left[f_i(t, u(t)) \int_0^\infty g_i(t, s) f_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \le a_i \int_0^\infty |f_i(t, u(t))| \mathrm{d}t + b_i,$$

(C8)$_\infty$ *there exist* $r > 0$ *and* $\alpha_i > 0$ *with* $r\alpha_i > H_i + a_i$ *such that for any* $u \in (C_l[0, \infty))^n$,

$$u_i(t) f_i(t, u(t)) \ge r\alpha_i |f_i(t, u(t))| \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, \infty).$$

Then, (1.2) has at least one solution in $(C_l[0, \infty))^n$.

Theorem 4.4 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(D1)-(D5), (C9)*$_\infty$ *and (C10)*$_\infty$ *where*

(C9)$_\infty$ *there exist constants* $a_i \ge 0$, $0 < \tau_i \le 1$ *and* b_i *such that for any* $u \in (C_l[0, \infty))^n$,

$$\int_0^\infty \left[f_i(t, u(t)) \int_0^\infty g_i(t, s) f_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \le a_i \left[\int_0^\infty |f_i(t, u(t))| \mathrm{d}t \right]^{\tau_i} + b_i,$$

(C10)$_\infty$ *there exist* $r > 0$ *and* $\beta_i > 0$ *such that for any* $u \in (C_l[0, \infty))^n$,

$$u_i(t) f_i(t, u(t)) \ge \beta_i ||u(t)|| \cdot |f_i(t, u(t))| \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, \infty).$$

Then, (1.2) has at least one solution in $(C_l[0, \infty))^n$.

Theorem 4.5 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(D1)-(D5), (C10)*$_\infty$, *(C11)*$_\infty$ *and (C12)*$_\infty$ *where*

(C11)$_\infty$ *there exist* $r > 0$, $\eta_i > 0$, $\gamma_i > 0$ *and* $\phi_i \in L^{\frac{\gamma_i + 1}{\gamma_i}}[0, \infty)$ *such that for any* $u \in (C_l[0, \infty))^n$,

$$||u(t)|| \geq \eta_i |f_i(t, u(t)|^{\gamma_i} + \phi_i(t) \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, \infty),$$

(C12)$_\infty$ *there exist* $a_i \geq 0$, $0 < \tau_i < \gamma_i + 1$, b_i, *and* $\psi_i \in L^{\frac{\gamma_i+1}{\gamma_i}}[0,\infty)$ *with* $\psi_i \geq 0$ *almost everywhere on* $[0, \infty)$, *such that for any* $u \in (C_l[0, \infty))^n$,

$$\int_0^\infty \left[f_i(t, u(t)) \int_0^\infty g_i(t, s) f_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \leq a_i \left[\int_0^\infty \psi_i(t) |f_i(t, u(t))| \mathrm{d}t \right]^{\tau_i} + b_i.$$

Also, $\varphi_i \in BC[0, \infty)$, $h_i \in L^{\frac{\gamma_i+1}{\gamma_i}}[0,\infty)$, $\psi_i \in BC[0, \infty)$ *and* $\int_0^\infty |g_i(t,s)|^{\frac{\gamma_i+1}{\gamma_i}} \mathrm{d}s \in BC[0,\infty)$.

Then, (1.2) has at least one solution in $(C_l[0, \infty))^n$.

We also have a remark similar to Remark 3.1.

Remark 4.2 In Theorem 4.5 the conditions (C10)$_\infty$ and (C11)$_\infty$ can be replaced by the following; this is evident from the proof.

$(\mathrm{C10})'_\infty$ There exist $r > 0$ and $\beta_i > 0$ such that for any $u \in (C_l[0, \infty))^n$,

$$u_i(t) f_i(t, u(t)) \geq \beta_i |u_i|_0 \cdot |f_i(t, u(t))| \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, \infty),$$

where we denote $|u_i|_0 = \sup_{t \in [0,\infty)} |u_i(t)|$.

$(\mathrm{C11})'_\infty$ There exist $r > 0$, $\eta_i > 0$, $\gamma_i > 0$ and $\phi_i \in L^{\frac{\gamma_i+1}{\gamma_i}}[0,\infty)$ such that for any $u \in (C_l[0, \infty))^n$,

$$|u_i|_0 \geq \eta_i |f_i(t, u(t))|^{\gamma_i} + \phi_i(t) \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, \infty).$$

5 Existence results for (1.2) in $(BC[0, \infty))^n$

Let the Banach space $B = (BC[0, \infty))^n$ be equipped with the norm:

$$||u|| = \max_{1 \leq i \leq n} \sup_{t \in [0,\infty)} |u_i(t)| = \max_{1 \leq i \leq n} |u_i|_0$$

where we let $|u_i|_0 = \sup_{t \in [0,\infty)} |u_i(t)|$, $1 < i < n$. Throughout, for $u \in B$ and $t \in [0, \infty)$ we shall denote

$$||u(t)|| = \max_{1 \leq i \leq n} |u_i(t)|.$$

Moreover, for each $1 \leq i \leq n$, let $1 \leq p_i \leq \infty$ be an integer and q_i be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$. For $x \in L^{p_i}[0,\infty)$, we shall define $||x||_{p_i}$ as in Section 4.

Our first result is a variation of an existence principle of Lee and O'Regan [25].

Theorem 5.1 *For each* $1 \leq i \leq n$, *assume (D2)'-(D4)' and (D6) hold where*

(D6) $h_i \in BC[0, \infty)$, *denote* $H_i \equiv \sup_{t \in [0, \infty)} |h_i(t)|$.

