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Positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders

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Abstract

In this article, we study the existence of positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders

$$\begin{cases} -D_{0^+}^{\alpha}u(t) = f(t, v(t)),\ 0 < t < 1, \\ D_{0^+}^{\beta}v(t) = g(t, u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{cases}$$

where $2 < \alpha \le 3$, $3 < \beta \le 4$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivative, and $f, g : [0, 1] \times [0, +\infty) \to [0, +\infty)$ are given continuous functions, $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$. Our analysis relies on fixed point theorems on cones. Some sufficient conditions for the existence of at least one or two positive solutions for the boundary value problem are established. As an application, examples are presented to illustrate the main results.

Keywords: Positive solution, coupled system, fractional Green?'?s function, fixed point theorem

1 Introduction

Fractional differential equations have been of great interest recently. It is caused by the both intensive development of the theory of fractional calculus itself and applications, see [1-6]. Recently, there are a large number of papers dealing with the existence of solutions of nonlinear fractional differential equations by the use of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory, Adomian decomposition method, etc.), see [7-21]. The articles [13-21] considered boundary value problems for fractional differential equations.

Yu and Jiang [20] examined the existence of positive solutions for the following problem

$$D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0,\ 0 < t < 1,$$
$$u(0) = u(1) = u'(0) = 0,$$

where $2 < \alpha \le 3$ is a real number, $f \in C([0,1] \times [0, +\infty); (0, +\infty))$ and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional differentiation. Using the properties of the Green function, they obtained some existence criteria for one or two positive solutions for singular and nonsingular boundary value problems by means of the Krasnosel'skii fixed point theorem and a mixed monotone method.

Xu et al. [21] considered the existence of positive solutions for the following problem

$$D_{0^+}^{\alpha}u(t) = f(t, u(t)),\ 0 < t < 1,$$
$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

where $3 < \alpha \leq 4$ is a real number, $f \in C([0, 1] \times [0, +\infty), (0, +\infty))$ and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional differentiation. Using the properties of the Green function, they gave some multiple positive solutions for singular and nonsingular boundary value problems, and also they gave uniqueness of solution for singular problem by means of Leray-Schauder nonlinear alternative, a fixed point theorem on cones and a mixed monotone method.

On the other hand, the study of coupled systems involving fractional differential equations is also important as such systems occur in various problems, see [22-30].

Bai and Fang [24] considered the existence of positive solutions of singular coupled system

$$\begin{cases} D^s u = f(t, v),\ 0 < t < 1, \\ D^p v = g(t, u),\ 0 < t < 1, \end{cases}$$

where $0 < s, p < 1$, and $f, g : [0, 1) \times [0, +\infty) \to [0, +\infty)$ are two given continuous functions, $\lim_{t \to 0^+} f(t, \cdot) = +\infty$, $\lim_{t \to 0^+} g(t, \cdot) = +\infty$ and D^s, D^p are two standard Riemann-Liouville fractional derivatives. They established the existence results by a nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem on a cone.

Su [25] discussed a boundary value problem for a coupled differential system of fractional order

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)),\ 0 < t < 1, \\ D^{\beta}v(t) = g(t, u(t), D^{\nu}u(t)),\ 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $\mu, \nu > 0$, $\alpha - \nu \geq 1$, $\beta - \mu \geq 1$, $f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions and D is the standard Riemann-Liouville fractional derivative. By means of Schauder fixed point theorem, an existence result for the solution was obtained.

From the above works, we can see a fact, although the coupled systems of fractional boundary value problems have been investigated by some authors, coupled systems due to mixed fractional orders are seldom considered. Motivated by all the works above, in this article we investigate the existence of positive solutions for a coupled system of nonlinear differential equations of mixed fractional orders

$$\begin{cases} -D_{0^+}^{\alpha}u(t) = f(t, v(t)),\ 0 < t < 1, \\ D_{0^+}^{\beta}v(t) = g(t, u(t)),\quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = v'(1) = 0, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $3 < \beta \leq 4$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivative, and $f, g : [0, 1] \times [0, +\infty) \to [0, +\infty)$ are given continuous functions, $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$. Our analysis relies on fixed point theorems on cones. Some sufficient conditions for the existence of at least one or two positive solutions for the boundary value problem are established. Finally, we present some examples to demonstrate our results.

