FINITE DIFFERENCE SCHEMES WITH MONOTONE OPERATORS

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Received 13 October 2003 and in revised form 10 December 2003

To the memory of my mother, Liliana

Several existence theorems are given for some second-order difference equations associated with maximal monotone operators in Hilbert spaces. Boundary conditions of monotone type are attached. The main tool used here is the theory of maximal monotone operators.

1. Introduction

In [1, 2], the authors proved the existence of the solution of the boundary value problem

$$p(t)u''(t)+r(t)u'(t)\in Au(t)+f(t), \quad \text{a.e. on } [0,T],\ T>0, \tag{1.1}$$

$$u'(0)\in\alpha\big(u(0)-a\big), \qquad u'(T)\in-\beta\big(u(T)-b\big), \tag{1.2}$$

where $A:D(A)\subseteq H\to H$, $\alpha:D(\alpha)\subseteq H\to H$, and $\beta:D(\beta)\subseteq H\to H$ are maximal monotone operators in the real Hilbert space H (satisfying some specific properties), a, b are given elements in the domain $D(A)$ of A, $f\in L^2(0,T;H)$, and $p,r:[0,T]\to\mathbb{R}$ are continuous functions, $p(t)\geq k>0$ for all $t\in[0,T]$.

Particular cases of this problem were considered before in [9, 10, 12, 15, 16]. If $p\equiv 1$, $r\equiv 0$, $f\equiv 0$, $T=\infty$, and the boundary conditions are $u(0)=a$ and $\sup\{\|u(t)\|, t\geq 0\}<\infty$ instead of (1.2), the solution $u(t)$ of (1.1), (1.2) defines a semigroup of nonlinear contractions $\{S_{1/2}(t),\ t\geq 0\}$ on the closure $\overline{D(A)}$ of $D(A)$ (see [9, 10]). This semigroup and its infinitesimal generator $A_{1/2}$ have some important properties (see [9, 10, 11, 12]).

A discretization of (1.1) is $p_i(u_{i+1}-2u_i+u_{i-1})+r_i(u_{i+1}-u_i)\in k_iAu_i+g_i$, $i=\overline{1,N}$, where N is a given natural number, $p_i,r_i,k_i>0$, $g_i\in H$. This leads to the finite difference scheme

$$(p_i+r_i)u_{i+1}-(2p_i+r_i)u_i+p_iu_{i-1}\in k_iAu_i+g_i, \quad i=\overline{1,N}, \tag{1.3}$$

$$u_1-u_0\in\alpha\big(u_0-a\big), \qquad u_{N+1}-u_N\in-\beta\big(u_{N+1}-b\big), \tag{1.4}$$

where $a,b\in H$ are given, $(p_i)_{i=\overline{1,N}}$, $(r_i)_{i=\overline{1,N}}$, and $(k_i)_{i=\overline{1,N}}$ are sequences of positive numbers, and $(g_i)_{i=\overline{1,N}}\in H^N$.

Advances in Difference Equations 2004:1 (2004) 11–22
2000 Mathematics Subject Classification: 39A12, 39A70, 47H05
URL: http://dx.doi.org/10.1155/S1687183904310046

In this paper, we study the existence and uniqueness of the solution of problem (1.3), (1.4) under various conditions on A, α, and β.

The case $p_i \equiv 1$, $r_i \equiv 0$, $g_i \equiv 0$ was discussed in [14] for the boundary conditions $u_0 = a$ and $u_{N+1} = b$. These boundary conditions can be seen as a particular case of (1.4) with $\alpha = \beta = \partial j$ (the subdifferential of j), where $j : H \to \overline{\mathbb{R}}$ is the lower-semicontinuous, convex, and proper function:

$$j(x) = \begin{cases} 0, & x = 0, \\ +\infty, & \text{otherwise}. \end{cases} \tag{1.5}$$

In [6, 8, 13, 14], one studies the existence, uniqueness, and asymptotic behavior of the solution of the difference equation

$$(p_i + r_i)u_{i+1} - (2p_i + r_i)u_i + p_i u_{i-1} \in k_i A u_i + g_i, \quad i \geq 1, \tag{1.6}$$

($p_i \equiv 1$, $r_i \equiv 0$ in [13, 14] and the general case in [6, 8]), subject to the boundary conditions

$$u_0 = a, \qquad \sup_{i \geq 0} ||u_i|| < \infty. \tag{1.7}$$

Here $\|\cdot\|$ is the norm of H. In [7], the author establishes the existence for problem (1.3), (1.4) under the hypothesis that A is also strongly monotone.

Other classes of difference or differential inclusions in abstract spaces are presented in [3, 4, 5].

In Section 2, we recall some notions and results that we need to show our main existence theorems. They are stated in Section 3 and represent the discrete version of some results obtained in [1, 2] for the continuous case.

2. Preliminary results

In this section, we recall some fundamental elements on nonlinear analysis we need in this paper.