For each $k = 1, 2, \ldots$, suppose there exists $u^k = (u_1^k, u_2^k, \ldots, u_n^k) \in (C[0,k])^n$ that satisfies

$$u_i^k(t) = h_i(t) + \int_0^k g_i(t,s) f_i(s, u_1^k(s), u_2^k(s), \ldots, u_n^k(s)) \mathrm{d}s, \quad t \in [0,k], \quad 1 \leq i \leq n. \tag{5.1}$$

Further, for $1 \le i \le n$ and $k = 1, 2,...$, there is a bounded set $B \subseteq \mathbb{R}$ such that $u_i^k(t) \in B$ for each $t \in [0, k]$. Then, (1.2) has a solution $u^* \in (BC[0, \infty))^n$ such that for $1 \le i \le n$, $u_i^*(t) \in \bar{B}$ for all $t \in [0, \infty)$.

Proof First we shall show that

$$\begin{cases} \text{for each } 1 \le i \le n \text{ and } \ell = 1, 2, \ldots, \text{ the sequence } \{u_i^k\}_{k\ge\ell} \\ \text{is uniformly bounded and equicontinuous on } [0, \ell]. \end{cases} \tag{5.2}$$

The uniform boundedness of $\{u_i^k\}_{k\ge\ell}$ follows immediately from the hypotheses; therefore, we only need to prove that $\{u_i^k\}_{k\ge\ell}$ is equicontinuous. Let $1 \le i \le n$. Since $u_i^k(t) \in B$ for each $t \in [0, k]$, there exists $\mu_B \in L^{q_i}[0, \infty)$ such that $|f_i(s,u^k(s))| \le \mu_B(s)$ for almost every $s \in [0, k]$.Fix t, $t' \in [0, \lambda]$. Then, from (5.1) we find that

$$\begin{aligned} \left|u_i^k(t) - u_i^k(t')\right| &\le |h_i(t) - h_i(t')| + \int_0^k \left|g_i^t(s) - g_i^{t'}(s)\right| \cdot |f_i(s, u^k(s))| \mathrm{d}s \\ &= |h_i(t) - h_i(t')| + \int_0^\infty 1_{[0,k]} \left|g_i^t(s) - g_i^{t'}(s)\right| \cdot |f_i(s, u^k(s))| \mathrm{d}s \\ &\le |h_i(t) - h_i(t')| + ||g_i^t - g_i^{t'}||_{p_i} \cdot ||\mu_B||_{q_i} \to 0 \end{aligned}$$

as $t \to t'$. Therefore, $\{u_i^k\}_{k\ge\ell}$ is equicontinuous on $[0, \lambda]$.

Let $1 \le i \le n$. Now, (5.2) and the Arzéla-Ascoli theorem yield a subsequence N_1 of $\mathbb{N} = \{1, 2,...\}$ and a function $z_i^1 \in C[0, 1]$ such that $u_i^k \to z_i^1$ uniformly on [0,1] as $k \to \infty$ in N_1. Let$N_2^* = N_1 \setminus \{1\}$. Then, (5.2) and the Arzéla-Ascoli theorem yield a subsequence N_2 of N_2^* and a function $z_i^2 \in C[0, 2]$ such that $u_i^k \to z_i^2$ uniformly on [0,2] as $k \to \infty$ in N_2. Note that $z_i^2 = z_i^1$ on [0,1] since $N_2 \subseteq N_1$. Continuing this process, we obtain subsequences of integers N_1, N_2,... with

$$N_1 \supseteq N_2 \supseteq \cdots \supseteq N_\ell \supseteq \cdots, \quad \text{where } N_\ell \subseteq \{\ell, \ell + 1, \ldots\}, \tag{5.3}$$

and functions $z_i^\ell \in C[0, \ell]$ such that

$$\begin{aligned} &u_i^k \to z_i^\ell \text{ uniformly on } [0, \ell] \text{ as } k \to \infty \text{ in } N_\ell, \\ &\text{and } z_i^{\ell+1} = z_i^\ell \text{ on } [0, \ell], \ \ell = 1, 2, \ldots. \end{aligned} \tag{5.4}$$

Let $1 \le i \le n$. Define a function $u_i^* : [0, \infty] \to \mathbb{R}$ by

$$u_i^*(t) = z_i^\ell(t), \quad t \in [0, \ell]. \tag{5.5}$$

Clearly, $u_i^* \in C[0, \infty)$ and $u_i^*(t) \in \bar{B}$ for each $t \in [0, \lambda]$. It remains to prove that $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ solves (1.2). Fix $t \in [0, \infty)$. Then, choose and fix λ such that $t \in [0, \lambda]$. Take $k \ge \lambda$. Now, from (5.1) we have

$$u_i^k(t) = h_i(t) + \int_0^k g_i(t, s) f_i(s, u_1^k(s), u_2^k(s), \ldots, u_n^k(s)) \mathrm{d}s, \quad t \in [0, \ell]$$

or equivalently

$$\begin{aligned}u_i^k(t) - h_i(t) - \int_0^\ell g_i(t,s)f_i(s,u_1^k(s),u_2^k(s),\ldots,u_n^k(s))\mathrm{d}s\\ = \int_l^k g_i(t,s)f_i(s,u_1^k(s),u_2^k(s),\ldots,u_n^k(s))\mathrm{d}s,\quad t\in[0,\ell].\end{aligned} \tag{5.6}$$

Since f_i is a L^{q_i}-Carathéodory function and $u_i^k(t) \in B$ for each $t \in [0, k]$, there exists $\mu_B \in L^{q_i}[0,\infty)$ such that

$$|g_i(t,s)f_i(s,u_1^k(s),u_2^k(s),\ldots,u_n^k(s))| \le |g_i^t(s)|\mu_B(s),\quad a.e.\ s\in[0,k]$$

and $|g_i^t|\mu_B \in L^1[0,\infty)$. Let $k \to \infty$ ($k \in N_\ell$) in (5.6). Since $u_i^k \to z_i^\ell$ uniformly on $[0, \ell]$, an application of Lebesgue-dominated convergence theorem gives

$$\left|z_i^\ell(t) - h_i(t) - \int_0^\ell g_i(t,s)f_i(s,z_1^\ell(s),z_2^\ell(s),\ldots,z_n^\ell(s))\mathrm{d}s\right| \le \int_l^\infty |g_i^t(s)|\mu_B(s)\mathrm{d}s,\quad t\in[0,\ell]$$

or equivalently (noting (5.5))

$$\left|u_i^*(t) - h_i(t) - \int_0^\ell g_i(t,s)f_i(s,u_1^*(s),u_2^*(s),\ldots,u_n^*(s))\mathrm{d}s\right| \le \int_l^\infty |g_i^t(s)|\mu_B(s)\mathrm{d}s,\quad t\in[0,\ell]. \tag{5.7}$$