The article is organized as follows. In Sect. 2, we shall give some definitions and lemmas to prove our main results. In Sect. 3, we establish existence results of at least one or two positive solutions for boundary value problem (1.1) by fixed point theorems on cones. In Sect. 4, examples are presented to illustrate the main results.

2 Preliminaries

For the convenience of readers, we give some background materials from fractional calculus theory to facilitate analysis of problem (1.1). These materials can be found in the recent literature, see [20,21,31-33].

Definition 2.1 [31] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 [31] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, \mathrm{d}s,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

From the definition of the Riemann-Liouville derivative, we can obtain the following statement.

Lemma 2.1 [31]*Let* $\alpha > 0$. *If we assume* $u \in C(0, 1) \cap L(0, 1)$, *then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *as unique solutions, where* n *is the smallest integer greater than or equal to* α.

Lemma 2.2 [31]*Assume that* $u \in C(0, 1) \cap L(0, 1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0, 1) \cap L(0, 1)$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \text{ for some } c_i \in \mathbb{R}, \quad i = 1, 2, \ldots, n,$$

where n *is the smallest integer greater than or equal to* α.

In the following, we present the Green function of fractional differential equation boundary value problem.

Lemma 2.3 [20]*Let* $h_1 \in C[0, 1]$ *and* $2 < \alpha \leq 3$. *The unique solution of problem*

$$-D_{0^+}^{\alpha} u(t) = h_1(t), \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1) = u'(0) = 0, \tag{2.2}$$

is

$$u(t) = \int_0^1 G_1(t,s)h_1(s)\,ds,$$

where

$$G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Here G_1 (t, s) is called the Green function of boundary value problem (2.1) *and* (2.2).

Lemma 2.4 [20]*The function $G_1(t, s)$ defined by* (2.3) *satisfies the following conditions:*

(A1) $G_1(t, s) = G_1(1 - s, 1 - t)$, *for* $t, s \in (0, 1)$;

(A2) $t^{\alpha-1}(1 - t)s(1 - s)^{\alpha-1} \le \Gamma(\alpha)G_1(t, s) \le (\alpha - 1)s(1 - s)^{\alpha-1}$, *for* $t, s \in (0, 1)$;

(A3) $G_1(t, s) > 0$, *for* $t, s \in (0, 1)$;

(A4) $t^{\alpha-1}(1 - t)s(1 - s)^{\alpha-1} \le \Gamma(\alpha)G_1(t, s) \le (\alpha - 1)(1 - t)\, t^{\alpha-1}$, *for* $t, s \in (0, 1)$.

Remark **2.1** Let $q_1(t) = t^{\alpha-1}(1 - t)$, $k_1(s) = s(1 - s)^{\alpha-1}$. Then

$$q_1(t)k_1(s) \le \Gamma(\alpha)G_1(t,s) \le (\alpha-1)k_1(s).$$

Lemma 2.5 [21]*Let $h_2 \in C[0, 1]$ and $3 < \beta \le 4$. The unique solution of problem*

$$D_{0^+}^{\beta}u(t) = h_2(t), \quad 0 < t < 1 \tag{2.4}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{2.5}$$

is

$$u(t) = \int_0^1 G_2(t,s)h_2(s)\,ds,$$

where

$$G_2(t,s) = \begin{cases} \dfrac{(t-s)^{\beta-1} + (1-s)^{\beta-2}t^{\beta-2}[(s-t) + (\beta-2)(1-t)s]}{\Gamma(\beta)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\beta-2}(1-s)^{\beta-2}[(s-t) + (\beta-2)(1-t)s]}{\Gamma(\beta)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.6}$$

Here $G_2(t, s)$ is called the Green function of boundary value problem (2.4) *and* (2.5).

Lemma 2.6 [21]*The function $G_2(t, s)$ defined by* (2.6) *satisfies the following conditions:*

(B1) $G_2(t, s) = G_2(1 - s, 1 - t)$, *for* $t, s \in (0, 1)$;

(B2) $(\beta - 2)t^{\beta-2}(1 - t)^2 s^2(1 - s)^{\beta-2} \le \Gamma(\beta)G_2(t, s) \le M_0 s^2(1 - s)^{\beta-2}$, *for* $t, s \in (0, 1)$;

(B3) $G_2(t, s) > 0$, *for* $t, s \in (0, 1)$;

(B4) $(\beta - 2)s^2(1 - s)^{\beta-2}t^{\beta-2}(1 - t)^2 \le \Gamma(\beta)G_2(t, s) \le M_0 t^{\beta-2}(1 - t)^2$, *for* $t, s \in (0, 1)$, *here* $M_0 = \max\{\beta - 1, (\beta - 2)^2\}$.