If H is a real Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, then the operator $A \subseteq H \times H$ (with the domain $D(A)$ and the range $R(A)$) is called a *monotone operator* if $(x - x', y - y') \geq 0$ for all $x, x' \in D(A)$, $y \in Ax$, and $y' \in Ax'$. The monotone operator $A \subseteq H \times H$ is said to be *maximal monotone* if it is not properly enclosed in a monotone operator. A basic result of Minty (see [11, Theorem 1.2, page 9]) asserts that A is maximal monotone if and only if A is monotone and the range of $A + \lambda I$ is the whole space H for all $\lambda > 0$ (or equivalently, for only one $\lambda_0 > 0$). It is also known that a maximal monotone and coercive operator A is surjective, that is, its range $R(A)$ is H.

For all $x \in D(A)$, we denote by $A^0 x$ *the element of least norm* in Ax:

$$||A^0 x|| = \inf \{\|y\|, \ y \in Ax\}. \tag{2.1}$$

If A is maximal monotone and $\|A^0 x\| \to \infty$ as $\|x\| \to \infty$, then A is surjective.

The operator $A \subseteq H \times H$ (possibly multivalued) is said to be *one to one* if $(Ax_1) \cap (Ax_2) \neq \Phi$ (with $x_1, x_2 \in D(A)$) implies $x_1 = x_2$.

If A and B are maximal monotone in H and their domains satisfy the condition $(\operatorname{int} D(A)) \cap D(B) \neq \Phi$, then $A+B$ is maximal monotone (see [11, Theorem 1.7, page 46]). If $A : D(A) \subseteq H \to H$ is maximal monotone, then A is *demiclosed*, that is, from $[x_n, y_n] \in A$, $x_n \rightharpoonup x$ and $y_n \to y$, then $[x, y] \in A$. Here and everywhere below, we denote by "$\rightharpoonup$" the weak convergence and by "$\to$" the strong convergence in H.

For every maximal monotone operator A and the scalar $\lambda > 0$, we may consider the single-valued and everywhere-defined operators J_λ and A_λ, namely, $J_\lambda = (I+\lambda A)^{-1}$ and $A_\lambda = (I - J_\lambda)/\lambda$. They are called the *resolvent* and the *Yosida approximation* of A, respectively. Obviously, we have $J_\lambda x + \lambda A_\lambda x = x$ for all $x \in H$ and for all $\lambda > 0$. Properties of these operators can be found in, for example, [11, Proposition 1.1, page 42] or [11, Proposition 3.2, page 73].

Recall now another result concerning the sum of two maximal monotone operators (see [11, Theorem 3.6, page 82]).

THEOREM 2.1. *If $A : D(A) \subseteq H \to H$ and $B : D(B) \subseteq H \to H$ are maximal monotone operators in H such that $D(A) \cap D(B) \neq \Phi$ and $(y, A_\lambda x) \geq 0$ for all $[x, y] \in B$ and for all $\lambda > 0$, then $A + B$ is maximal monotone.*

We end this section with some remarks on problem (1.3), (1.4). Denoting

$$\theta_i = \frac{p_i}{p_i + r_i}, \quad c_i = \frac{k_i}{p_i + r_i}, \quad f_i = \frac{g_i}{p_i + r_i}, \quad i = \overline{1,N}, \tag{2.2}$$

problem (1.3), (1.4) becomes

$$\begin{gathered} u_{i+1} - (1+\theta_i)u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, \quad i = \overline{1,N}, \\ u_1 - u_0 \in \alpha(u_0 - a), \qquad u_{N+1} - u_N \in -\beta(u_{N+1} - b). \end{gathered} \tag{2.3}$$

If $p_i, r_i, k_i > 0$, $i = \overline{1,N}$, then $\theta_i \in (0,1)$ and $c_i > 0$ for all $i = \overline{1,N}$.

Let $(a_i)_{i=\overline{1,N}}$ be the finite sequence given by

$$a_0 = 1, \qquad a_i = \frac{1}{\theta_1 \cdots \theta_i}, \quad i = \overline{1,N}, \tag{2.4}$$

and let $\mathcal{L}$ be the product space $H^N = H \times \cdots \times H$ (N factors) endowed with the scalar product

$$\left\langle (u_i)_{i=\overline{1,N}}, (v_i)_{i=\overline{1,N}} \right\rangle = \sum_{i=1}^{N} a_i (u_i, v_i). \tag{2.5}$$

It is clear that H^N and $\mathcal{L}$ coincide as sets and their norms are equivalent. Observe that

$$a_i \theta_i = a_{i-1}, \quad i = \overline{1,N}. \tag{2.6}$$

Consider the operator B in $H^N \times H^N$:

$$\begin{gathered} B\left((u_i)_{i=\overline{1,N}}\right) = \left(-u_{i+1} + (1+\theta_i)u_i - \theta_i u_{i-1}\right)_{i=\overline{1,N}}, \\ D(B) = \left\{(u_i)_{i=\overline{1,N}} \in H^N,\ u_1 - u_0 \in \alpha(u_0 - a),\ u_{N+1} - u_N \in -\beta(u_{N+1} - b)\right\}. \end{gathered} \tag{2.7}$$

This operator is not necessarily monotone in H^N, but we have the following auxiliary result (see [7, Proposition 2.1]).

