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Multiplicity results for a generalized Sturm-Liouville dynamical system on time scales

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Abstract

By applying the fixed point theorem in cones, some new and general results on the existence of positive solution to second order generalized Sturm-Liouville dynamical system on time scale $\mathbb{T}$

$$\begin{cases} u_1^{\Delta\Delta}(t) + h_1(t)f_1(t, u_1(t), u_2(t), u_1^{\Delta}(t), u_2^{\Delta}(t)) = 0, \\ u_2^{\Delta\Delta}(t) + h_2(t)f_2(t, u_1(t), u_2(t), u_1^{\Delta}(t), u_2^{\Delta}(t)) = 0, \end{cases} t \in [t_1, t_2]_{\mathbb{T}}, \\ au_i(t_1) - bu_i^{\Delta}(t_1) = \sum_{k=1}^{m-2} a_k u_i(\xi_k), \\ cu_i(\sigma(t_2)) + du_i^{\Delta}(\sigma(t_2)) = \sum_{k=1}^{m-2} b_k u_i(\xi_k), \quad (i = 1, 2),$$

are obtained. The first-order Δ-derivatives are involved in the nonlinear terms explicitly.

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1 Introduction

Time scales theory allows us to handle systematically the continuous and discrete problems and to setup a certain structure, which is to play the role of generalizing continuous and discrete [1,2], as well as more general systems simultaneously. The study of time scales has led to several important applications, for example, in the study of insect population models, phytoremediation of metals, wound healing, and epidemic models [3-5].

Dynamic systems have been the subject of numerous investigations and are also used of various real processes and phenomena in physics, such as, dynamics of gas flow, Newtonian fluid mechanics, nuclear physics, boundary layer theory, and so on, which during their evolutionary processes experience an abrupt change of state at certain moments of time, we refer to Wang [6], Zhang [7] and the references therein. Recently, there have seen a significant development and have made substantial progress on discussing the solutions for the dynamic systems on time scales, the main research theories used is the fixed point theorems in cones. Furthermore, noting an operation on the space of functions from one state space to other state space has to be defined, generalizing the continuous and discrete operations [8]. Applying the operation on the space of functions to discuss the solutions for the dynamical systems on time scales, it can bridge the gap between the continuous and discrete, mainly, can avoid the respective study in the solutions of

continuous and discrete systems. That is to say, the field of dynamic systems on time scales contains and extends the classical dynamical systems for differential and difference. For the related results, we refer to [9-12]. Specially, the first-order derivatives are involved in the nonlinear terms explicitly in dynamic system, which describe more complicated phenomena and possess some value of applications. But very little work has been done on the existence of solutions to this variety of dynamical systems on time scales. Based on the known results of boundary value problems for the differential and difference equation, the lower order derivatives are involved in the nonlinear terms explicitly. This study is devoted to proving the existence of solutions for a generalized Sturm-Liouville dynamical system on time scales $\mathbb{T}$

$$\begin{cases} u_1^{\Delta\Delta}(t) + h_1(t)f_1(t, u_1(t), u_2(t), \ u_1^{\Delta}(t), u_2^{\Delta}(t)) = 0, \\ u_2^{\Delta\Delta}(t) + h_2(t)f_2(t, u_1(t), u_2(t), u_1^{\Delta}(t), u_2^{\Delta}(t)) = 0, \end{cases} t \in [t_1, t_2]_{\mathbb{T}}, \\ au_i(t_1) - bu_i^{\Delta}(t_1) = \sum_{k=1}^{m-2} a_k u_i(\xi_k), \\ cu_i(\sigma(t_2)) + du_i^{\Delta}(\sigma(t_2)) = \sum_{k=1}^{m-2} b_k u_i(\xi_k), \end{cases} (i = 1, 2), \qquad (S)$$

where $\sigma(\cdot)$ is the forward jump operator, a, b, c, d, a_k, b_k, ξ_k, h_i and f_i $(i = 1, 2)$ satisfy
(A_1) a, b, c, $d \in [0, \infty)$, a_k, $b_k \in (0, \infty)(k = 1, 2, ..., m\text{-}2)$, $a > \sum_{k=1}^{m-2} a_k$, $c > \sum_{k=1}^{m-2} b_k$, $t_1 < \xi_1 < ... < \xi_{m-2} < t_2$;

(A_2) $h_i \in C_{rd}([t_1, t_2]_{\mathbb{T}}, [0, \infty))$ $(i = 1, 2)$ and there exists $t_0 \in (t_1, t_1)_{\mathbb{T}}$ such that $h_i(t_0) > 0$, $f_i \in C([t_1, t_2]_{\mathbb{T}} \times [0, \infty)^2 \times (-\infty, \infty)^2, [0, \infty))$ $(i = 1, 2)$.

A solution $u(t) := (u_1(t), u_2(t))$ of the system (S) is positive if, for each $i = 1, 2$, $u_i(t) \geq 0$ for all $t \in [t_1, t_2]_{\mathbb{T}}$, and there is at least one nontrivial component of $u(t)$.

We would like to mention some results of Li and Sun [9], which motivated us to consider our problem (S). In [9], the authors have studied the existence criteria of at least triple nonnegative solutions for a dynamical system on a measure chain

$$\begin{cases} u_1^{\Delta\Delta}(t) + f_1(t, u_1(\sigma(t)), u_2(\sigma(t))) = 0, \\ u_2^{\Delta\Delta}(t) + f_2(t, u_1(\sigma(t)), u_2(\sigma(t))) = 0, \end{cases} t \in [a, b], \\ \alpha u_i(a) - \beta u_i^{\Delta}(a) = 0, \quad \gamma u_i(\sigma(b)) + \delta u_i^{\Delta}(\sigma(b)) = 0 \ (i = 1, 2), \end{cases}$$

where α, β, γ, $\delta \geq 0$, $f_i : [a, b] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous, the main tools used is the fixed point theorems in cones. In addition, Sun et al. [11] considered a discrete system with parameter

$$\begin{cases} \Delta^2 u_i(k) + \lambda h_i(k)f_i(u_1(k), u_2(k), ..., u_n(k)) = 0, \quad k \in [0, T], \\ u_i(0) = u_i(T + 2) = 0 \quad (i = 1, 2, ..., n), \end{cases}$$

where $\lambda > 0$ is a constant, T and $n \geq 2$ are two fixed positive integers. They have established the existence of one positive solution by using the theory of fixed point index. Inspired by [6,7,9,13,14] et al, this study establishes some new and more general results for the existence of multiple positive solutions to the dynamical system (S). The results are even new for the general time scales as well as in special case of the continuous and discrete dynamical systems. Our results extend the known results of Li and Sun [9] $(a_k = 0, b_k = 0 \ (k = 1, 2, ..., m\text{ - }2))$ from another point of view, our Theorem 3.2 improve the main results of Shao and Zhang [12]. Specially, taking $\mathbb{T} = \mathbb{R}$, our

Theorem 3.2 extend the main results of Zhang and Liu [7]; taking $\mathbb{T} = \mathbb{Z}$, our Theorem 3.3 improve the main results of Sun et al. [11].