Remark **2.2** *Let* $q_2(t) = t^{\beta-2}(1 - t)^2$, $k_2(s) = s^2(1 - s)^{\beta-2}$. Then

$$(\beta-2)q_2(t)k_2(s) \le \Gamma(\beta)G_2(t,s) \le M_0k_2(s).$$

The following two lemmas are fundamental in the proofs of our main results.

Lemma 2.7 [32]*Let E be a Banach space, and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either*

(D1) $||Sw|| \leq ||w||$, $w \in P \cap \partial\Omega_1$, $||Sw|| \geq ||w||$, $w \in P \cap \partial\Omega_2$, *or*

(D2) $||Sw|| \geq ||w||$, $w \in P \cap \partial\Omega_1$, $||Sw|| \leq ||w||$, $w \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.8 [33]*Let E be a Banach space, and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 and Ω_3 are open subsets of E with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \subset \bar{\Omega}_2 \subset \Omega_3$, and let $S : P \cap (\bar{\Omega}_3 \backslash \Omega_1) \to P$ be a completely continuous operator such that*

(E1) $||Sw|| \geq ||w||$, $\forall\, w \in P \cap \partial\Omega_1$;

(E2) $||Sw|| \leq ||w||$, $Sw \neq w$, $\forall\, w \in P \cap \partial\Omega_2$;

(E3) $||Sw|| \geq ||w||$, $\forall\, w \in P \cap \partial\Omega_3$.

Then S has two fixed points w_1 and w_2 in $P \cap (\bar{\Omega}_3 \backslash \Omega_1)$with $w_1 \in (\bar{\Omega}_2 \backslash \Omega_1)$ and $w_2 \in (\bar{\Omega}_3 \backslash \Omega_2)$.

3 Main results

In this section, we establish the existence results of positive solutions for boundary value problem (1.1).

Consider the following coupled system of integral equations:

$$\begin{cases} u(t) = \int_0^1 G_1(t,s) f(s, v(s))\, \mathrm{d}s, \\ v(t) = \int_0^1 G_2(t,s) g(s, u(s))\, \mathrm{d}s. \end{cases} \tag{3.1}$$

Lemma 3.1 *Suppose that f, g : [0, 1] × [0, +∞) → [0, +∞) are continuous. Then (u, v) ∈ C[0, 1] × C[0, 1] is a solution of* (1.1) *if and only if (u, v) ∈ C[0, 1] × C[0, 1] is a solution of system* (3.1).

This proof is similar to that of Lemma 3.3 in [25], so is omitted.

From (3.1), we can get the following integral equation

$$u(t) = \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,r) g(r, u(r))\, \mathrm{d}r\right) \mathrm{d}s, \quad t \in [0,1].$$

Let Banach space $E = C[0, 1]$ be endowed with the norm $||u|| = \max_{0 \leq t \leq 1} |u(t)|$. De ne the cone $P \subset E$ by

$$P = \left\{u \in E : u(t) \geq \frac{q_1(t)}{\alpha - 1} ||u||, \quad t \in [0,1]\right\}.$$

We define an operator $T : P \to E$ as follows

$$Tu(t) = \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,r) g(r, u(r))\, \mathrm{d}r\right) \mathrm{d}s, \quad t \in [0,1].$$

Lemma 3.2 *$T : P \to P$ is completely continuous.*

Proof. For $u \in P$, $0 \le t \le 1$, by Lemma 2.4,

$$\begin{aligned} ||Tu|| &= \max_{0\le t\le 1} |Tu(t)| \\ &= \max_{0\le t\le 1} \left| \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,r) g(r,u(r))\,\mathrm{d}r\right) \mathrm{d}s \right| \\ &\le \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 k_1(s) f\left(s, \int_0^1 G_2(s,r) g(r,u(r))\,\mathrm{d}r\right) \mathrm{d}s, \end{aligned}$$

$$\begin{aligned} Tu(t) &= \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,r) g(r,u(r))\,\mathrm{d}r\right) \mathrm{d}s \\ &\ge \int_0^1 \frac{q_1(t)k_1(s)}{\Gamma(\alpha)} f\left(s, \int_0^1 G_2(s,r) g(r,u(r))\,\mathrm{d}r\right) \mathrm{d}s \\ &\ge \frac{q_1(t)}{\alpha-1} ||Tu||. \end{aligned}$$

Thus we have $T(P) \subset P$.