Proposition 2.2. *The operator B given above is maximal monotone in $\mathcal{L}$.*

Recall here an existence theorem from [7], which we use in the sequel.

Theorem 2.3. *Assume that A, α, and β are maximal monotone operators in H with $0 \in D(A) \cap D(\alpha) \cap D(\beta)$, A is also strongly monotone and*

$$(A_\lambda x - A_\lambda y, z) \geq 0 \tag{2.8}$$

for all $z \in \alpha(x-y)$ (with $x-y \in D(\alpha)$) and for all $z \in \beta(x-y)$ (with $x-y \in D(\beta)$). If $\theta_i \in (0,1)$, $c_i > 0$, $f_i \in H$, $i = \overline{1,N}$, and $a,b \in H$, then problem (2.3) has a unique solution $(u_i)_{i=\overline{1,N}} \in D(A)^N$.

3. Existence theorems

Let H be a real Hilbert space with the norm $\|\cdot\|$ and the scalar product $(\cdot,\cdot)$. Consider the maximal monotone operators $A : D(A) \subseteq H \to H$, $\alpha : D(\alpha) \subseteq H \to H$, and $\beta : D(\beta) \subseteq H \to H$ satisfying the properties

$$0 \in D(A) \cap D(\alpha) \cap D(\beta), \quad 0 \in \alpha(0) \cap \beta(0), \tag{3.1}$$

$$(A_\lambda x - A_\lambda y, z) \geq 0 \quad \forall z \in \alpha(x-y) \text{ with } x-y \in D(\alpha), \tag{3.2}$$

$$(A_\lambda x - A_\lambda y, z) \leq 0 \quad \forall z \in -\beta(x-y) \text{ with } x-y \in D(\beta). \tag{3.3}$$

Consider the difference inclusion (1.3), (1.4). As we have already discussed, problem (1.3), (1.4) has the equivalent form (2.3).

We first study the existence of the solution to problem (1.3), (1.4) in the case $a = b = 0$, supposing that

$$(A_\lambda x, z) \geq 0 \quad \forall z \in \alpha(x) \text{ with } x \in D(\alpha),\ z \in \beta(x) \text{ with } x \in D(\beta), \tag{3.4}$$

and

$$R(\alpha) \text{ is bounded}, \qquad ||\beta^0(x)|| \longrightarrow \infty \quad \text{as } \|x\| \longrightarrow \infty, \tag{3.5}$$

or

$$R(\beta) \text{ is bounded}, \qquad ||\alpha^0(x)|| \longrightarrow \infty \quad \text{as } \|x\| \longrightarrow \infty. \tag{3.6}$$

Theorem 3.1. *Let A, α, and β be maximal monotone operators in the real Hilbert space H such that (3.1), (3.4), and (3.5) or (3.6) hold. If $p_i, r_i, k_i > 0$, $i = \overline{1,N}$, and $(g_i)_{i=\overline{1,N}} \in H^N$, then the boundary value problem*

$$\begin{gathered} (p_i + r_i)u_{i+1} - (2p_i + r_i)u_i + p_i u_{i-1} \in k_i A u_i + g_i, \quad i = \overline{1,N}, \\ u_1 - u_0 \in \alpha(u_0), \qquad u_{N+1} - u_N \in -\beta(u_{N+1}), \end{gathered} \tag{3.7}$$

has at least one solution $(u_i)_{i=\overline{1,N}} \in D(A)^N$. The solution is unique up to an additive constant. If A or α is one to one, then the solution is unique. If A is, in addition, strongly monotone, then again uniqueness is obtained.

Proof. We use the form (2.3) of the problem (3.7), where $a = b = 0$. By Proposition 2.2, we know that the operator

$$B\Big((u_i)_{i=\overline{1,N}}\Big) = \big(-u_{i+1} + (1+\theta_i)u_i - \theta_i u_{i-1}\big)_{i=\overline{1,N}},$$
$$D(B) = \Big\{(u_i)_{i=\overline{1,N}} \in H^N,\ u_1 - u_0 \in \alpha(u_0),\ u_{N+1} - u_N \in -\beta(u_{N+1})\Big\} \tag{3.8}$$

is maximal monotone in $\mathscr{L}$. Denote by $|\cdot|$ the norm in $\mathscr{L}$. We show that

$$\Big|B\Big((u_i)_{i=\overline{1,N}}\Big)\Big| \longrightarrow \infty \quad \text{as } \Big|(u_i)_{i=\overline{1,N}}\Big| \longrightarrow \infty. \tag{3.9}$$