Finally, letting $\ell \to \infty$ in (5.7) and use the fact $|g_i^t|\mu_B \in L^1[0,\infty)$ to get

$$u_i^*(t) - h_i(t) - \int_0^\infty g_i(t,s)f_i(s,u_1^*(s),u_2^*(s),\ldots,u_n^*(s))\mathrm{d}s = 0,\quad t\in[0,\infty).$$

Hence, $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ is a solution of (1.2). □

It is noted that one of the conditions in Theorem 5.1, namely, (5.1) has a solution in $(C[0, k])^n$, which has already been discussed in Section 3. As such, our subsequent Theorems 5.2-5.5 will make use of Theorem 5.1 and the technique used in Section 3. These results are parallel to Theorems 3.2-3.5 and 4.2-4.5.

Theorem 5.2 *Let (D2)-(D4) and (D6) be satisfied for each* $1 \le i \le n$. *Moreover, suppose the following conditions hold for each* $1 \le i \le n$ *and each* $w \in \{1, 2,...\}$:

$(C5)_w$ *there exist* $B_i > 0$ *such that for any* $u \in (C[0, w])^n$,

$$\int_0^w \left[f_i(t,u(t))\int_0^w g_i(t,s)f_i(s,u(s))\mathrm{d}s\right]\mathrm{d}t \le B_i,$$

$(C6)_w$ *there exist* $r > 0$ *and* $\alpha_i > 0$ *with* $r\alpha_i > H_i$ *(*H_i *as in (D6)) such that for any* $u \in (C[0, w])^n$,

$$u_i(t)f_i(t,u(t)) \ge r\alpha_i|f_i(t,u(t))| \text{ for } ||u(t)||\, r \text{ and a.e. } t\in[0,w].$$

Then, (1.2) has at least one solution in $(BC[0, \infty))^n$.

Proof We shall apply Theorem 5.1. To do so, for $w = 1, 2,...,$ we shall show that the system

$$u_i(t) = h_i(t) + \int_0^w g_i(t,s)f_i(s,u(s))\mathrm{d}s, \quad t \in [0,w], \quad 1 \le i \le n \tag{5.8}$$

has a solution in $(C[0, w])^n$. Obviously, (5.8) is just (1.1) with $T = w$. Let $w \in \{1, 2,...\}$ be fixed.

Let $u = (u_1, u_2,..., u_n) \in (C[0,w])^n$ be any solution of $(3.1)_\lambda$ (with $T = w$) where $\lambda \in (0, 1)$. We shall model after the proof of Theorem 3.2 with $T = w$ and H_i given in (D6). As in (3.9), define

$$I = \{t \in [0,w] : ||u(t)|| \le r\} \quad \text{and} \quad J = \{t \in [0,w] \ : \ ||u(t)|| > r\}.$$

Let $1 \le i \le n$. If $t \in I$, then by (D2) there exists $\mu_{r,i} \in L^1[0, \infty)$ such that

$$\int_I |f_i(t,u(t))|\mathrm{d}t \le \int_I \mu_{r,i}(t)\mathrm{d}t \le \int_0^\infty \mu_{r,i}(t)\mathrm{d}t = ||\mu_{r,i}||_1$$

[which is the analog of (3.10)]. Proceeding as in the proof of Theorem 3.2, we then obtain the analog of (3.14) as

$$\int_J |f_i(t,u(t))|\mathrm{d}t \le \frac{(H_i + r)||\mu_{r,i}||_1 + B_i}{r\alpha_i - H_i} \equiv k_i \quad (\text{independent of}\, w).$$

Further, the analog of (3.15) appears as

$$\begin{aligned} |u_i(t)| &\le \sup_{t\in[0,w]} |h_i(t)| + \left(\sup_{t\in[0,w]} \operatorname{ess}\sup_{s\in[0,w]} |g_i(t,s)|\right)(||\mu_{r,i}||_1 + k_i) \\ &\le H_i + \left(\sup_{t\in[0,\infty)} \operatorname{ess}\sup_{s\in[0,\infty)} |g_i(t,s)|\right)(||\mu_{r,i}||_1 + k_i) \equiv l_i \quad (\text{independent of}\, w), \quad t \in [0,w]. \end{aligned} \tag{5.9}$$

Hence, $||u|| \le \max_{1\le i\le n} l_i = L$ and we conclude from Theorem 3.1 that (5.8) has a solution u^* in $(C[0, w])^n$. Using similar arguments as in getting (5.9), we find $|u_i^*(t)| \le l_i$for each $t \in [0, w]$. All the conditions of Theorem 5.1 are now satisfied, it follows that (1.2) has at least one solution in $(BC[0, \infty))^n$. □

The proof of subsequent Theorems 5.3-5.5 will model after the proof of Theorem 5.2, and will employ similar arguments as in the proof of Theorems 3.3-3.5. As such, we shall present the results and omit the proof.