The operator $T : P \to P$ is continuous in view of continuity of $G(t, s)$, $f(t, u)$, and $g(t, u)$. For any bounded set M, $T(M)$ is uniformly bounded and equicontinuous. This proof is similar to that of Lemma 2.1.1 in [20], so is omitted. By means of Arzela-Ascoli Theorem, $T : P \to P$ is completely continuous. This completes the proof.

We consider the following hypotheses in what follows.

(A_1) $\lim_{u\to 0^+}\sup_{t\in[0,1]} \dfrac{f(t,u)}{u} = 0,\ \lim_{u\to 0^+}\sup_{t\in[0,1]} \dfrac{g(t,u)}{u} = 0;$

(A_2) $\lim_{u\to +\infty}\inf_{t\in[0,1]} \dfrac{f(t,u)}{u} = +\infty,\ \lim_{u\to +\infty}\inf_{t\in[0,1]} \dfrac{g(t,u)}{u} = +\infty;$

(A_3) $\lim_{u\to 0^+}\inf_{t\in[0,1]} \dfrac{f(t,u)}{u} = +\infty,\ \lim_{u\to 0^+}\inf_{t\in[0,1]} \dfrac{g(t,u)}{u} = +\infty;$

(A_4) $\lim_{u\to +\infty}\sup_{t\in[0,1]} \dfrac{f(t,u)}{u} = 0,\ \lim_{u\to +\infty}\sup_{t\in[0,1]} \dfrac{g(t,u)}{u} = 0;$

(A_5) $f(t, u)$ and $g(t, u)$ are two increasing functions with respect to u, and there exists $N > 0$ such that

$$n_1 f\left(t, \int_0^1 n_2 g(r,N)\,\mathrm{d}r\right) < N, \quad \text{for } t \in [0,1],$$

where $n_1 = \max_{0\le t,s\le 1} G_1(t, s)$, $n_2 = \max_{0\le t,s\le 1} G_2(t, s)$.

Theorem 3.1 *Assume that hypotheses* (A_1) *and* (A_2) *hold. Then the boundary value problem* (1.1) *has at least one positive solution* (u, v).

Proof. By hypothesis (A_1), we see that there exists $p_1 \in (0, 1)$ such that

$$f(t,u) \le \lambda_1 u, \quad g(t,u) \le \lambda_2 u, \quad \text{for } (t,u) \in [0,1] \times (0,p_1), \tag{3.2}$$

where $\lambda_1, \lambda_2 > 0$ and satisfy

$$\frac{\lambda_1(\alpha-1)}{\Gamma(\alpha)} \int_0^1 k_1(s)\,\mathrm{d}s \le 1, \quad \frac{\lambda_2 M_0}{\Gamma(\beta)} \int_0^1 k_2(s)\,\mathrm{d}s \le 1. \tag{3.3}$$

For $u \in P$ with $||u|| = \frac{p_1}{2}$, we have

$$\int_0^1 G_2(s,r)g(r,u(r))\,\mathrm{d}r \le \int_0^1 \frac{M_0k_2(r)}{\Gamma(\beta)}g(r,u(r))\,\mathrm{d}r \le \frac{\lambda_2||u||}{\Gamma(\beta)}\int_0^1 M_0k_2(r)\,\mathrm{d}r \le ||u|| = \frac{p_1}{2} < p_1,$$

then by (3.2) and (3.3), we get

$$\begin{aligned}||Tu|| &\le \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 k_1(s)f\left(s,\int_0^1 G_2(s,r)g(r,u(r))\,\mathrm{d}r\right)\mathrm{d}s\\
&\le \frac{\lambda_1(\alpha-1)}{\Gamma(\alpha)}\int_0^1 k_1(s)\int_0^1 G_2(s,r)g(r,u(r))\,\mathrm{d}r\,\mathrm{d}s\\
&\le \lambda_1\lambda_2||u||\frac{M_0(\alpha-1)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 k_1(s)\int_0^1 k_2(r)\,\mathrm{d}r\,\mathrm{d}s\\
&\le ||u||.\end{aligned}$$