Suppose by contradiction that $(u_i^n)_{i=\overline{1,N}} \in D(B)$ such that $|(u_i^n)_{i=\overline{1,N}}| \to \infty$ as $n \to \infty$ and $|B((u_i^n)_{i=\overline{1,N}})| \le C_1$. If $(a_i)_{i=\overline{1,N}}$ is the sequence given in (2.4), this means that

$$\sum_{i=1}^{N} a_i \|u_i^n\|^2 \longrightarrow \infty, \qquad \sum_{i=1}^{N} a_i \|u_{i+1}^n - u_i^n - \theta_i(u_i^n - u_{i-1}^n)\|^2 \le C_1. \tag{3.10}$$

Assume that (3.5) holds. By the boundary conditions in (3.7), we obtain that $u_1^n - u_0^n$ is bounded, say $\|u_1^n - u_0^n\| \le C_2$, for all $n \in \mathbb{N}$ and

$$\|u_{N+1}^n - u_N^n\| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty \text{ if } \|u_{N+1}^n\| \longrightarrow \infty. \tag{3.11}$$

The equality $a_i(u_{i+1}^n - u_i^n) = u_1^n - u_0^n + \sum_{k=1}^{i}[a_k(u_{k+1}^n - u_k^n) - a_{k-1}(u_k^n - u_{k-1}^n)]$ implies that $\|a_i(u_{i+1}^n - u_i^n)\| \le C_2 + C_3|B((u_i)_{i=\overline{1,N}})|$ and in view of (3.10), we get $\|a_i(u_{i+1}^n - u_i^n)\| \le C_4$, $i = \overline{1,N}$, $n \in \mathbb{N}$. In particular, $\|a_N(u_{N+1}^n - u_N^n)\| \le C_4$ for all $n \in \mathbb{N}$ and from (3.11), we infer that $\|u_{N+1}^n\| \le C_5$ for all $n \in \mathbb{N}$.

Using the boundedness of u_{N+1}^n and $a_k(u_{k+1}^n - u_k^n)$ and the identity

$$u_i^n = u_{N+1}^n - \sum_{k=i}^{N}(u_{k+1}^n - u_k^n), \quad i = \overline{1,N}, \tag{3.12}$$

one arrives at $\|u_i^n\| \le C_6$, hence $\sum_{i=1}^{N} a_i\|u_i^n\|^2 \le C_7$ for all $n \in \mathbb{N}$. But this is in contradiction with (3.10) and therefore (3.9) is true. This shows that B is coercive.

Next we show that

$$\Big\langle B\Big((u_i)_{i=\overline{1,N}}\Big), (A_\lambda u_i)_{i=\overline{1,N}}\Big\rangle \ge 0 \quad \forall (u_i)_{i=\overline{1,N}} \in D(B),\ \lambda > 0. \tag{3.13}$$

Indeed,

$$\begin{aligned}\left\langle B\left(\left(u_i\right)_{i=\overline{1,N}}\right), \left(A_\lambda u_i\right)_{i=\overline{1,N}}\right\rangle &= -\sum_{i=1}^{N}\left[a_i\left(u_{i+1}-u_i, A_\lambda u_i\right) - a_{i-1}\left(u_i - u_{i-1}, A_\lambda u_{i-1}\right)\right] \\ &\quad + \sum_{i=1}^{N} a_{i-1}\left(u_i - u_{i-1}, A_\lambda u_i - A_\lambda u_{i-1}\right) \\ &\geq -a_N\left(u_{N+1}-u_N, A_\lambda u_N\right) + \left(u_1 - u_0, A_\lambda u_0\right).\end{aligned} \tag{3.14}$$

Hypothesis (3.4) for $x = u_0$ and $z = u_1 - u_0$ gives us $(u_1 - u_0, A_\lambda u_0) \geq 0$, while (3.4) for $x = u_{N+1}$ and $z = -u_{N+1} + u_N$ implies that $-(u_{N+1} - u_N, A_\lambda u_N) = (u_{N+1} - u_N, A_\lambda u_{N+1} - A_\lambda u_N) - (u_{N+1} - u_N, A_\lambda u_{N+1}) \geq 0$. Thus, by (3.14), inequality (3.13) follows.

Let $\mathcal{A} : D(A)^N \to H^N$ be the operator

$$\mathcal{A}\left(\left(u_i\right)_{i=\overline{1,N}}\right) = \left(c_1 v_1, \ldots, c_N v_N\right), \quad v_i \in A u_i, \quad u_i \in D(A),\ i = \overline{1,N}. \tag{3.15}$$

Since $(0,\ldots,0) \in D(\mathcal{A}) \cap D(B)$ and (3.13) takes place, we deduce with the aid of Theorem 2.1 and Proposition 2.2 the maximal monotonicity of $B + \mathcal{A}$ in $\mathcal{L}$. Next, we can easily show that $\langle B((u_i)_{i=\overline{1,N}}), \mathcal{A}((u_i)_{i=\overline{1,N}})\rangle \geq 0$, so $|(B+\mathcal{A})(u_i)_{i=\overline{1,N}}| \geq |B((u_i)_{i=\overline{1,N}})|$, and from (3.9), one obtains the coercivity of $B + \mathcal{A}$. This shows that $B + \mathcal{A}$ is surjective, that is, for all $(f_i)_{i=\overline{1,N}} \in H^N$, there exists $(u_i)_{i=\overline{1,N}} \in D(\mathcal{A}) \cap D(B)$ such that $(B+\mathcal{A})((u_i)_{i=\overline{1,N}}) = (-f_i)_{i=\overline{1,N}}$. But this is the abstract form of (3.7). Thus the existence is proved.