The rest of the paper is organized as follows. Section 2 provides some background material for discussing the generalized Sturm-Liouville dynamical system (S). An important lemma and a criterion for the existence of three positive solutions to the dynamical system (S) are established, the results are tested on an example, and a general result for dynamical system is considered in Section 3.

2 Preliminaries

Let γ and θ be nonnegative continuous convex functionals on a cone P, α be a nonnegative continuous concave functional on P, β be a nonnegative continuous functional on P, and m_1, m_2, m_3 and m_4 be positive numbers. We define the following convex sets

$$P(\gamma, m_4) = \{u \in P : \gamma(u) < m_4\},$$
$$P(\gamma, \alpha, m_2, m_4) = \{u \in P : m_2 \le \alpha(u), \gamma(u) \le m_4\},$$
$$P(\gamma, \theta, \alpha, m_2, m_3, m_4) = \{u \in P : m_2 \le \alpha(u), \theta(u) \le m_3, \gamma(u) \le m_4\},$$

and a closed set

$$Q(\gamma, \beta, m_1, m_4) = \{u \in P : m_1 \le \beta(u), \gamma(u) \le m_4\}.$$

To prove our main results, we state the Avery and Peterson fixed point theorem [15].

Lemma 2.1 *Let P be a cone in a real Banach space $\mathcal{B}$. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and β be a nonnegative continuous functional on P satisfying $\beta(\lambda u) \le \lambda\beta(u)$ for $0 \le \lambda \le 1$, such that for some positive numbers ε and m_4,*

$$\alpha(u) \le \beta(u) \text{ and } ||u|| \le \varepsilon\gamma(u) \text{ for all } u \in \overline{P(\gamma, m_4)}.$$

Suppose $T : \overline{P(\gamma, m_4)} \to \overline{P(\gamma, m_4)}$ is completely continuous and there are positive numbers m_1, m_2 and m_3 with $m_1 < m_2$ such that

(B_1) $\{u \in P(\gamma, \theta, \alpha, m_2, m_3, m_4): \alpha(u) > m_2\} \ne \varnothing$, $\alpha(Tu) > m_2$ for $u \in P(\gamma, \theta, \alpha, m_2, m_3, m_4)$;

(B_2) $\alpha(Tu) > m_2$ for $u \in P(\gamma, \alpha, m_2, m_4)$ with $\theta(Tu) > m_3$;

(B_3) $0 \notin Q(\gamma, \beta, m_1, m_4)$ and $\beta(Tu) < m_1$ for $u \in Q(\gamma, \beta, m_1, m_4)$ with $\beta(u) = m_1$.

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, m_4)}$ such that

$$\gamma(u_j) \le m_4, \quad (j = 1, 2, 3);$$
$$m_2 < \alpha(u_1); \quad m_1 < \beta(u_2) \text{ with } \alpha(u_2) < m_2; \quad \beta(u_3) < m_1.$$

In order to discuss conditions for the existence of at least three positive solutions to the generalized Sturm-Liouville dynamical system (S), we firstly define $x(t) = a(t - t_1) + b$, $y(t) = c(\sigma(t_2) - t) + d$, $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$. It is easy to see that $x(t)$ and $y(t)$ are solutions of the problem $-x^{\Delta\Delta}(t) = 0$, $x(t_1) = b$, $x^{\Delta}(t_1) = a$ and $-y^{\Delta\Delta}(t) = 0$, $y(\sigma(t_2)) = d$, $y^{\Delta}(\sigma(t_2)) = -c$, respectively. Let $G(t, s)$ be Green's function of the boundary value problem

$$\begin{cases} u^{\Delta\Delta}(t) = 0, \quad t \in [t_1, t_2]_{\mathbb{T}}, \\ au(t_1) - bu^{\Delta}(t_1) = 0, \quad cu(\sigma(t_2)) + du^{\Delta}(\sigma(t_2)) = 0. \end{cases}$$

It is known [16] that

$$G(t,s) = \frac{1}{r} \begin{cases} x(t)\gamma(\sigma(s)), & t \leq s, \\ x(\sigma(s))\gamma(t), & \sigma(s) \leq t, \end{cases} \tag{2.1}$$

for $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$ and $s \in [t_1, t_2]_{\mathbb{T}}$, where $r = bc + ad + ac(\sigma(t_2) - t_1)$.
Define

$$\eta = \min\left\{t \in \mathbb{T} : t \geq \frac{3t_1 + \sigma(t_2)}{4}\right\}, \quad \omega = \max\left\{t \in \mathbb{T} : t \leq \frac{t_1 + 3\sigma(t_2)}{4}\right\}.$$

Obviously, η and ω exist, and satisfy

$$\frac{3t_1 + \sigma(t_2)}{4} \leq \eta < \omega \leq \frac{t_1 + 3\sigma(t_2)}{4}.$$

Further,

if $\sigma(\omega) = t_2$ and $d = 0$, then $t_2 < \sigma(t_2)$.

For the Green's function $G(t, s)$, we have

$$0 \leq G(t,s) \leq G(\sigma(s), s), \quad (t,s) \in [t_1, \sigma(t_2)]_{\mathbb{T}} \times [t_1, t_2]_{\mathbb{T}},$$

since

$$\frac{G(t,s)}{G(\sigma(s),s)} = \begin{cases} \dfrac{x(t)}{x(\sigma(s))}, & t \leq s, \\ \dfrac{\gamma(t)}{\gamma(\sigma(s))}, & \sigma(s) \leq t. \end{cases}$$

Furthermore, we obtain

$$G(t,s) \geq \bar{k}G(\sigma(s), s),$$

Where $\bar{k} = \min_{(s,t) \in [t_1, \sigma(t_2)]_{\mathbb{T}} \times \left[\frac{3t_1 + \sigma(t_2)}{4}, \frac{t_1 + 3\sigma(t_2)}{4}\right]_{\mathbb{T}}} \dfrac{G(t,s)}{G(\sigma(s),s)}$,

$$G(t,s) \geq kG(\sigma(s), s), \tag{2.2}$$

Where $k = \min_{(s,t) \in [t_1, \sigma(t_2)]_{\mathbb{T}} \times [\eta, \sigma(\omega)]_{\mathbb{T}}} \dfrac{G(t,s)}{G(\sigma(s),s)}$. Obviously, $0 < \bar{k} \leq k < 1$.