Theorem 5.3 *Let (D2)-(D4) and (D6) be satisfied for each* $1 \le i \le n$*. Moreover, suppose the following conditions hold for each* $1 \le i \le n$ *and each* $w \in \{1, 2,...\}$:

$(C7)_w$ *there exist constants* $a_i \ge 0$ *and* b_i *such that for any* $u \in (C[0, w])^n$,

$$\int_0^w \left[f_i(t,u(t)) \int_0^w g_i(t,s)f_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t \le a_i \int_0^w \left|f_i(t,u(t))\right| \mathrm{d}t + b_i,$$

$(C8)_w$ *there exist* $r > 0$ *and* $\alpha_i > 0$ *with* $r\alpha_i > H_i + a_i$ *(*H_i *as in (D6)) such that for any* $u \in (C[0, w])^n$,

$$u_i(t)f_i(t,u(t)) \ge r\alpha_i|f_i(t,u(t))| \textit{ for } ||u(t)|| > r \textit{ and a.e. } t \in [0,w].$$

Then, (1.2) has at least one solution in $(BC[0, \infty))^n$.

Theorem 5.4 *Let (D2)-(D4) and (D6) be satisfied for each* $1 \le i \le n$*. Moreover, suppose the following conditions hold for each* $1 \le i \le n$ *and each* $w \in \{1, 2,...\}$:

$(C9)_w$ *there exist constants* $a_i \geq 0$, $0 < \tau_i \leq 1$ *and* b_i *such that for any* $u \in (C[0, w])^n$,

$$\int_0^w \left[f_i(t, u(t)) \int_0^w g_i(t, s) f_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \leq a_i \left[\int_0^w \left| f_i(t, u(t)) \right| \mathrm{d}t \right]^{\tau_i} + b_i,$$

$(C10)_w$ *there exist* $r > 0$ *and* $\beta_i > 0$ *such that for any* $u \in (C[0, w])^n$,

$$u_i(t) f_i(t, u(t)) \geq \beta_i ||u(t)|| \cdot ||f_i(t, u(t))| \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, w].$$

Then, (1.2) has at least one solution in $(BC[0, \infty))^n$.

Theorem 5.5 *Let (D2)-(D4) and (D6) be satisfied for each* $1 \leq i \leq n$. *Moreover, suppose the following conditions hold for each* $1 \leq i \leq n$ *and each* $w \in \{1, 2, \ldots\}$: $(C10)_w$,

$(C11)_w$ *there exist* $r > 0$, $\eta_i > 0$, $\gamma_i > 0$ *and* $\phi_i \in L^{\frac{\gamma_i + 1}{\gamma_i}}[0, w]$ *such that for any* $u \in (C[0, w])^n$,

$$||u(t)|| \geq \eta_i |f_i(t, u(t)|^{\gamma_i} + \phi_i(t) \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, w],$$

$(C12)_w$ *there exist* $a_i \geq 0$, $0 < \tau_i < \gamma_i + 1$, b_i, *and* $\psi_i \in L^{\frac{\gamma_i + 1}{\gamma_i}}[0, w]$ *with* $\psi_i \geq 0$ *almost everywhere on* $[0, w]$, *such that for any* $u \in (C[0, w])^n$,

$$\int_0^w \left[f_i(t, u(t)) \int_0^w g_i(t, s) f_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \leq a_i \left[\int_0^w \psi_i(t) |f_i(t, u(t))| \mathrm{d}t \right]^{\tau_i} + b_i.$$

Also, $\varphi_i \in C[0, w]$, $h_i \in L^{\frac{\gamma_i + 1}{\gamma_i}}[0, w]$, $\psi_i \in C[0, w]$ *and* $\int_0^w |g_i(t, s)|^{\frac{\gamma_i + 1}{\gamma_i}} \mathrm{d}s \in C[0, w]$.

Then, (1.2) has at least one solution in $(BC[0, \infty))^n$.

We also have a remark similar to Remark 3.1.

Remark 5.1 In Theorem 5.5 the conditions $(C10)_w$ and $(C11)_w$ can be replaced by the following, this is evident from the proof.

$(C10)'_w$ There exist $r > 0$ and $\beta_i > 0$ such that for any $u \in (C[0, w])^n$,

$$u_i(t) f_i(t, u(t)) \geq \beta_i |u_i|_0 \cdot |f_i(t, u(t))| \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, w],$$

where we denote $|u_i|_0 = \sup_{t \in [0, w]} |u_i(t)|$.

$(C11)'_w$ There exist $r > 0$, $\eta_i > 0$, $\gamma_i > 0$ and $\phi_i \in L^{\frac{\gamma_i + 1}{\gamma_i}}[0, w]$ such that for any $u \in (C[0, w])^n$,

$$|u_i|_0 \geq \eta_i |f_i(t, u(t)|^{\gamma_i} + \phi_i(t) \text{ for } ||u(t)|| > r \text{ and a.e. } t \in [0, w].$$

6 Existence of constant-sign solutions

In this section, we shall establish the existence of *constant-sign* solutions of the systems (1.1) and (1.2), in $(C[0, T])^n$, $(C_l[0, \infty))^n$ and $(BC[0, \infty))^n$. Once again, we shall employ an argument originated from Brezis and Browder [11].

Throughout, let $\theta_i \in \{-1, 1\}$, $1 \leq i \leq n$ be fixed. For each $1 \leq j \leq n$, we define

$$[0, \infty)_j = \begin{cases} [0, \infty), & \theta_j = 1 \\ (-\infty, 0], & \theta_j = -1. \end{cases}$$

6.1 System (1.1)

Our first result is "parallel" to Theorem 3.2.

Theorem 6.1 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1), (C2)-(C4) with* $p_i = \infty$ *and* $q_i = 1$, *(C5), (C6) and (E1)-(E3) where*

(E1) $\theta_i h_i(t) \ge 0$ *for* $t \in [0, T]$,

(E2) $g_i(t, s) \ge 0$ *for* $s, t \in [0, T]$,

(E3) $\theta_i f_i(t, u) \ge 0$ *for* $(t,u) \in [0,T] \times \prod_{j=1}^{n} [0,\infty)_j$.

Then, (1.1) has at least one constant-sign solution in $(C[0, T])^n$.