We show now that the difference of the two solutions $(u_i)_{i=\overline{1,N}}$ and $(v_i)_{i=\overline{1,N}}$ of (3.7) is a constant. Put $w_i = u_i - v_i$, $i = \overline{0,N+1}$. Subtracting the corresponding equations of (2.3) for u_i and v_i, multiplying by $a_i w_i$, and summing from $i = 1$ to $i = N$, one arrives with the aid of the monotonicity of A at

$$\sum_{i=1}^{N}\left[a_i\left(w_{i+1} - w_i, w_i\right) - a_i\theta_i\left(w_i - w_{i-1}, w_i\right)\right] \geq 0 \tag{3.16}$$

or, in view of (2.6), at

$$\sum_{i=1}^{N}\left[a_i\left(w_{i+1} - w_i, w_i\right) - a_{i-1}\left(w_i - w_{i-1}, w_{i-1}\right)\right] \geq \sum_{i=1}^{N} a_{i-1}\left\|w_i - w_{i-1}\right\|^2. \tag{3.17}$$

By the boundary conditions in (2.3), we have

$$\sum_{i=1}^{N} a_{i-1}\left\|w_i - w_{i-1}\right\|^2 \leq a_N\left(w_{N+1} - w_N, w_N\right) - \left(w_1 - w_0, w_0\right) \leq 0, \tag{3.18}$$

so $w_0 = w_1 = \cdots = w_N$. This implies that $u_i = v_i + C$, $i = \overline{0,N}$, where $C \in H$ is a constant. If A or α is one to one, then the uniqueness follows easily. If A is maximal monotone and strongly monotone, then we obtain

$$\sum_{i=1}^{N} a_i\left\|w_i\right\|^2 + \sum_{i=1}^{N} a_{i-1}\left\|w_i - w_{i-1}\right\|^2 \leq 0, \tag{3.19}$$

so the solution is unique and the proof is complete. □

Now we replace (3.4) by (3.2) and (3.3) and remove (3.5) and (3.6). Adding the boundedness of the domain of β, we can state the following result.

Theorem 3.2. *Let A, α, and β be maximal monotone operators in H such that $D(\beta)$ is bounded and (3.1), (3.2), (3.3) hold. If $a,b \in H$, $(g_i)_{i=\overline{1,N}} \in H^N$, and $p_i,r_i,k_i > 0$, $i = \overline{1,N}$, then problem (1.3), (1.4) admits at least one solution $(u_i)_{i=\overline{1,N}} \in D(A)^N$ and the difference between two solutions is constant. If A or α is one to one, then the solution is unique. If A is also strongly monotone, then again uniqueness is obtained.*

Proof. We use again the equivalent form (2.3) of problem (1.3), (1.4) and the maximal monotone operator $\mathcal{A}$ given by (3.15). If A_λ and $\mathcal{A}_\lambda$ are the Yosida approximations of A and $\mathcal{A}$, respectively, then $\mathcal{A}_\lambda((u_i)_{i=\overline{1,N}}) = (c_1A_\lambda u_1,\dots,c_NA_\lambda u_N)$ for all $(u_i)_{i=\overline{1,N}} \in H^N$. By Proposition 2.2, $B+\mathcal{A}_\lambda$ is maximal monotone in $\mathcal{L}$, therefore, $R(B+\mathcal{A}_\lambda+\lambda I) = \mathcal{L}$, that is, for all $(f_i)_{i=\overline{1,N}} \in H^N$, for all $\lambda > 0$, the problem

$$\begin{aligned} &u_{i+1}^\lambda - (1+\theta_i)u_i^\lambda + \theta_i u_{i-1}^\lambda = c_iA_\lambda u_i^\lambda + \lambda u_i^\lambda + f_i, \quad i = \overline{1,N}, \\ &u_1^\lambda - u_0^\lambda \in \alpha(u_0^\lambda - a), \qquad u_{N+1}^\lambda - u_N^\lambda \in -\beta(u_{N+1}^\lambda - b), \end{aligned} \tag{3.20}$$

has a unique solution $(u_i^\lambda)_{i=\overline{1,N}} \in H^N$. (The uniqueness follows from Theorem 2.3 for the strongly monotone operator $\mathcal{A}_\lambda + \lambda I$.)