Denote

$$E = \begin{vmatrix} -\sum_{k=1}^{m-2} a_k x(\xi_k) & r - \sum_{k=1}^{m-2} a_k \gamma(\xi_k) \\ r - \sum_{k=1}^{m-2} b_k x(\xi_k) & -\sum_{k=1}^{m-2} b_k \gamma(\xi_k) \end{vmatrix}.$$

Lemma 2.2 *Assume* (A_1) *holds. If* $E \neq 0$, *then for* $g_i \in C_{rd}([t_1, t_2]_{\mathbb{T}})$ $(i = 1, 2)$, *the system*

$$\begin{cases} u_1^{\Delta\Delta}(t) + g_1(t) = 0, \\ u_2^{\Delta\Delta}(t) + g_2(t) = 0, \end{cases} t \in [t_1, t_2]_{\mathbb{T}}, \\ au_i(t_1) - bu_i^{\Delta}(t_1) = \sum_{k=1}^{m-2} a_k u_i(\xi_k), \\ cu_i(\sigma(t_2)) + du_i^{\Delta}(\sigma(t_2)) = \sum_{k=1}^{m-2} b_k u_i(\xi_k), \end{cases} (i = 1, 2) \tag{2.3}$$

has a unique solution

$$u(t) = (u_1(t), u_2(t)) = \left(\int_{t_1}^{t_2} G(t,s)g_1(s)\Delta s + x(t)A(g_1) + y(t)B(g_1), \right.$$

$$\left. \int_{t_1}^{t_2} G(t,s)g_2(s)\Delta s + x(t)A(g_2) + y(t)B(g_2) \right),$$

where

$$A(g_i) = \frac{1}{E} \begin{vmatrix} \sum_{k=1}^{m-2} a_k \int_{t_1}^{t_2} G(\xi_k,s)g_i(s)\Delta s & r - \sum_{k=1}^{m-2} a_k y(\xi_k) \\ \sum_{k=1}^{m-2} b_k \int_{t_1}^{t_2} G(\xi_k,s)g_i(s)\Delta s & - \sum_{k=1}^{m-2} b_k y(\xi_k) \end{vmatrix},$$

$$B(g_i) = \frac{1}{E} \begin{vmatrix} - \sum_{k=1}^{m-2} a_k x(\xi_k) & \sum_{k=1}^{m-2} a_k \int_{t_1}^{\sigma(t_2)} G(\xi_k,s)g_i(s)\Delta s \\ r - \sum_{k=1}^{m-2} b_k x(\xi_k) & \sum_{k=1}^{m-2} b_k \int_{t_1}^{\sigma(t_2)} G(\xi_k,s)g_i(s)\Delta s \end{vmatrix} \quad (i = 1, 2).$$

Proof. The proof is similar to Lemma 2.2 [17], we omit it. □

In addition, we give a necessary hypothesis:

(A_3) $E < 0$, $r - \sum_{k=1}^{m-2} a_k y(\xi_k) > 0$, $r - \sum_{k=1}^{m-2} b_k x(\xi_k) > 0$.

Lemma 2.3 *If (A_1) and (A_3) hold, then for $g_i \in C_{rd}([t_1, t_2]_{\mathbb{T}})$ with $g_i \geq 0$ $(i = 1, 2)$, there exists $\tau_1 \in (0, 1)$ such that the unique solution $u = (u_1, u_2)$ of the system (2.3) satisfies*

$$u_i(t) \geq 0 \ (i = 1, 2), \quad t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$$

$$\inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} u_1(t) + \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} u_2(t) \geq \tau_1 \left(\sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_1(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_2(t)| \right). \quad (2.4)$$

Proof. The proof is similar to Lemma 2.2 [18], we omit it. □

3 Main results

Let $E = C_{rd}^1[t_1, \sigma(t_2)]_{\mathbb{T}}$ with the sup norm. Set $\|u\|_0 = \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_1(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_2(t)|$, $\|u^\Delta\|_0 = \sup_{t \in [t_1, t_2]_{\mathbb{T}}} |u_1^\Delta(t)| + \sup_{t \in [t_1, t_2]_{\mathbb{T}}} |u_2^\Delta(t)|$, Let $\mathcal{B} = E \times E$, the norm of $u \in \mathcal{B}$ is defined by

$$\|u\| = \max\{\|u\|_0, \|u^\Delta\|_0\},$$

then $\mathcal{B}$ is a Banach space. The cone $P \subset \mathcal{B}$ is defined by

$$P = \left\{ \begin{array}{l} u = (u_1, u_2 \in \mathcal{B}) : u_i(t) \text{ is nonnegative and concave on } [t_1, \sigma(t_2)]_{\mathbb{T}}, \\ u_i \text{ satisfies } au_i(t_1) - bu_i^\Delta(t_1) = \sum_{k=2}^{m-2} a_k u_i(\xi_k), \\ cu_i(\sigma(t_2)) + du_i^\Delta(\sigma(t_2)) = \sum_{k=2}^{m-2} b_k u_i(\xi_k) \quad (i = 1, 2), \\ \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} u_1(t) + \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} u_2(t) \\ \geq \tau_1 \left(\sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_1(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_2(t)| \right) \end{array} \right\}. \quad (3.1)$$

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals β and γ, and the nonnegative continuous functional θ be defined on the cone P by

$$\alpha(u) = \inf_{t \in [\eta\sigma(\omega)]_{\mathbb{T}}} |u_1(t)| + \inf_{t \in [\eta\sigma(\omega)]_{\mathbb{T}}} |u_2(t)|, \quad \gamma(u) = \sup_{t \in [t_1, t_2]_{\mathbb{T}}} |u_1^{\Delta}(t)| + \sup_{t \in [t_1, t_2]_{\mathbb{T}}} |u_2^{\Delta}(t)|,$$

$$\theta(u) = \beta(u) = \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_1(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_2(t)|. \tag{3.2}$$

Since the first-order Δ-derivatives are involved in the nonlinear terms explicitly to the system (S), we give the following lemma.

Lemma 3.1 *Assume* (A_1) *holds. If* $u = (u_1, u_2) \in P$, *then*

$$||u||_0 \le \tau_2 ||u^{\Delta}||_0,$$

where

$$\tau_2 = \max \left\{ \frac{a(t_0 - t_1) + b}{a - \sum_{k=1}^{m-2} a_k}, \frac{c(\sigma(t_2) - t_0) + d}{c - \sum_{k=1}^{m-2} b_k} \right\}.$$

Proof. In fact, it stuffices to prove that

$$\sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_1(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_2(t)| \le \tau_2 \Big(\sup_{t \in [t_1, t_2]_{\mathbb{T}}} |u_1^{\Delta}(t)| + \sup_{t \in [t_1, t_2]_{\mathbb{T}}} |u_2^{\Delta}(t)| \Big)$$

holds. Suppose to contrary, there exists t_0 such that $u_1(t_0) + u_2(t_0) > \tau_2(|u_1^{\Delta}(t_0)| + |u_2^{\Delta}(t_0)|)$. Since $u = (u_1, u_2) \in P$, we choose that D such that $D \subseteq [t_1, t_2]_{\mathbb{T}}^{\kappa}$, and $[t_1, t_2]_{\mathbb{T}}^{\kappa} \setminus D$ is countable and contains no right-scattered elements of $[t_1, t_2]_{\mathbb{T}}$, $u_i(t)$ ($i = 1, 2$) is continuous on $[t_1, t_2]_{\mathbb{T}}$ and differentiable in each $t \in D$, so $u_i(t)$ ($i = 1, 2$) is pre-differentiable with D. Let $t_0 \in \mathbb{T}$ such that $t_1 < t_0 < t_2$, and the compact intervals $U_1, U_2 \subset [t_1, t_2]_{\mathbb{T}}$ such that U_1 has endpoints t_1, t_0, U_2 has endpoints $t_0, \sigma(t_2)$.