Hence, if we choose $\Omega_1 = \{u \in E : ||u|| < \frac{p_1}{2}\}$, then

$$||Tu|| \le ||u||, \quad \text{for } u \in P \cap \partial\Omega_1. \tag{3.4}$$

From hypothesis (A_2), there exist positive constants μ_1, μ_2, C_1, and C_2 such that

$$f(t,u) \ge \mu_1 u - C_1, \quad g(t,u) \ge \mu_2 u - C_2, \quad \text{for } (t,u) \in [0,1]\times[0,+\infty), \tag{3.5}$$

where μ_1 and μ_2 satisfy

$$\mu_1\int_0^1 G_1(l,s)q_2(s)\,\mathrm{d}s \ge 1, \quad \frac{\mu_2(\beta-2)}{(\alpha-1)\Gamma(\beta)}\int_0^1 q_1(r)k_2(r)\,\mathrm{d}r \ge 2, \tag{3.6}$$

For $u \in P$ and $l \in (0, 1)$, then by (3.5) and (3.6), we have

$$\begin{aligned}Tu(l) &= \int_0^1 G_1(l,s)f\left(s,\int_0^1 G_2(s,r)g(r,u(r))\,\mathrm{d}r\right)\mathrm{d}s\\
&\ge \int_0^1 G_1(l,s)\left(\mu_1\int_0^1 G_2(s,r)g(r,u(r))\,\mathrm{d}r - C_1\right)\mathrm{d}s\\
&= \mu_1\int_0^1 G_1(l,s)\int_0^1 G_2(s,r)g(r,u(r))\,\mathrm{d}r\,\mathrm{d}s - C_1\int_0^1 G_1(l,s)\,\mathrm{d}s\\
&\ge \mu_1\int_0^1 G_1(l,s)\int_0^1 G_2(s,r)(\mu_2 u(r) - C_2)\,\mathrm{d}r\,\mathrm{d}s - C_1\int_0^1 G_1(l,s)\mathrm{d}s\\
&= \mu_1\mu_2\int_0^1 G_1(l,s)\int_0^1 G_2(s,r)u(r)\,\mathrm{d}r\,\mathrm{d}s - C(l)\\
&\ge \mu_1\mu_2\frac{\beta-2}{(\alpha-1)\Gamma(\beta)}||u||\int_0^1 G_1(l,s)q_2(s)\int_0^1 q_1(r)k_2(r)\,\mathrm{d}r\,\mathrm{d}s - C(l)\\
&\ge 2||u|| - C(l),\end{aligned}$$

where

$$\begin{aligned} C(l) &= \mu_1 C_2 \int_0^1 G_1(l,s) \int_0^1 G_2(s,r)\,\mathrm{d}r\,\mathrm{d}s + C_1 \int_0^1 G_1(l,s)\,\mathrm{d}s \\ &\le \frac{\mu_1 C_2 M_0}{\Gamma(\beta)} \int_0^1 G_1(l,s) \int_0^1 k_2(r)\,\mathrm{d}r\,\mathrm{d}s + C_1 \int_0^1 G_1(l,s)\,\mathrm{d}s \\ &= C_3, \end{aligned}$$

so,

$$Tu(l) \ge 2||u|| - C_3.$$

Thus, if we set $p_2 > \max\{p_1, C_3\}$ and $\Omega_2 = \{u \in E : ||u|| < p_2\}$, then

$$||Tu|| \ge ||u||, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.7}$$

Now, from (3.4), (3.7), and Lemma 2.7, we guarantee that T has a fix point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$, and clearly (u, v) is a positive solution of (1.1). The proof is completed.

Theorem 3.2 *Assume that hypotheses* (A_3) *and* (A_4) *hold. Then the boundary value problem* (1.1) *has at least one positive solution* (u, v).

Proof. By hypothesis (A_3), we see that there exists $p \in (0, 1)$ such that

$$f(t,u) \ge \eta_1 u, \quad g(t,u) \ge \eta_2 u, \quad \text{for } (t,u) \in [0,1] \times (0,p), \tag{3.8}$$

where η_1, $s_2 > 0$ and satisfy

$$\eta_1 \int_0^1 G_1(l,s) q_2(s)\,\mathrm{d}s \ge 1, \quad \frac{\eta_2(\beta-2)}{(\alpha-1)\Gamma(\beta)} \int_0^1 q_1(r) k_2(r)\mathrm{d}r \ge 1, \tag{3.9}$$

From $g(t, 0) \equiv 0$ and the continuity of g, then there exists $p_3 \in (0, 1)$ such that

$$g(t,u) \le \frac{p}{M_0 \int_0^1 k_2(r)\,\mathrm{d}r}, \quad \text{for } (t,u) \in [0,1] \times (0,p_3).$$