We first prove that $(u_i^\lambda)_{i=\overline{1,N}}$ is bounded in H with respect to λ. To do this, we multiply (3.20) by $a_iu_i^\lambda$ and sum up from $i=1$ to $i=N$. Without any loss of generality, suppose that $0 \in A0$. If not, we put $\widetilde{A} = A + A^0 0$ and $\widetilde{f}_i = f_i - c_iA^0 0$ instead of A and f_i, respectively, where A^0x denotes the element of least norm in Ax. Since A_λ is monotone, $A_\lambda 0 = 0$, and $a_i\theta_i = a_{i-1}$, we derive

$$\begin{aligned} &\sum_{i=1}^N a_i(u_{i+1}^\lambda - u_i^\lambda, u_i^\lambda) - \sum_{i=1}^N a_{i-1}(u_i^\lambda - u_{i-1}^\lambda, u_{i-1}^\lambda) \\ &\qquad \geq \sum_{i=1}^N a_{i-1}\|u_i^\lambda - u_{i-1}^\lambda\|^2 + \lambda\sum_{i=1}^N a_i\|u_i^\lambda\|^2 + \sum_{i=1}^N a_i(f_i,u_i^\lambda), \end{aligned} \tag{3.21}$$

hence

$$\sum_{i=1}^N a_{i-1}\|u_i^\lambda - u_{i-1}^\lambda\|^2 \leq a_N(u_{N+1}^\lambda - u_N^\lambda, u_N^\lambda) - (u_1^\lambda - u_0^\lambda, u_0^\lambda) - \sum_{i=1}^N a_i(f_i,u_i^\lambda). \tag{3.22}$$

Since $u_1^\lambda - u_0^\lambda \in \alpha(u_0^\lambda - a)$, $0 \in \alpha(0)$, and α is monotone, we infer

$$-(u_1^\lambda - u_0^\lambda, u_0^\lambda) \leq -(u_1^\lambda - u_0^\lambda, a) \leq \|a\| \cdot \|u_1^\lambda - u_0^\lambda\| \tag{3.23}$$

and, similarly,

$$(u_{N+1}^\lambda - u_N^\lambda, u_N^\lambda) \leq -\|u_{N+1}^\lambda - u_N^\lambda\|^2 + (u_{N+1}^\lambda - u_N^\lambda, u_{N+1}^\lambda - b) + (u_{N+1}^\lambda - u_N^\lambda, b), \tag{3.24}$$

so

$$(u_{N+1}^\lambda - u_N^\lambda, u_N^\lambda) \leq \|b\| \cdot \|u_{N+1}^\lambda - u_N^\lambda\|. \tag{3.25}$$

Now (3.22), (3.23), and (3.25) yield

$$\sum_{i=1}^{N} a_{i-1}\left|\left|u_i^\lambda - u_{i-1}^\lambda\right|\right|^2 \leq a_N\|b\| \cdot \left|\left|u_{N+1}^\lambda - u_N^\lambda\right|\right| + \|a\| \cdot \left|\left|u_1^\lambda - u_0^\lambda\right|\right| \\ + \left(\sum_{i=1}^{N} a_i\left|\left|f_i\right|\right|^2\right)^{1/2}\left(\sum_{i=1}^{N} a_i\left|\left|u_i^\lambda\right|\right|^2\right)^{1/2}. \tag{3.26}$$

The hypothesis that $D(\beta)$ is bounded and the boundary conditions imply the boundedness of u_{N+1}^λ with respect to λ. Using this, together with the estimates

$$\left|\left|u_k^\lambda\right|\right| \leq \left(\sum_{i=1}^{N} a_i\left|\left|u_i^\lambda\right|\right|^2\right)^{1/2}, \\ \left|\left|u_k^\lambda - u_{k-1}^\lambda\right|\right| \leq \left(\sum_{i=1}^{N} a_{i-1}\left|\left|u_i^\lambda - u_{i-1}^\lambda\right|\right|^2\right)^{1/2} \tag{3.27}$$

for $k = \overline{1,N}$ in (3.26), one deduces

$$\sum_{i=1}^{N} a_{i-1}\left|\left|u_i^\lambda - u_{i-1}^\lambda\right|\right|^2 \leq C_1 + C_2\left(\sum_{i=1}^{N} a_{i-1}\left|\left|u_i^\lambda - u_{i-1}^\lambda\right|\right|^2\right)^{1/2} + C_3\left(\sum_{i=1}^{N} a_i\left|\left|u_i^\lambda\right|\right|^2\right)^{1/2}, \tag{3.28}$$

with $C_1, C_2, C_3 > 0$ independent of λ.