For the endpoint t_1, from the mean value theorem on time scales [1, p25], one has that

$$\sup_{t \in U_1^{\kappa} \cap D} |u_i^{\Delta}(t)| \ge \frac{|u_i(t_0) - u_i(t_1)|}{|t_0 - t_1|} = \frac{u_i(t_0) - \frac{1}{a}(bu_i^{\Delta}(t_1) + \sum_{k=1}^{m-2} a_k u_i(\xi_i))}{t_0 - t_1} \quad (i = 1, 2),$$

$u_i(t)$ ($i = 1, 2$) is concave on $U_1^{\kappa} \cap D$, it follows that $u_i^{\Delta}(t_1) = \sup_{t \in U_1^{\kappa} \cap D} |u_i^{\Delta}(t)|$ ($i = 1, 2$).

We have

$$\sup_{t \in U_1^{\kappa} \cap D} |u_i^{\Delta}(t)| \ge \frac{a - \sum_{k=1}^{m-2} a_k}{a(t_0 - t_1) + b} u_i(t_0) \quad (i = 1, 2),$$

so we have

$$\sup_{t \in U_1^{\kappa} \cap D} |u_1^{\Delta}(t)| + \sup_{t \in U_1^{\kappa} \cap D} |u_2^{\Delta}(t)| \ge \frac{a - \sum_{k=1}^{m-2} a_k}{a(t_0 - t_1) + b} (u_1(t_0) + u_2(t_0)).$$

For the endpoint $\sigma(t_2)$, one has that

$$\sup_{t \in U_2^{\kappa} \cap D} |u_i^{\Delta}(t)| \ge \frac{|u_i(\sigma(t_2)) - u_i(t_0)|}{|\sigma(t_2) - t_0|} = \frac{u_i(t_0) - u_i(\sigma(t_2))}{\sigma(t_2) - t_0}$$

$$= \frac{u_i(t_0) - \frac{1}{c}(-du_i^{\Delta}(\sigma(t_2)) + \sum_{k=1}^{m-2} b_k u_i(\xi_i))}{\sigma(t_2) - t_0} \quad (i = 1, 2),$$

again, $u_i(t)$ ($i = 1, 2$) is concave on $U_1^{\kappa} \cap D$, it follows that $u_i^{\Delta}(\sigma(t_2)) = -\sup_{t \in U_2^{\kappa} \cap D} |u_i^{\Delta}(t)|$ ($i = 1, 2$). We have

$$\sup_{t \in U_2^\kappa \cap D} |u_i^\Delta(t)| \geq \frac{c - \sum_{k=1}^{m-2} b_k}{c(\sigma(t_2) - t_0) + d} u_i(t_0) \quad (i = 1, 2),$$

we get

$$\sup_{t \in U_2^\kappa \cap D} |u_1^\Delta(t)| + \sup_{t \in U_2^\kappa \cap D} |u_2^\Delta(t)| \geq \frac{c - \sum_{k=1}^{m-2} b_k}{c(\sigma(t_2) - t_0) + d} (u_1(t_0) + u_2(t_0)).$$

From the discussion results of the above cases, we can obtain that for $t = t_0$,

$$|u_1^\Delta(t_0)| + |u_2^\Delta(t_0)| \geq \min\left\{ \frac{a - \sum_{k=1}^{m-2} a_k}{a(t_0 - t_1) + b}, \frac{c - \sum_{k=1}^{m-2} b_k}{c(\sigma(t_2) - t_0) + d} \right\} (u_1(t_0) + u_2(t_0))$$

$$> \tau_2 \min\left\{ \frac{a - \sum_{k=1}^{m-2} a_k}{a(t_0 - t_1) + b}, \frac{c - \sum_{k=1}^{m-2} b_k}{c(\sigma(t_2) - t_0) + d} \right\} (|u_1^\Delta(t_0)| + |u_2^\Delta(t_0)|)$$

$$= |u_1^\Delta(t_0)| + |u_2^\Delta(t_0)|,$$

this is a contradiction. The proof is complete. □

To prove our main results, we recommend notation

$$\lambda_1^- = \left(a - \sum_{k=1}^{m-2} a_k\right)\left(c(\sigma(t_2))^2 + d(\sigma(t_2) + \sigma^2(t_2)) - \sum_{k=1}^{m-2} b_k \xi_k^2\right)$$

$$+ \left(c - \sum_{k=1}^{m-2} b_k\right)\left(\sum_{k=1}^{m-2} a_k \xi_k^2 + b(t_1 + \sigma(t_1)) - at_1^2\right),$$

$$\lambda_2^- = \left(\sum_{k=1}^{m-2} a_k \xi_k^2 + b(t_1 + \sigma(t_1)) - at_1^2\right)\left(\sum_{k=1}^{m-2} b_k \xi_k - c\sigma(t_2) - d\right)$$

$$+ \left(c(\sigma(t_2))^2 + d(\sigma(t_2) + \sigma^2(t_2)) - \sum_{k=1}^{m-2} b_k \xi_k^2\right)\left(\sum_{k=1}^{m-2} a_k \xi_k + b - at_1\right),$$

$$\bar{\lambda} = \left(a - \sum_{k=1}^{m-2} a_k\right)\left(c\sigma(t_2) + d - \sum_{k=1}^{m-2} b_k \xi_k\right) + \left(c - \sum_{k=1}^{m-2} b_k\right)\left(\sum_{k=1}^{m-2} a_k \xi_k + b - at_1\right),$$

$$\lambda_1 = \frac{\lambda_1^-}{\bar{\lambda}}, \qquad \lambda_2 = \frac{\lambda_2^-}{\bar{\lambda}},$$

where $\sigma^2(t_2) = \sigma(\sigma(t_2))$, and for $i = 1, 2$,

$$L_i = \max\left\{ \left| x^\Delta(t_1) \int_{t_1}^{t_2} \frac{1}{r}\gamma(\sigma(s))h_i(s)\Delta s + x^\Delta(t_1)A(h_i) + \gamma^\Delta(t_1)B(h_i) \right|, \right.$$

$$\left. \left| \gamma^\Delta(t_2) \int_{t_1}^{t_2} \frac{1}{r}x(\sigma(s))h_i(s)\Delta s + x^\Delta(t_2)A(h_i) + \gamma^\Delta(t_2)B(h_i) \right| \right\},$$

$$M_i = \min\left\{ \int_{t_1}^{t_2} G(\eta, s)h_i(s)\Delta s + x(\eta)A(h_i) + \gamma(\eta)B(h_i), \right.$$

$$\left. \int_{t_1}^{t_2} G(\sigma(\omega), s)h_i(s)\Delta s + x(\sigma(\omega))A(h_i) + \gamma(\sigma(\omega))B(h_i) \right\},$$

$$N_i = \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} \left(\int_{t_1}^{t_2} G(t, s)h_i(s)\Delta s + x(t)A(h_i) + \gamma(t)B(h_i) \right).$$