Proof First, we shall show that the system

$$u_i(t) = h_i(t) + \int_0^T g_i(t,s) f_i^*(s,u(s))\mathrm{d}s, \quad t \in [0,T], \ 1 \le i \le n \tag{6.1}$$

has a solution in $(C[0, T])^n$, where,

$$f_i^*(t, u_1, \ldots, u_n) = f_i(t, v_1, \ldots, v_n), \quad t \in [0,T], \quad 1 \le i \le n \tag{6.2}$$

where for $1 \le j \le n$,

$$v_j = \begin{cases} u_j, \ \theta_j u_j \ge 0 \\ 0, \ \theta_j u_j \le 0. \end{cases}$$

Clearly, $f_i^*(t,u) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and f_i^*satisfies (C2).

We shall employ Theorem 3.1, so let $u = (u_1, u_2, \ldots, u_n) \in (C[0, T])^n$ be any solution of

$$u_i(t) = \lambda \left(h_i(t) + \int_0^T g_i(t,s) f_i^*(s,u(s))\mathrm{d}s \right), \quad t \in [0,T], \quad 1 \le i \le n \quad (6.3)_\lambda$$

where $\lambda \in (0, 1)$. Using (E1)-(E3), we have for $t \in [0, T]$ and $1 \le i \le n$,

$$\theta_i u_i(t) = \lambda \left(\theta_i h_i(t) + \int_0^T g_i(t,s) \theta_i f_i^*(s,u(s))\mathrm{d}s \right) \ge 0.$$

Hence, u is a *constant-sign* solution of $(6.3)_\lambda$, and it follows that

$$f_i^*(t,u(t)) = f_i(t,u(t)), \quad t \in [0,T], \quad 1 \le i \le n. \tag{6.4}$$

Noting (6.4), we see that $(6.3)_\lambda$ is the same as $(3.1)_\lambda$. Therefore, using a similar technique as in the proof of Theorem 3.2, we obtain (3.15) and subsequently $||u|| \le \max_{1 \le i \le n} l_i \equiv L$. It now follows from Theorem 3.1 (with $M = L + 1$) that (6.1) has a solution $u^* \in (C[0, T])^n$.

Noting (E1)-(E3), we have for $t \in [0, T]$ and $1 \le i \le n$,

$$\theta_i u_i^*(t) = \theta_i h_i(t) + \int_0^T g_i(t,s) \theta_i f_i^*(s,u^*(s))\mathrm{d}s \ge 0.$$

Thus, u^* is of *constant sign*. From (6.2), it is then clear that

$$f_i^*(t,u^*(t)) = f_i(t,u^*(t)), \quad t \in [0,T], \quad 1 \le i \le n.$$

Hence, u^* is actually a solution of (1.1). This completes the proof of the theorem. □

Based on the proof of Theorem 6.1, we can develop parallel results to Theorems 3.3-3.11 as follows.

Theorem 6.2 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1), (C2)-(C4) with* $p_i = \infty$ *and* $q_i = 1$, *(C7), (C8) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Theorem 6.3 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1), (C2)-(C4) with* $p_i = \infty$ *and* $q_i = 1$, *(C9), (C10) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Theorem 6.4 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1), (C2)-(C4) with* $p_i = \infty$ *and* $q_i = 1$, *(C10)-(C12) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Theorem 6.5 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C5), (C10), (C13) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Theorem 6.6 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C7), (C10), (C13) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Theorem 6.7 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C10), (C13), (C14) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Theorem 6.8 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C10), (C13), (C15) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Theorem 6.9 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C10), (C13), (C16) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Theorem 6.10 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(C1)-(C4), (C10), (C13), (C17) and (E1)-(E3). Then, (1.1) has at least one constant-sign solution in* $(C[0, T])^n$.

Remark 6.1 Similar to Remarks 3.1 and 3.2, in Theorem 6.4 the conditions (C10) and (C11) can be replaced by (C10)' and (C11)'; whereas in Theorems 6.5-6.10, (C10) and (C13) can be replaced by (C10)' and (C13)'.

6.2 System (1.2)

We shall first obtain the existence of constant-sign solutions of (1.2) in $(C_l[0, \infty))^n$. The first result is "parallel" to Theorem 4.2.

Theorem 6.11 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(D1)-(D5),* $(C5)_\infty$, $(C6)_\infty$ *and* $(E1)_\infty$*-*$(E3)_\infty$ *where*

$(E1)_\infty$ $\theta_i h_i(t) \ge 0$ *for* $t \in [0, \infty)$,

$(E2)_\infty$ $g_i(t, s) \ge 0$ *for* $s, t \in [0, \infty)$,

(E3)$\theta_i f_i(t,u) \ge 0$ *for* $(t, u) \in [0, \infty) \times \prod_{j=1}^{n} [0, \infty)_j$.

Then, (1.2) has at least one constant-sign solution in $(C_l[0, \infty))^n$.

Proof First, we shall show that the system

$$u_i(t) = h_i(t) + \int_0^\infty g_i(t,s) f_i^*(s, u(s)) \mathrm{d}s, \quad t \in [0, \infty), \quad 1 \le i \le n \tag{6.5}$$

has a solution in $(C_l[0, \infty))^n$. Here,

$$f_i^*(t, u_1, \ldots, u_n) = f_i(t, v_1, \ldots, v_n), \quad t \in [0, \infty), \quad 1 \le i \le n \tag{6.6}$$

where

$$v_j = \begin{cases} u_j, \ \theta_j u_j \ge 0 \\ 0, \ \theta_j u_j \le 0. \end{cases}$$

Clearly, $f_i^*(t, u) : [0, \infty] \times \mathbb{R}^n \to \mathbb{R}$ and f_i^* satisfies (D2).