For each $i = \overline{1,N}$, we have $u_i^\lambda = u_0^\lambda + \sum_{k=1}^{i}(u_k^\lambda - u_{k-1}^\lambda)$, so

$$\left|\left|u_i^\lambda\right|\right| \leq \left|\left|u_0^\lambda\right|\right| + \left(\sum_{k=1}^{N} \frac{1}{a_{k-1}}\right)\left(\sum_{k=1}^{N} a_{k-1}\left|\left|u_k^\lambda - u_{k-1}^\lambda\right|\right|^2\right)^{1/2}, \quad i = \overline{1,N}. \tag{3.29}$$

From the boundary conditions, it follows that

$$\left|\left|u_0^\lambda\right|\right|^2 \leq \|a\| \cdot \left|\left|u_0^\lambda\right|\right| + \|a\|\left(\sum_{i=1}^{N} a_{i-1}\left|\left|u_i^\lambda - u_{i-1}^\lambda\right|\right|^2\right)^{1/2}, \tag{3.30}$$

and thus

$$\left|\left|u_0^\lambda\right|\right| \leq \|a\| + \|a\|^{1/2}\left(\sum_{i=1}^{N} a_{i-1}\left|\left|u_i^\lambda - u_{i-1}^\lambda\right|\right|^2\right)^{1/4}. \tag{3.31}$$

Inequalities (3.29) and (3.31) imply that

$$\left|\left|u_i^\lambda\right|\right| \leq C_4 + C_5\left(\sum_{i=1}^{N} a_{i-1}\left|\left|u_i^\lambda - u_{i-1}^\lambda\right|\right|^2\right)^{1/2}, \quad i = \overline{1,N}, \tag{3.32}$$

which, together with (3.28), leads to the boundedness

$$\sum_{i=1}^{N} a_{i-1}\left|\left|u_i^\lambda - u_{i-1}^\lambda\right|\right|^2 \leq C_6 \quad \forall \lambda > 0. \tag{3.33}$$

Now (3.31) and (3.32) show that $\|u_i^\lambda\| \le C_7$, $i = \overline{0,N}$ and $\lambda > 0$, and therefore,

$$\sum_{i=1}^{N} a_i \|u_i^\lambda\|^2 \le C_8 \quad \forall \lambda > 0. \tag{3.34}$$

All the constants $C_j > 0$ ($j = 1, \dots, 13$) here and below are independent of λ.

Multiplying (3.20) by $a_i A_\lambda u_i^\lambda$ and summing from 1 to N, we get via (2.6)

$$\begin{aligned}
&\sum_{i=1}^{N} \left[a_i (u_{i+1}^\lambda - u_i^\lambda, A_\lambda u_i^\lambda) - a_{i-1}(u_i^\lambda - u_{i-1}^\lambda, A_\lambda u_{i-1}^\lambda)\right] - \sum_{i=1}^{N} a_{i-1}(u_i^\lambda - u_{i-1}^\lambda, A_\lambda u_i^\lambda - A_\lambda u_{i-1}^\lambda) \\
&\qquad = \sum_{i=1}^{N} a_i c_i \|A_\lambda u_i^\lambda\|^2 + \lambda \sum_{i=1}^{N} a_i (u_i^\lambda, A_\lambda u_i^\lambda) + \sum_{i=1}^{N} a_i (f_i, A_\lambda u_i^\lambda).
\end{aligned} \tag{3.35}$$

Let $c = \inf\{c_i,\ i = \overline{1,N}\}$. Then

$$c \sum_{i=1}^{N} a_i \|A_\lambda u_i^\lambda\|^2 \le a_N (u_{N+1}^\lambda - u_N^\lambda, A_\lambda u_N^\lambda) - (u_1^\lambda - u_0^\lambda, A_\lambda u_0^\lambda) - \sum_{i=1}^{N} a_i (f_i, A_\lambda u_i^\lambda). \tag{3.36}$$

We observe that assumptions (3.2) and (3.3) and the boundary conditions yield

$$\begin{aligned}
(u_{N+1}^\lambda - u_N^\lambda, A_\lambda u_N^\lambda) &\le \|A^0 b\| \cdot \|u_{N+1}^\lambda - u_N^\lambda\|, \\
-(u_1^\lambda - u_0^\lambda, A_\lambda u_0^\lambda) &\le \|A^0 a\| \cdot \|u_1^\lambda - u_0^\lambda\|,
\end{aligned} \tag{3.37}$$

therefore, (3.36) implies

$$\begin{aligned}
c \sum_{i=1}^{N} a_i \|A_\lambda u_i^\lambda\|^2 &\le a_N \|A^0 b\| \cdot \|u_{N+1}^\lambda - u_N^\lambda\| + \|A^0 a\| \cdot \|u_1^\lambda - u_0^\lambda\| \\
&\quad + \left(\sum_{i=1}^{N} a_i \|f_i\|^2\right)^{1/2} \left(\sum_{i=1}^{N} a_i \|A_\lambda u_i^\lambda\|^2\right)^{1/2}.
\end{aligned} \tag{3.38}$$

In view of (3.29), (3.31), and the boundedness of u_{N+1}^λ, this means that

$$\sum_{i=1}^{N} a_i \|A_\lambda u_i^\lambda\|^2 \le C_9 + C_{10} \left(\sum_{i=1}^{N} a_i \|A_\lambda u_i^\lambda\|^2\right)^{1/2} + C_{11} \left(\sum_{i=1}^{N} a_{i-1} \|u_i^\lambda - u_{i-1}^\lambda\|^2\right)^{1/2}. \tag{3.39}$$