Theorem 3.2 *Suppose* (A_1)-(A_3) *hold and* $f_i(t, 0, 0, 0, 0) \not\equiv 0$ $(i = 1, 2)$ *for* $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$. *Assume that there exist positive numbers* m_1, m_2 *and* m_4 *such that* m_1

$$< m_2 \le \frac{\tau_1^2(\lambda_1^2 + 4\lambda_2)}{8|\sigma(t_2) + \sigma^2(t_2) - \lambda_1|} m_4 \text{and}$$

(C_1) *if* $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$, $v_1, v_2 \ge 0$, $v_1 + v_2 \in [0, \tau_2 m_4]$ *and* $w_1 + w_2 \in [-m_4, m_4]$, *then*

$$f_i(t, v_1, v_2, w_1, w_2) \le \frac{m_4}{2L_i} \quad (i = 1, 2);$$

(C_2) *there exists* $i_0 \in \{1, 2\}$, *such that*

$$f_{i_0}(t, v_1, v_2, w_1, w_2) > \frac{m_2}{2M_{i_0}},$$

for $t \in [\eta, \sigma(\omega)]_{\mathbb{T}}$, $v_1, v_2 \ge 0$, $v_1 + v_2 \in [m_2, m_2 \tau_1^{-2}]$ *and* $w_1 + w_2 \in [-m_4, m_4]$;

(C_3) *if* $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$, $v_1, v_2 \ge 0$, $v_1 + v_2 \in [0, m_1]$ *and* $w_1 + w_2 \in [-m_4, m_4]$, *then*

$$f_i(t, v_1, v_2, w_1, w_2) \le \frac{m_1}{2N_i} \quad (i = 1, 2).$$

Then the system (S) has at least three positive solutions u_1, u_2, *and* u_3 *satisfying*

$$\gamma(u_j) \le m_4 (j = 1, 2, 3);$$
$$m_2 < \alpha(u_1); \quad m_1 < \beta(u_2) \text{ with } \alpha(u_2) < m_2; \quad \beta(u_3) < m_1. \tag{3.3}$$

Proof. Define the operator T by

$$(Tu)(t) = ((T_1)(t), (T_2)(t)),$$

where $(T_i)(t) = \int_{t_1}^{t_2} G(t, s) h_i(s) f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s)) \Delta s + x(t) A(h_i f_i) + \gamma(t) B(h_i f_i)$ $(i = 1, 2)$

for $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$. For $u = (u_1, u_2) \in P$, from Lemma 2.2 we note that the system (S) has a solution u if and only if u is the fixed point of $u(t) = (Tu)(t)$. Hypothesis (A_3) implies that $A(h_i f_i) \ge 0$, $B(h_i f_i) \ge 0$ $(i = 1, 2)$. By Equation 2.4, we have $(T_i)(t) \ge 0$ $(i = 1, 2)$ for $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$, $(T_i)(t)$ $(i = 1, 2)$ is concave on $[t_1, \sigma(t_2)]_{\mathbb{T}}$ and satisfies $cT_i(\sigma(t_2)) + dT_i^\Delta(\sigma(t_2)) = \sum_{k=1}^{m-2} b_k T_i(\xi_k)$, $cT_i(\sigma(t_2)) + dT_i^\Delta(\sigma(t_2)) = \sum_{k=1}^{m-2} b_k T_i(\xi_k)$, $(i = 1, 2)$ Further we can obtain that $\inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} T_1(t) + \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} T_2(t) \ge \tau_1(\sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |T_1(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |T_2(t)|$, since for $i = 1, 2$,

$$\sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |(T_i)(t)|$$

$$= \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} \left(\int_{t_1}^{t_2} G(t, s) h_i(s) f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s)) \Delta s + x(t) A(h_i f_i) + \gamma(t) B(h_i f_i) \right)$$

$$\le \int_{t_1}^{t_2} G(\sigma(s), s) h_i(s) f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s)) \Delta s + x(\sigma(t_2)) A(h_i f_i) + \gamma(t_1) B(h_i f_i),$$

on the other hand, for $t \in [\eta, \sigma(\omega)]_{\mathbb{T}}$, Equation 2.2 implies that there exists $\tau_1 \in (0, 1)$ such that

$$(T_i)(t) = \int_{t_1}^{t_2} G(t,s)h_i(s)f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s))\Delta s + x(t)A(h_if_i) + \gamma(t)B(h_if_i)$$

$$\geq \int_{t_1}^{t_2} \tau_1 G(\sigma(s), s)h_i(s)f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s))\Delta s + x\left(\frac{3t_1 + \sigma(t_2)}{4}\right)A(h_if_i)$$

$$+ \gamma\left(\frac{t_1 + 3\sigma(t_2)}{4}\right)B(h_if_i)$$

$$\geq \tau_1\left(\int_{t_1}^{t_2} G\sigma(s), s)h_i(s)f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s))\Delta s + x(\sigma(t_2))A(h_if_i) + \gamma(t_1)B(h_if_i)\right)$$

$$\geq \tau_1 \sup_{t\in[t_1,\sigma(t_2)]_{\mathbb{T}}} |(T_i)(t)|.$$

Thus, we can conclude that $Tu \in P$. Furthermore, it is easy to check that the operator T is completely continuous by using Arzela-Ascoli lemma, and so the conditions of Lemma 2.1 hold with respect to T. α, γ, θ, and β are defined by Equation 3.2, we also note that α, γ, θ, $\beta : P \to [0, \infty)$ are continuous nonnegative functionals such that $\alpha(u) \leq \theta(u) = \beta(u)$ for all $u \in P$, β satisfies $\beta(\lambda u) \leq \lambda\beta(u)$ for $0 \leq \lambda \leq 1$, and $|u| \leq \varepsilon\gamma(u)$ for all $u \in \overline{P(\gamma, m_4)}$, where $\varepsilon = \max\{\tau_2, 1\}$. We now show that if assumption (C_1) is satisfied, then

$$T : \overline{P(\gamma, m_4)} \to \overline{P(\gamma, m_4)}. \tag{3.4}$$

In fact, for $u = (u_1, u_2) \in \overline{P(\gamma, m_4)}$, we have $\gamma(u) = \sup_{t\in[t_1,t_2]_{\mathbb{T}}} |u_1^\Delta(t)| + \sup_{t\in[t_1,t_2]_{\mathbb{T}}} |u_2^\Delta(t)| \leq m_4$. By Lemma 3.1, it holds $\sup_{t\in[t_1,\sigma(t_2)]_{\mathbb{T}}} |u_1(t)| + \sup_{t\in[t_1,\sigma(t_2)]_{\mathbb{T}}} |u_2(t)| \leq \tau_2(\sup_{t\in[t_1,t_2]_{\mathbb{T}}} |u_1^\Delta(t)| + \sup_{t\in[t_1,t_2]_{\mathbb{T}}} |u_2^\Delta(t)|) \leq \tau_2 m_4$, and so assumption (C_1) implies $f_i(t, u_1(t), u_2(t), u_1^\Delta(t), u_1^\Delta(t)) \leq \frac{m_4}{2L_i}$ $(i = 1, 2)$, for $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$. On the other hand, for $u \in P$, we have proved that $Tu = (T_1, T_2) \in P$, it follows that