We shall employ Theorem 4.1, so let $u = (u_1, u_2,..., u_n) \in (C_l[0, \infty))^n$ be any solution of

$$u_i(t) = \lambda \left(h_i(t) + \int_0^\infty g_i(t,s) f_i^*(s, u(s)) \mathrm{d}s \right), \quad t \in [0, \infty), \quad 1 \le i \le n \quad (6.7)_\lambda$$

where $\lambda \in (0, 1)$. Then, using a similar technique as in the proof of Theorem 6.1 (and also Theorem 4.2), we can show that (1.2) has a constant-sign solution $u^* \in (C_l[0, \infty))^n$. □

Remark 6.2 Similar to Remark 4.1, in Theorem 6.11 the conditions (D2)-(D5) can be replaced by (D2)'-(D5)'.

Based on the proof of Theorem 6.11, we can develop parallel results to Theorems 4.3-4.5 as follows.

Theorem 6.12 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(D1)-(D5), (C7)*$_\infty$*, (C8)*$_\infty$ *and (E1)*$_\infty$*-(E3)*$_\infty$*. Then, (1.2) has at least one constant-sign solution in* $(C_l[0, \infty))^n$.

Theorem 6.13 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(D1)-(D5), (C9)*$_\infty$*, (C10)*$_\infty$ *and (E1)*$_\infty$*-(E3)*$_\infty$*. Then, (1.2) has at least one constant-sign solution in* $(C_l[0, \infty))^n$.

Theorem 6.14 *Let the following conditions be satisfied for each* $1 \le i \le n$: *(D1)-(D5), (C10)*$_\infty$*-(C12)*$_\infty$ *and (E1)*$_\infty$*-(E3)*$_\infty$*. Then, (1.2) has at least one constant-sign solution in* $(C_l[0, \infty))^n$.

Remark 6.3 Similar to Remark 4.2, in Theorem 6.14 the conditions (C10)$_\infty$ and (C11)$_\infty$ can be replaced by $(\mathrm{C10})'_\infty$ and $(\mathrm{C11})'_\infty$.

We shall now obtain the existence of constant-sign solutions of (1.2) in $(BC[0, \infty))^n$. The first result is 'parallel' to Theorem 5.1.

Theorem 6.15 *For each* $1 \le i \le n$, *assume (D2)'-(D4)' and (D6). For each* $k = 1, 2,...,$ *suppose there exists a constant-sign* $u^k = (u_1^k, u_2^k, \ldots, u_n^k) \in (C[0, k])^n$*that satisfies*

$$u_i^k(t) = h_i(t) + \int_0^k g_i(t,s) f_i(s, u_1^k(s), u_2^k(s), \ldots, u_n^k(s)) \mathrm{d}s, \quad t \in [0, k], \quad 1 \le i \le n. \tag{6.8}$$

Further, for $1 \le i \le n$ *and* $k = 1, 2,\ldots,$ *there is a bounded set* $B \subseteq \mathbb{R}$ *such that* $u_i^k(t) \in B$ *for each* $t \in [0, k]$. *Then, (1.2) has a constant-sign solution* $u^* \in (BC[0, \infty))^n$ *such that for* $1 \le i \le n$, $u_i^*(t) \in \bar{B}$ *for all* $t \in [0, \infty)$.

Proof Using a similar technique as in the proof of Theorem 5.1, we can show that (5.2) holds. Let $1 \le i \le n$. Together with the Arzéla-Ascoli theorem, we obtain subsequences of integers $N_1, N_2,\ldots$ satisfying (5.3), and functions $z_i^\ell \in C[0, \ell]$ such that (5.4) holds. Define a function $u_i^* : [0, \infty) \to \mathbb{R}$ by (5.5), i.e.,

$$u_i^*(t) = z_i^\ell(t), \quad t \in [0, \ell].$$

Since $\theta_i u_i^k \ge 0$, we have $\theta_i z_i^\ell \ge 0$ and so $\theta_i u_i^* \ge 0$. Hence, u_i^* is of constant sign. The rest of the proof is the same as that of Theorem 5.1. □

The next result is "parallel" to Theorem 5.2.

Theorem 6.16 *Let (D2)-(D4) and (D6) be satisfied for each* $1 \le i \le n$. *Moreover, suppose the following conditions hold for each* $1 \le i \le n$ *and each* $w \in \{1, 2,\ldots,\}$: *(C5)*$_w$, *(C6)*$_w$ *and (E1)*$_w$ *- (E3)*$_w$ *where*

(E1)$_w$ $\theta_i h_i(t) \ge 0$ *for* $t \in [0, w]$,

(E2)$_w$ $g_i(t, s) \ge 0$ *for* $s, t \in [0, w]$,

(E3)$_w$ $\theta_i f_i(t,u) \ge 0$ *for* $(t, u) \in [0, w] \times \prod_{j=1}^n [0, \infty)_j$.

Then, (1.2) has at least one constant-sign solution in $(BC[0, \infty))^n$.

Proof We shall apply Theorem 6.15. To do so, for $w = 1, 2,\ldots,$ we shall show that the system (5.8) has a constant-sign solution u^* in $(C[0, w])^n$. The proof of this is similar to that of Theorem 6.1 (with $T = w$) and Theorem 5.2. As in (5.9) we have $|u_i^*(t)| \le l_i$ for each $t \in [0, w]$ and $1 \le i \le n$. All the conditions of Theorem 6.15 are now satisfied and the conclusion is immediate. □

Based on the proof of Theorem 6.16, we can develop parallel results to Theorems 5.3-5.5 as follows:

Theorem 6.17 *Let (D2)-(D4) and (D6) be satisfied for each* $1 \le i \le n$. *Moreover, suppose the following conditions hold for each* $1 \le i \le n$ *and each* $w \in \{1, 2,\ldots\}$: *(C7)*$_w$, *(C8)*$_w$ *and (E1)*$_w$*-(E3)*$_w$. *Then, (1.2) has at least one constant-sign solution in* $(BC[0, \infty))^n$.