For $u \in P$ with $||u|| = p_3$, we have

$$\int_0^1 G_2(s,r) g(r,u)(r))\mathrm{d}r \le \int_0^1 G_2(s,r) \frac{p}{M_0 \int_0^1 k_2(r)\mathrm{d}r}\mathrm{d}r < p,$$

for $l \in (0, 1)$, by (3.8) and (3.9), we get

$$\begin{aligned} Tu(l) &= \int_0^1 G_1(l,s) f\left(s, \int_0^1 G_2(s,r) g(r,u(r))\,\mathrm{d}r\right)\mathrm{d}s \\ &\ge \eta_1 \int_0^1 G_1(l,s) \int_0^1 G_2(s,r) g(r,u(r))\mathrm{d}r\,\mathrm{d}s \\ &\ge \eta_1\eta_2 \int_0^1 G_1(l,s) \int_0^1 G_2(s,r) u(r)\mathrm{d}r\,\mathrm{d}s \\ &\ge \eta_1\eta_2 \frac{\beta-2}{(\alpha-1)\Gamma(\beta)} ||u|| \int_0^1 G_1(l,s) q_2(s) \int_0^1 q_1(r) k_2(r)\mathrm{d}r\,\mathrm{d}s \\ &\ge ||u||, \end{aligned}$$

Hence, if we choose $\Omega_3 = \{u \in E : ||u|| < p_3\}$, then

$$||Tu|| \leq ||u||, \quad \text{for } u \in P \cap \partial\Omega_3. \tag{3.10}$$

From hypothesis (A_4), there exist positive constants δ_1, δ_2, C_4, and C_5 such that

$$f(t, u) \geq \delta_1 u + C_4, \quad g(t, u) \geq \delta_2 u + C_5, \quad \text{for } (t, u) \in [0, 1] \times [0, +\infty), \tag{3.11}$$

where δ_1 and δ_2 satisfy

$$\frac{\delta_1(\alpha - 1)}{\Gamma(\alpha)} \int_0^1 k_1(s)\,\mathrm{d}s \leq \frac{1}{2}, \quad \frac{\delta_2 M_0}{\Gamma(\beta)} \int_0^1 k_2(r)\,\mathrm{d}r \leq \frac{1}{2}. \tag{3.12}$$

Then by (3.11) and (3.12), we have

$$\begin{aligned}
||Tu|| &\leq \frac{\alpha - 1}{\Gamma(\alpha)} = \int_0^1 k_1(s) f\left(s, \int_0^1 G_2(s, r) g(r, u(r))\mathrm{d}r\right)\mathrm{d}s \\
&\leq \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^1 k_1(s) \left(\delta_1 \int_0^1 G_2(s, r) g(r, u(r))\mathrm{d}r + C_4\right)\mathrm{d}s \\
&= \frac{\delta_1(\alpha - 1)}{\Gamma(\alpha)} \int_0^1 k_1(s) \int_0^1 G_2(s, r) g(r, u(r))\,\mathrm{d}r\,\mathrm{d}s + \frac{C_4(\alpha - 1)}{\Gamma(\alpha)} \int_0^1 k_1(s)\mathrm{d}s \\
&\leq \frac{\delta_1(\alpha - 1)}{\Gamma(\alpha)} \int_0^1 k_1(s) \int_0^1 \frac{M_0 k_2(r)}{\Gamma(\beta)} (\delta_2 u(r) + C_5)\mathrm{d}r\,\mathrm{d}s + \frac{C_4(\alpha - 1)}{\Gamma(\alpha)} \int_0^1 k_1(s)\mathrm{d}s \\
&\leq \frac{\delta_1 \delta_2 M_0(\alpha - 1)}{\Gamma(\alpha)\Gamma(\beta)} ||u|| \int_0^1 k_1(s) \int_0^1 k_2(r)\,\mathrm{d}r\,\mathrm{d}s - C_6 \\
&\leq \frac{1}{4}||u|| + C_6,
\end{aligned}$$

where

$$C_6 = \frac{\delta_1 C_5 M_0(\alpha - 1)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 k_1(s) \int_0^1 k_2(r)\mathrm{d}r\,\mathrm{d}s + \frac{C_4(\alpha - 1)}{\Gamma(\alpha)} \int_0^1 k_1(s)\,\mathrm{d}s.$$

Thus, if we set $p_4 > \max\{2p_3, 2C_6\}$ and $\Omega_4 = \{u \in E : ||u|| < p_4\}$, then

$$||Tu|| \leq ||u||, \quad \text{for } u \in P \cap \partial\Omega_4. \tag{3.13}$$

Now, from (3.10), (3.13), and Lemma 2.7, we guarantee that T has a fix point $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$, and clearly (u, v) is a positive solution of (1.1). The proof is completed.