$$\sup_{t\in[t_1,t_2]_{\mathbb{T}}} |(T_i)^\Delta(t)|$$

$$= \sup_{t\in[t_1,t_2]_{\mathbb{T}}} \left| \gamma^\Delta(t) \int_{t_1}^{t} \frac{1}{r}x(\sigma(s))h_i(s)f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s))\Delta s \right.$$

$$+ x^\Delta(t) \int_{t}^{t_2} \frac{1}{r}\gamma(\sigma(s))h_i(s)f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s)) \triangle s + x^\Delta(t)A(h_if_i)$$

$$+ \gamma^\Delta(t)B(h_if_i) \Big|$$

$$\leq \frac{m_4}{2L_i} \max\left\{ \left| x^\Delta(t_1) \int_{t_1}^{t_2} \frac{1}{r}\gamma(\sigma(s))h_i(s) \triangle s + x^\Delta(t_1)A(h_i) + \gamma^\Delta(t_1)B(h_i) \right|, \right.$$

$$\left. \left| \gamma^\Delta(t_2) \int_{t_1}^{t_2} \frac{1}{r}x(\sigma(s))h_i(s)\Delta s + x^\Delta(t_2)A(h_i) + \gamma^\Delta(t_2)B(h_i) \right| \right\} = \frac{m_4}{2} \quad (i = 1, 2).$$

Thus,

$$\gamma(Tu) = \sup_{t\in[t_1,t_2]_{\mathbb{T}}} |(T_1)^\Delta(t)| + \sup_{t\in[t_1,t_2]_{\mathbb{T}}} |(T_2)^\Delta(t)| \leq m_4.$$

Therefore, Equation 3.4 holds.

To check condition (B_1) in Lemma 2.1, we choose $u_0 = (u_{10}, u_{20})$, taking $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$, $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$. where $\bar{m} = \dfrac{4m_2}{\tau_1^2(\lambda_1^2 + 4\lambda_2)} > 0$. It is not difficult to verify that $u_{i0}(t) \geq 0$ $(i = 1, 2)$, $u_{i0}(t)$ $(i = 1, 2)$ is concave on $[t_1, \sigma(t_2)]_{\mathbb{T}}$ and satisfies $au_{i0}(t_1) - bu_{i0}^{\Delta}(t_1) = \sum_{k=1}^{m-2} a_k u_{i0}(\xi_k), cu_{i0}(\sigma(t_2)) + du_{i0}^{\Delta}(\sigma(t_2)) = \sum_{k=1}^{m-2} b_k u_{i0}(\xi_k)$ $(i = 1, 2)$. By the properties of $u_{i0}(t)$ $(i = 1, 2)$, we can obtain

$$
\begin{aligned}
\alpha(u_0) &= \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} |u_{10}(t)| + \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} |u_{20}(t)| \\
&= \min\{u_{10}(\eta), u_{10}(\sigma(\omega))\} + \min\{u_{20}(\eta), u_{20}(\sigma(\omega))\} \\
&\geq \tau_1\left(\frac{m_2}{\tau_1^2} + \frac{m_2}{\tau_1^2}\right) > m_2,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
\theta(u_0) &= \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_{10}(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_{20}(t)| = u_{10}\left(\frac{\lambda_1}{2}\right) + u_{20}\left(\frac{\lambda_1}{2}\right) = \frac{m_2}{\tau_1^2} + \frac{m_2}{\tau_1^2} > m_2, \\
\gamma(u_0) &= \sup_{t \in [t_1, t_2]_{\mathbb{T}}} |u_{10}^{\Delta}(t)| + \sup_{t \in [t_1, t_2]_{\mathbb{T}}} |u_{20}^{\Delta}(t)| \\
&= \max\{|u_{10}^{\Delta}(t_1)|, |u_{10}^{\Delta}(\sigma(t_2))|\} + \max\{|u_{20}^{\Delta}(t_1)|, |u_{20}^{\Delta}(\sigma(t_2))|\} \leq m_4.
\end{aligned}
\tag{3.6}
$$

Equations 3.5 and 3.6 imply that $\inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} u_{10}(t) + \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} u_{20}(t) \geq \tau_1(\sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_{10}(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |u_{20}(t)|$, thus we get $u_0 \in P$ and so $\alpha(u_0) > m_2$, $\theta(u_0) > m_2$, $\gamma(u_0) \leq m_4$ Further, $u_0 \in P(\gamma, \theta, \alpha, m_2, m_2\tau_1^{-2}, m_4)$ and $\{u_0 \in P(\gamma, \theta, \alpha, m_2, m_2\tau_1^{-2}, m_4) : \alpha(u_0) > m_2\} \neq \emptyset$. For $u \in P(\gamma, \theta, \alpha, m_2, m_2\tau_1^{-2}, m_4)$, there is $m_2 \leq u_1(t) + u_2(t) \leq m_2\tau_1^{-2}, |u_1^{\Delta}(t) + u_2^{\Delta}(t)| \leq m_4$ for $t \in [\eta, \sigma(\omega)]_{\mathbb{T}}$. Hence by assumption (C_2), one has that $f_{i_0}(t, u_1(t), u_2(t), u_1^{\Delta}(t), u_2^{\Delta}(t)) > \dfrac{m_2}{2M_{i_0}}$ $(i_0 = 1, 2)$ for $t \in [\eta, \sigma(\omega)]_{\mathbb{T}}$. So we obtain that, for $i_0 = 1, 2$

$$
\begin{aligned}
&\inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} |T_{i_0}(t)| = \min\{(T_{i_0})(\eta), (T_{i_0})(\sigma(\omega))\} \\
&= \min\left\{ \int_{t_1}^{t_2} G(\eta, s)h_{i_0}(s)f_{i_0}(s, u_1(s), u_2(s), u_1^{\Delta}(s), u_2^{\Delta}(s))\Delta s + x(\eta)A(h_{i_0}f_{i_0}) \right. \\
&\quad + \gamma(\eta)B(h_{i_0}f_{i_0}), \int_{t_1}^{t_2} G(\sigma(\omega), s)h_{i_0}(s)f_{i_0}(s, u_1(s), u_2(s), u_1^{\Delta}(s), u_2^{\Delta}(s))\Delta s \\
&\quad \left. + x(\sigma(\omega))A(h_{i_0}f_{i_0}) + \gamma(\sigma(\omega))B(h_{i_0}f_{i_0})\right\} \\
&\geq \frac{m_2}{2M_{i_0}} \min\left\{ \int_{t_1}^{t_2} G(\eta, s)h_{i_0}(s)\Delta s + x(\eta)A(h_{i_0}) + \gamma(\eta)B(h_{i_0}), \right. \\
&\quad \left. \int_{t_1}^{t_2} G(\sigma(\omega), s)h_{i_0}(s)\Delta s + x(\sigma(\omega))A(h_{i_0}) + \gamma(\sigma(\omega))B(h_{i_0}) \right\} = \frac{m_2}{2}.
\end{aligned}
$$

Thus,

$$
\alpha(Tu) = \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} |T_1(t)| + \inf_{t \in [\eta, \sigma(\omega)]_{\mathbb{T}}} |T_2(t)| > m_2.
$$

So we get $\alpha(Tu) > m_2$ for $u = (u_1, u_2) \in P(\gamma, \theta, \alpha, m_2, \frac{m_2}{\tau_1}, m_4)$ and condition (B_1) in Lemma 2.1 is satisfied.