Theorem 6.18 *Let (D2)-(D4) and (D6) be satisfied for each* $1 \le i \le n$. *Moreover, suppose the following conditions hold for each* $1 \le i \le n$ *and each* $w \in \{1, 2,\ldots\}$: *(C9)*$_w$, *(C10)*$_w$ *and (E1)*$_w$*-(E3)*$_w$. *Then, (1.2) has at least one constant-sign solution in* $(BC[0, \infty))^n$.

Theorem 6.19 *Let (D2)-(D4) and (D6) be satisfied for each* $1 \le i \le n$. *Moreover, suppose the following conditions hold for each* $1 \le i \le n$ *and each* $w \in \{1, 2,\ldots\}$: *(C11)*$_w$, *(C12)*$_w$ *and (E1)*$_w$*-(E3)w. Then, (1.2) has at least one constant-sign solution in* $(BC[0, \infty))^n$.

Remark 6.4 Similar to Remark 5.1, in Theorem 6.19 the conditions (C10)$_w$ and (C11)$_w$ can be replaced by $(\mathrm{C10})'_w$ and $(\mathrm{C11})'_w$.

7 Examples

We shall now illustrate the results obtained through some examples.

Example 7.1 In system (1.1), consider the following f_i, $1 \le i \le n$:

$$f_i(t,u) = \begin{cases} \kappa_i(t,u), & u \in P \\ 0, & \text{otherwise.} \end{cases} \tag{7.1}$$

Here,

$$P = \{u \in (C[0,T])^n : u_1(t), u_2(t), \ldots, u_n(t) > c \text{ for all } t \in [0,T]\}$$

where $c > 0$ is a given constant, and κ_i is such that

(a) the map $u \alpha f_i(t, u)$ is continuous for almost all $t \in [0, T]$;
(b) the map $t \alpha f_i(t, u)$ is measurable for all $u \in \mathbb{R}^n$;
(c) for any $r > 0$, there exists $\mu_{r,i} \in L^1[0, T]$ such that $|u| \le r$ implies $|\kappa_i(t, u)| \le \mu_{r,i}(t)$ for almost all $t \in [0, T]$;
(d) for any $u \in P$, $u_i(t)\kappa_i(t, u(t)) \ge 0$ for all $t \in [0, T]$.

Next, suppose for each $1 \le i \le n$,

$$h_i \in C[0,T] \text{ with } H_i \equiv \sup_{t\in[0,T]} |h_i(t)| < c. \tag{7.2}$$

Clearly, conditions (C1) and (C2) with $q_i = 1$ are fulfilled. We shall check that condition (C6) is satisfied. Pick $r > c$ and $\alpha_i = \frac{c}{r}$, $1 \le i \le n$. Then, from (7.2) we have $r\alpha_i = c > H_i$.

Let $u \in P$. Then, from (7.1) we have $f_i(t, u) = \kappa_i(t, u)$. Consider $||u(t)|| > r$ where $t \in [0, T]$. If $||u(t)|| = |u_i(t)|$, then noting (d) we have

$$\begin{aligned} u_i(t)f_i(t,u(t)) &= |u_i(t)| \cdot |f_i(t,u(t))| = ||u(t)|| \cdot |f_i(t,u(t))| \\ &> r|f_i(t,u(t))| \\ &> r \cdot \frac{c}{r} \cdot |f_i(t,u(t))| \\ &= r\alpha_i |f_i(t,u(t))|. \end{aligned} \tag{7.3}$$

If $||u(t)|| = |u_k(t)|$ for some $k \ne i$, then

$$\begin{aligned} u_i(t)f_i(t,u(t)) &= |u_i(t)| \cdot |f_i(t,u(t))| = r \cdot \frac{|u_i(t)|}{r} \cdot |f_i(t,u(t))| \\ &> r \cdot \frac{c}{r} \cdot |f_i(t,u(t))| \\ &= r\alpha_i |f_i(t,u(t))|. \end{aligned} \tag{7.4}$$

Therefore, from (7.3) and (7.4) we see that condition (C6) holds for $u \in P$.

For $u \in (C[0, T])^n \backslash P$, we have $f_i(t, u) = 0$ and (C6) is trivially true. Hence, we have shown that condition (C6) is satisfied.

The next example considers a convolution kernel $g_i(t, s)$ which arises in nonlinear diffusion and percolation problems; the particular case when $n = 1$ has been investigated by Bushell and Okrasiński [26].

Example 7.2 Consider system (1.1) with (7.1), (7.2), and for $1 \le i \le n$,

$$g_i(t,s) = (t-s)^{\gamma_i - 1} \tag{7.5}$$

where $\gamma_i > 1$.

Clearly, g_i satisfies (C3) and (C4) with $p_i = \infty$. Next, we shall check condition (C5). For $u \in P$ (P is given in Example 7.1), we have

$$\begin{aligned}\int_0^T \left[f_i(t,u(t)) \int_0^T g_i(t,s) f_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t &= \int_0^T \left[\kappa_i(t,u(t)) \int_0^T (t-s)^{\gamma_i - 1}\kappa_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t \\ &\le T^{\gamma_i - 1} \int_0^T \left[\kappa_i(t,u(t)) \int_0^T \kappa_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t \\ &\le B_i \end{aligned} \tag{7.6}$$

since $\kappa_i(t, u)$ satisfies (c) (note (c) is stated in Example 7.1). This shows that condition (C5) holds for $u \in P$. For $u \in (C[0, T])^n \backslash P$, we have $f_i(t, u) = 0$ and (C5) is trivially true. Therefore, condition (C5) is satisfied.

It now follows from Theorem 3.2 that the system (1.1) with (7.1), (7.2) and (7.5) has at least one solution in $(C[0, T])^n$.

The next example considers an $g_i(t, s)$ of which the particular case when $n = 1$ originates from the well known Emden differential equation.

Example 7.3 Consider system (1.1) with (7.1), (7.2), and for $1 \le i \le n$,

$$g_i(t,s) = (t-s)s^{\gamma_i} \tag{7.7}$$

where $\gamma_i \ge 0$.