Theorem 3.3 *Assume that hypotheses* (A_2), (A_3), *and* (A_5) *hold. Then the boundary value problem* (1.1) *has at least two positive solutions* (u_1, v_1) *and* (u_2, v_2).

Proof. Set $B_N = \{u \in E : ||u|| < N\}$. From (A_5), for $u \in P \cap \partial B_N$, then we have

$$\begin{aligned}
||Tu|| &= \max_{0\le t\le 1} |Tu(t)| \\
&= \max_{0\le t\le 1} \left| \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_2(s,r) g(r, u(r))\,\mathrm{d}r\right) \mathrm{d}s \right| \\
&\le n_1 \int_0^1 f\left(s, \int_0^1 G_2(s,r) g(r,u(r))\,\mathrm{d}r\right) \mathrm{d}s \\
&\le n_1 \int_0^1 f\left(s, \int_0^1 n_2 g(r,u(r))\,\mathrm{d}r\right) \mathrm{d}s \\
&< n_1 \int_0^1 f\left(s, \int_0^1 n_2 g(r,N))\,\mathrm{d}r\right) \mathrm{d}s < N.
\end{aligned}$$

Thus, $||Tu|| < ||u||$, $\forall\ u \in P \cap \partial B_N$. By (A_2) and (A_3), we can get

$$\begin{aligned}
&||Tu|| \ge ||u||, \quad \forall u \in P \cap \partial\Omega_2, \\
&||Tu|| \ge ||u||, \quad \forall u \in P \cap \partial\Omega_3.
\end{aligned}$$

So, we can choose p_2, p_3, and N such that $p_3 < N < p_2$ and satisfy the above three inequalities. By Lemma 2.8, we guarantee that T has two fix points $u_1 \in P \cap (\bar{\Omega}_2 \backslash B_N)$ and $u_2 \in P \cap (\overline{B_N} \backslash \Omega_3)$. Then the boundary value problem (1.1) at least two positive solutions (u_1, v_1) and (u_2, v_2). This completes the proof.

In fact, from (3.1), we can also obtain the following integral equation

$$v(t) = \int_0^1 G_2(t,s) g\left(s, \int_0^1 G_1(s,r) f(r, v(r))\,\mathrm{d}r\right) \mathrm{d}s, \quad t \in [0,1].$$

Define the cone $P' \subset E$ by

$$P' = \left\{ v \in E : v(t) \ge \frac{(\alpha-2)q_2(t)}{M_0} ||v||,\ t \in [0,1] \right\}.$$

We define an operator T': $P' \to E$ as follows

$$T'v(t) = \int_0^1 G_2(t,s) g\left(s, \int_0^1 G_1(s,r) f(r,v(r))\mathrm{d}r\right) \mathrm{d}s, \quad t \in [0,1].$$

For $v \in P'$, $0 \le t \le 1$, by Lemma 2.6,

$$\begin{aligned}
||T'v|| &= \max_{0\le t\le 1} |T'v(t)| \\
&= \max_{0\le t\le 1} \left| \int_0^1 G_2(t,s) g\left(s, \int_0^1 G_1(s,r) f(r, v(r))\,\mathrm{d}r\right) \mathrm{d}s \right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^1 M_0 k_2(s) g\left(s, \int_0^1 G_1(s,r) f(r,v(r))\mathrm{d}r\right) \mathrm{d}s,
\end{aligned}$$

$$
\begin{aligned}
T'v(t) &= \int_0^1 G_2(t,s)g\left(s, \int_0^1 G_1(s,r)f(r,v(r))\,\mathrm{d}r\right)\mathrm{d}s \\
&\geq \int_0^1 \frac{(\alpha-2)}{\Gamma(\alpha)} q_2(t)k_2(s)g\left(s, \int_0^1 G_1(s,r)f(r,v(r))\,\mathrm{d}r\right)\mathrm{d}s \\
&\geq \frac{(\alpha-2)q_2(t)}{M_0}||Tv||.
\end{aligned}
$$

Thus we have $T'(P') \subset P'$.