We now prove that condition (B_2) in Lemma 2.1 holds. In fact, if $u = (u_1, u_2) \in P(\gamma, \alpha, m_2, m_4)$ with $\theta(Tu) > \dfrac{m_2}{\tau_1}$, then

$$\alpha(Tu) \geq \tau_1 \theta(Tu) > m_2.$$

Finally, we assert that condition (B_3) in Lemma 2.1 also holds. $(0, 0) \notin Q(\gamma, \beta, m_1, m_4)$, since $\beta((0, 0)) = 0 < m_1$. Assume that $u = (u_1, u_2) \in Q(\gamma, \beta, m_1, m_4)$ with $\beta(u) = m_1$, assumption (C_3) implies that

$$
\begin{aligned}
&\sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |T_i(t)| \\
&= \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} \int_{t_1}^{t_2} G(t, s) h_i(s) f_i(s, u_1(s), u_2(s), u_1^\Delta(s), u_2^\Delta(s)) \Delta s + x(t) A(h_i f_i) + y(t) B(h_i f_i) \\
&\leq \frac{m_1}{2N_i} \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} \left(\int_{t_1}^{t_2} G(t, s) h_i(s) \Delta s + x(t) A(h_i) + y(t) B(h_i) \right) = \frac{m_1}{2} \quad (i = 1, 2).
\end{aligned}
$$

Hence,

$$\beta(Tu) = \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |T_1(t)| + \sup_{t \in [t_1, \sigma(t_2)]_{\mathbb{T}}} |T_2(t)| \leq m_1.$$

Therefore, the conditions of Lemma 2.1 are fulfilled, thus the system (S) has at least three positive solutions u_1, u_2, and u_3, such that Equation 3.3 hold. □

From the above argument, we could consider naturally the general dynamic system on time scales $\mathbb{T}$.

$$
\begin{cases}
u_1^{\Delta\Delta}(t) + h_1(t) f_1(t, u_1(t), \ldots, u_n(t), u_1^\Delta(t), \ldots, u_n^\Delta(t)) = 0, \\
\ldots, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [t_1, t_2]_{\mathbb{T}}, \\
u_n^{\Delta\Delta}(t) + h_n(t) f_n(t, u_1(t), \ldots, u_n(t), u_1^\Delta(t), \ldots, u_n^\Delta(t)) = 0, \\
a u_i(t_1) - b u_i^\Delta(t_1) = \sum_{k=1}^{m-2} a_k u_i(\xi_k), \\
\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \ldots, n), \\
c u_i(\sigma(t_2)) + d u_i^\Delta(\sigma(t_2)) = \sum_{k=1}^{m-2} b_k u_i(\xi_k),
\end{cases}
\qquad (GS)
$$

where a, b, c, d, a_k, b_k, ξ_k ($k = 1, 2, \ldots, m - 2$) satisfy (A_1) and h_i, f_i ($i = 1, 2$) fulfill the condition

(A_2') $h_i \in C_{rd}([t_1, t_2]_{\mathbb{T}}, [0, \infty))$ ($i = 1, 2, \ldots, n$) and there exists $t_0 \in (t_1, t_1)_{\mathbb{T}}$ such that $h_i(t_0) > 0, f_i \in C([t_1, t_2]_{\mathbb{T}} \times [0, \infty)^n \times (-\infty, \infty)^n, [0, \infty))$ ($i = 1, 2, \ldots, n$).

We obtain a result, the proof of the result is omitted.

Theorem 3.3 *Suppose* (A_1), (A_2') *and* (A_3) *hold, and* $f_i(t, 0, \ldots, 0) \not\equiv 0$ ($i = 1, 2, \ldots, n$) *for* $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$. *Assume that there exist positive numbers* $m_1', m_2', m_4', \tau_1', \tau_2'$ *such that*

$$m_1' < m_2' \leq \frac{(\tau_1')^2(\lambda_1^2 + 4\lambda_2)}{4n|\sigma(t_2) + \sigma^2(t_2) - \lambda_1|} m_4', \quad 0 < \tau_1', \tau_2' < 1 \text{and}$$

(C_4) *if* $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$, v_1, v_2, $\ldots$, $v_n \geq 0$, $v_1 + v_2 + \cdots + v_n \in [-\tau_2', m_4']$ *and* $w_1 + w_2 + \cdots w_n \in [-m_4', m_4']$, *then*

$$f_i(t, v_1, \ldots, v_n, w_1, \ldots, w_n) \leq \frac{m_4'}{nL_i'} \quad (i = 1, 2, \ldots, n);$$

(C_5) there exists $i_0 \in \{1, 2, ..., n\}$, such that

$$f_{i_0}(t, v_1, \ldots, v_n, w_1, \ldots, w_n) > \frac{m'_2}{nM'_{i_0}},$$

for $t \in [\eta, \sigma(\omega)]_{\mathbb{T}}$, v_1, v_2, ..., $v_n \geq 0$, $v_1 + v_2 + \cdots + v_n \in [m'_2, m'_2(\tau'_1)^{-2}]$ and $w_1 + w_2 + \cdots w_n \in [-m'_4, m'_4]$;

(C_6) if $t \in [t_1, \sigma(t_2)]_{\mathbb{T}}$, v_1, v_2, ..., $v_n \geq 0$, $v_1 + v_2 + \cdots + v_n \in [0, m'_1]$ and $w_1 + w_2 + \cdots w_n \in [-m'_4, m'_4]$, then

$$f_i(t, v_1, \ldots, v_n, w_1, \ldots, w_n) \leq \frac{m'_1}{nN'_i} \quad (i = 1, 2, \ldots, n),$$

where L'_1, M'_{i_0}, N'_i $(i, i_0 = 1, 2, ..., n)$ are positive constants and are similar to L_i, M_{i_0}, N_i $(i, i_0 = 1, 2, ..., n)$, respectively.