Clearly, g_i satisfies (C3) and (C4) with $p_i = \infty$. Next, we see that condition (C5) is satisfied. In fact, for $u \in P$, corresponding to (7.6) we have

$$\begin{aligned}\int_0^T \left[f_i(t,u(t)) \int_0^T g_i(t,s) f_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t &= \int_0^T \left[\kappa_i(t,u(t)) \int_0^T (t-s)s^{\gamma_i}\kappa_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t \\ &\le T^{\gamma_i + 1} \int_0^T \left[\kappa_i(t,u(t)) \int_0^T \kappa_i(s,u(s))\mathrm{d}s \right] \mathrm{d}t \\ &\le B_i. \end{aligned} \tag{7.8}$$

Hence, by Theorem 3.2 the system (1.1) with (7.1), (7.2) and (7.7) has at least one solution in $(C[0, T])^n$.

Our next example illustrates the existence of a *positive* solution in $(C[0, T])^n$, this is the particular case of constant-sign solution usually considered in the literature.

Example 7.4 Let $\theta_i = 1$, $1 \le i \le n$. Consider system (1.1) with (7.1), (7.2), and for $1 \le i \le n$,

$$h_i(t) \ge 0, \quad t \in [0,T]. \tag{7.9}$$

Clearly, condition (E1) is met, and noting (d) in Example 7.1 condition (E3) is also fulfilled. Moreover, both $g_i(t, s)$ in (7.5) and (7.7) satisfy condition (E2). From Examples 7.1-7.3, we see that all the conditions of Theorem 6.1 are met. Hence, we conclude that

the system (1.1) with (7.1), (7.2), (7.9) and (7.5).

and

the system (1.1) with (7.1), (7.2), (7.9) and (7.7).

each of which has at least one *positive* solution in $(C[0, T])^n$.

Example 7.5 In system (1.2), consider the following f_i, $1 \le i \le n$:

$$f_i(t, u) = \begin{cases} \kappa_i(t, u), & u \in P_\infty \\ 0, & \text{otherwise.} \end{cases} \tag{7.10}$$

Here,

$$P_\infty = \{u \in (C_l[0, \infty))^n : u_1(t), u_2(t), \ldots, u_n(t) > c \text{ for all } t \in [0, \infty)\}$$

where $c > 0$ is a given constant, and κ_i is such that

(a)$_\infty$ the map $u \alpha f_i(t, u)$ is continuous for almost all $t \in [0, \infty)$;
(b)$_\infty$ the map $t \alpha f_i(t, u)$ is measurable for all $u \in \mathbb{R}^n$;
(c)$_\infty$ for any $r > 0$, there exists $\mu_{r,i} \in L^1[0, \infty)$ such that $|u| \le r$ implies $|\kappa_i(t, u)| \le \mu_{r,i}(t)$ for almost all $t \in [0, \infty)$;
(d)$_\infty$ for any $u \in P_\infty$, $u_i(t)\ \kappa_i(t, u(t)) \ge 0$ for all $t \in [0, \infty)$.

Next, suppose for each $1 \le i \le n$,

$$h_i \in C_l[0, \infty) \text{ with } H_i \equiv \sup_{t \in [0,\infty)} |h_i(t)| < c. \tag{7.11}$$

Clearly, conditions (D1) and (D2) are satisfied. Moreover, using a similar technique as in Example 7.1, we see that condition (C6)$_\infty$ is satisfied.

Example 7.6 Consider system (1.2) with (7.10), (7.11), and for $1 \le i \le n$,

$$g_i(t, s) = \frac{1}{s+1} + \frac{1}{(1+t)^{\gamma_i}} \tag{7.12}$$

where $\gamma_i \ge 1$.

Clearly, g_i satisfies (D3), (D4) and (D5) (take $\tilde{g}_i(s) = \frac{1}{s+1}$). Next, we shall check condition (C5)$_\infty$. For $u \in P_\infty$ (P_∞ is given in Example 7.5), we have

$$\begin{aligned} &\int_0^\infty \left[f_i(t, u(t)) \int_0^\infty g_i(t, s) f_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \\ &= \int_0^\infty \left[\kappa_i(t, u(t)) \int_0^\infty \left(\frac{1}{s+1} + \frac{1}{(1+t)^{\gamma i}} \right) \kappa_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \\ &\le 2 \int_0^\infty \left[\kappa_i(t, u(t)) \int_0^\infty \kappa_i(s, u(s)) \mathrm{d}s \right] \mathrm{d}t \le B_i \end{aligned} \tag{7.13}$$

since $\kappa_i(t, u)$ satisfies (c)$_\infty$ (note (c)$_\infty$ is stated in Example 7.5). This shows that condition (C5)$_\infty$ holds for $u \in P_\infty$. For $u \in (C_l[0, \infty))^n \backslash P_\infty$, we have $f_i(t, u) = 0$ and (C5)$_\infty$ is trivially true. Hence, condition (C5)$_\infty$ is satisfied.

We can now conclude from Theorem 4.2 that the system (1.2) with (7.10), (7.11) and (7.12) has at least one solution in $(C_l[0, \infty))^n$.

The next example shows the existence of a *positive* solution in $(C_l[0, \infty))^n$, this is the special case of constant-sign solution usually considered in the literature.

Example 7.7 Let $\theta_i = 1$, $1 \le i \le n$. Consider system (1.2) with (7.10)-(7.12), and for $1 \le i \le n$,

$$h_i(t) \geq 0, \quad t \in [0, \infty). \tag{7.14}$$

Clearly, conditions $(E1)_\infty$-$(E3)_\infty$ are satisfied. Noting Examples 7.5 and 7.6, we see that all the conditions of Theorem 6.11 are met. Hence, the system (1.2) with (7.11)-(7.12) has at least one *positive* solution in $(C_l[0, \infty))^n$.

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Authors' contributions
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