The operator T': $P' \to P'$ is continuous in view of continuity of $G(t, s)$, $f(t, u)$, and $g(t, u)$. For any bounded set M', $T'(M')$ is uniformly bounded and equicontinuous. This proof is similar to that of Lemma 3.1 in [21], so is omitted. By means of Arzela-Ascoli Theorem, T': $P' \to P'$ is completely continuous.

Remark **3.1** Theorems 3.1 and 3.2 also hold for the boundary value problem (1.1).

Proof. This proof is similar to that of Theorems 3.1 and 3.2, so is omitted.

Theorem 3.4 *If conditions* (A_5) *in the Theorem 3.3 is replaced by*

(A_5')*f*(t, u) *and* $g(t, u)$ *are two increasing functions with respect to u, and there exists* $N' > 0$ *such that*

$$
n_2 g\left(t, \int_0^1 n_1 f(r, N')\, dr\right) < N', \quad \text{for } t \in [0,1],
$$

where $n_1 = \max_{0 \le t,s \le 1} G_1(t, s)$, $n_2 = \max_{0 \le t,s \le 1} G_2(t, s)$.

Then the conclusion of Theorem 3.3 also holds.

Proof. This proof is similar to that of Theorem 3.3, so is omitted.

Remark **3.2** In this article, conditions $f(t, 0) \equiv 0$ and $g(t, 0) \equiv 0$ are too strong for the boundary value problem (1.1). So, we will give some new existence criteria for the boundary value problem (1.1) without conditions $f(t, 0) \equiv 0$ and $g(t, 0) \equiv 0$ in a new paper.

4 Examples

In this section, we will present examples to illustrate the main results.

Example 4.1 Consider the system of nonlinear differential equations

$$
\begin{cases}
-D_{0^+}^{\frac{5}{2}} u(t) = v(v+t-1),\ 0 < t < 1, \\
D_{0^+}^{\frac{7}{2}} v(t) = u(u+t-1), \quad 0 < t < 1, \\
u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = v'(1) = 0.
\end{cases}
\tag{4.1}
$$

Choose $f(t, v) = v(v + t - 1)$, $g(t, u) = u(u + t - 1)$. Then

$$
\lim_{u\to 0^+} \sup_{t\in[0,1]} \frac{f(t,u)}{u} = \lim_{u\to 0^+} \sup_{t\in[0,1]} \frac{g(t,u)}{u} = \lim_{u\to 0^+} \sup_{t\in[0,1]} (u+t-1) = 0,
$$

$$
\lim_{u\to +\infty} \inf_{t\in[0,1]} \frac{f(t,u)}{u} = \lim_{u\to +\infty} \inf_{t\in[0,1]} \frac{g(t,u)}{u} = \lim_{u\to +\infty} \inf_{t\in[0,1]} (u+t-1) = +\infty.
$$

So (A_1) and (A_2) hold. Thus, by Theorem 3.1, the boundary value problem (4.1) has a positive solution.

Example 4.2 Discuss the system of nonlinear differential equations

$$\begin{cases} -D_{0^+}^{\frac{7}{3}} u(t) = v^{\frac{1}{2}}(t+1), \ 0 < t < 1, \\ D_{0^+}^{\frac{7}{2}} v(t) = u^{\frac{1}{2}}(t+1), \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = v'(1) = 0. \end{cases} \tag{4.2}$$

Choose $f(t, v) = v^{\frac{1}{2}}(t+1)$, $g(t, u) = u^{\frac{1}{2}}(t+1)$. Then

$$\lim_{u \to 0^+} \inf_{t \in [0,1]} \frac{f(t,u)}{u} = \lim_{u \to 0^+} \inf_{t \in [0,1]} \frac{g(t,u)}{u} = \lim_{u \to 0^+} \inf_{t \in [0,1]} \frac{t+1}{u^{\frac{1}{2}}} = +\infty,$$

$$\lim_{u \to +\infty} \sup_{t \in [0,1]} \frac{f(t,u)}{u} = \lim_{u \to +\infty} \sup_{t \in [0,1]} \frac{g(t,u)}{u} = \lim_{u \to +\infty} \sup_{t \in [0,1]} \frac{t+1}{u^{\frac{1}{2}}} = 0.$$

So (A_3) and (A_4) hold. Thus, by Theorem 3.2, the boundary value problem (4.2) has a positive solution.

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Authors' contributions
The work presented here was carried out in collaboration between all authors.
YZ carried out the design of the study, the statistical analysis and drafted the manuscript.
SS and ZH conceived, instructed the design of the study and polished the manuscript.
WF participated discussion. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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