Then the system (GS) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\gamma(u_j) \leq m_4 (j = 1, 2, 3);$$

$$m_2 < \alpha(u_1); \quad m_1 < \beta(u_2) \text{ with } \alpha(u_2) < m_2; \quad \beta(u_3) < m_1.$$

Let $\mathbb{T} = \{0\} \cup \{\frac{1}{2^{n+1}}\} \cup [\frac{1}{2}, 1]$ $(n = 1, 2, ...)$, $h_i(t) = 1$, $(i = 1, 2)$, $m = 4$. Consider the system

$$
\begin{cases}
u_1^{\Delta\Delta}(t) + f_1(t, u_1(t), u_2(t), u_1^\Delta(t), u_2^\Delta(t)) = 0, \\
u_2^{\Delta\Delta}(t) + f_2(t, u_1(t), u_2(t), u_1^\Delta(t), u_2^\Delta(t)) = 0,
\end{cases} \; t \in [0, 1]_{\mathbb{T}}
$$
$$
4u_i(0) - 2u_i^\Delta(0) = u_i\left(\frac{1}{4}\right) + u_i\left(\frac{1}{2}\right),
$$
$$
4u_i(\sigma(1)) + 2u_i^\Delta(\sigma(1)) = u_i\left(\frac{1}{4}\right) + u_i\left(\frac{1}{2}\right), \quad (i = 1, 2),
$$

(3.7)

where

$$f_i(t, v_1, v_2, w_1, w_2) = \begin{cases} 10^{-4}t + 5(v_1 + v_2)^8 + 10^{-3}\left(\frac{w_1+w_2}{2.5\times10^5}\right)^3, & v_1 + v_2 < 4, \\ 10^{-4}t + 327680 + 10^{-3}\left(\frac{w_1+w_2}{2.5\times10^5}\right)^3, & v_1 + v_2 \geq 4, \end{cases} \quad (i = 1, 2).$$

It is easy to check that hypotheses (A_1)-(A_3) hold and f(t, 0, 0, 0, 0) ≢ 0 on $[0, 1]_{\mathbb{T}}$. By some calculations, we have $\tau_1 = \frac{1}{2}$, $\tau_2 = \frac{3}{2}$, $\lambda_1 = 1$, $\lambda_2 = \frac{37}{32}L_1 + L_2 = \frac{25}{48}$, $M_1 + M_2 \approx \frac{31}{3}$, $N_1 + N_2 = \frac{104}{7}$. If we choose $m_1 = \frac{1}{2}$, $m_2 = 1$, $m_4 = 2.5 \times 10^5$, then $f_i(t, v_1, v_2, w_1, w_2) \leq \frac{m_4}{2L_i} = 4.8 \times 10^2$ $(i = 1, 2)$ for $t \in [0, 1]_{\mathbb{T}}$, $v_1 + v_2 \in= [0, \frac{75}{2}]$, $w_1 + w_2 \in [-2.5 \times 10^5, 2.5 \times 10^5]$; $f_{i_0}(t, v_1, v_2, w_1, w_2) > \frac{m_2}{M_{i_0}} \approx 0.09$ $(i_0 \in \{1, 2\})$ for $t \in [\frac{1}{4}, \frac{3}{4}]_{\mathbb{T}}$ $v_1 + v_2 \in [1, 4]$, $w_1 + w_2 \in [-2.5 \times 10^5, 2.5 \times 10^5]$; $f_i(t, v_1, v_2, w_1, w_2) \leq \frac{m_1}{2N_i} \approx 0.03$ $(i = 1, 2)$ for $t \in [0, 1]_{\mathbb{T}}$, $v_1 + v_2 \in [0, \frac{1}{2}]$, $w_1 + w_2 \in [-2.5 \times 10^5, 2.5 \times 10^5]$. Thus all hypotheses of Theorem 3.2 hold. So the system (3.7) has at least three positive solutions $u_1(t)$, $u_2(t)$ and $u_3(t)$ such that

$$\max_{0 \le t \le 1} u_3(t) < \frac{1}{2} \text{ with } \min_{\frac{1}{4} \le t \le \frac{3}{4}} u_2(t) < 1, \max_{0 \le t \le 1} u_3(t) < \frac{1}{2} \text{ hold.}$$

Competing interests
The author declares that they have no competing interests.

References
1. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston. (2001)
2. Bohner, M, Peterson, A: Advances in Dynamic Equations on Time Scales. Birkhäusere, Boston. (2003)
3. Jones, MA, Song, B, Thomas, DM: Controlling wound healing through debridement. Math Comput Modelling. **40**, 1057–1064 (2004). doi:10.1016/j.mcm.2003.09.041
4. Atici, FM, Biles, DC, Lebedinsky, A: An appilcation of time scales to economics. Math Comput Model. **43**, 718–726 (2006). doi:10.1016/j.mcm.2005.08.014
5. Tisdell, C, Zaidi, A: Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling. Nonlinear Anal. **68**, 3504–3524 (2008). doi:10.1016/j.na.2007.03.043
6. Wang, H: On the number of positive solutions of nonlinear systems. J Math Anal Appl. **281**, 287–306 (2003)
7. Zhang, X, Liu, L: A necessary and sufficient condition of existence of positive solutions for nonlinear singular differential systems. J Math Anal Appl. **327**, 400–414 (2007). doi:10.1016/j.jmaa.2006.04.031
8. Agarwal, RP, O'Regan, D, Wong, PJY: Positive Solutions of Differential, Difference and Integral Equations. Kluwer, Dordrecht. (1999)
9. Li, W, Sun, H: Multiple positive solutions for nonlinear dynamical systems on a measure chain. J Comput Appl Math. **162**, 421–430 (2004). doi:10.1016/j.cam.2003.08.032
10. Choi, S, Im, D, Koo, N: Stability of linear dynamic systems on time scales. Adv Differ Equ. 670203 (2008)
11. Sun, J, Zhao, Y, Li, W: Existence of positive solution for second-order nonlinear discrete system with parameter. Math Comput Model. **41**, 493–499 (2005). doi:10.1016/j.mcm.2004.04.006
12. Shao, N, Zhang, Y: Existence of positive solutions for semipositone dynamic system on time scales. Electron J Differ Equ. **114**, 1–10 (2008)
13. Atici, FM, Guseinov, GSh: On Green's functions and positive solutions for boundary value problem on time scales. J Comput Appl Math. **141**, 75–99 (2002). doi:10.1016/S0377-0427(01)00437-X
14. Zhang, Y: A multiplicity result for a generalized Sturm-Liouville problem on time scales. J Differ Equ Appl. **16**, 963–974 (2010). doi:10.1080/10236190802660476
15. Avery, RI, Peterson, AC: Three positive fixed points of nonlinear operators on ordered Banach spaces. Comput Math Appl. **42**, 313–322 (2001). doi:10.1016/S0898-1221(01)00156-0
16. Erbe, L, Peterson, A: Positive solutions for a nonlinear differential equation on a measure chain. Boundary Value Problems and Related Topics. Math Comput Modelling. **32**(5-6), 571–585 (2000)
17. Ma, R: Multiple positive solutions for nonlinear *m*-point boundary value problem. Appl Math Comput. **148**, 249–262 (2004). doi:10.1016/S0096-3003(02)00843-3
18. Zhang, Y, Sun, H: Three positive solutions for a generalized Sturm-Liouville multipoint BVP with dependence on the first order derivative. Dyn Syst Appl. **17**, 313–3240 (2008)