According to (3.33), this leads to

$$\sum_{i=1}^{N} a_i \|A_\lambda u_i^\lambda\|^2 \le C_{12}. \tag{3.40}$$

We prove now that $u_i^\lambda - u_{i-1}^\lambda$ is a Cauchy sequence with respect to λ. Subtracting (3.20) with ν in place of λ from the original equation (3.20), multiplying the result by $a_i(u_i^\lambda - u_i^\nu)$, and summing up from $i=1$ to $i=N$, we find, with the aid of the equality $x = J_\lambda x + \lambda A_\lambda x$,

$$\begin{aligned}
&\sum_{i=1}^{N} a_i\left(u_{i+1}^\lambda - u_{i+1}^\nu - u_i^\lambda + u_i^\nu, u_i^\lambda - u_i^\nu\right) - \sum_{i=1}^{N} a_{i-1}\left(u_i^\lambda - u_i^\nu - u_{i-1}^\lambda + u_{i-1}^\nu, u_{i-1}^\lambda - u_{i-1}^\nu\right)\\
&\qquad = \sum_{i=1}^{N} a_{i-1}\left\|u_i^\lambda - u_i^\nu - u_{i-1}^\lambda + u_{i-1}^\nu\right\|^2 + \sum_{i=1}^{N} a_i c_i\left(A_\lambda u_i^\lambda - A_\nu u_i^\nu, J_\lambda u_i^\lambda - J_\nu u_i^\nu\right)\\
&\qquad\quad + \sum_{i=1}^{N} a_i c_i\left(A_\lambda u_i^\lambda - A_\nu u_i^\nu, \lambda A_\lambda u_i^\lambda - \nu A_\nu u_i^\nu\right) + \sum_{i=1}^{N} a_i\left(\lambda u_i^\lambda - \nu u_i^\nu, u_i^\lambda - u_i^\nu\right),
\end{aligned} \tag{3.41}$$

hence

$$\begin{aligned}
&\sum_{i=1}^{N} a_{i-1}\left\|u_i^\lambda - u_i^\nu - u_{i-1}^\lambda + u_{i-1}^\nu\right\|^2\\
&\qquad \le a_N\left(u_{N+1}^\lambda - u_{N+1}^\nu - u_N^\lambda + u_N^\nu, u_N^\lambda - u_N^\nu\right) - \left(u_1^\lambda - u_1^\nu - u_0^\lambda + u_0^\nu, u_0^\lambda - u_0^\nu\right)\\
&\qquad\quad + (\lambda+\nu)\sum_{i=1}^{N} a_i c_i\left(A_\lambda u_i^\lambda, A_\nu u_i^\nu\right) + (\lambda+\nu)\sum_{i=1}^{N} a_i\left(u_i^\lambda, u_i^\nu\right).
\end{aligned} \tag{3.42}$$

The boundary conditions in (3.20) and the upper bounds (3.34) and (3.40) imply

$$\sum_{i=1}^{N} a_{i-1}\left\|u_i^\lambda - u_i^\nu - u_{i-1}^\lambda + u_{i-1}^\nu\right\|^2 \le C_{13}(\lambda+\nu), \tag{3.43}$$

and therefore, $u_i^\lambda - u_{i-1}^\lambda$ is a strongly convergent sequence in H.

Let $u_i^\lambda \rightharpoonup u_i$, $i=\overline{1,N}$ (on a subsequence denoted again by λ). Then $u_i^\lambda - u_{i-1}^\lambda \to u_i - u_{i-1}$, so $B((u_i^\lambda)_{i=\overline{1,N}}) \to B((u_i)_{i=\overline{1,N}})$. In addition, we have $J_\lambda u_i^\lambda (= u_i^\lambda - \lambda A_\lambda u_i^\lambda) \rightharpoonup u_i$ as $\lambda \to 0$, $i=\overline{1,N}$.

Since A is demiclosed, this enables us to pass to the limit as $\lambda \to 0$ in (3.20) written under the form

$$-B\Big(\left(u_i^\lambda\right)_{i=\overline{1,N}}\Big) - \lambda\left(u_i^\lambda\right)_{i=\overline{1,N}} - \left(f_i\right)_{i=\overline{1,N}} \in \mathcal{A}\Big(\left(J_\lambda u_i^\lambda\right)_{i=\overline{1,N}}\Big), \tag{3.44}$$

and one obtains that $(u_i)_{i=\overline{1,N}}$ verifies problem (2.3). The uniqueness follows like in Theorem 3.1. The proof is complete. □

We now replace the boundedness of $D(\beta)$ by the conditions

$$\inf\left\{-\frac{(y,x)}{\|x\|},\ y \in -\beta(x)\right\} \longrightarrow \infty \quad \text{as } \|x\| \longrightarrow \infty, \tag{3.45}$$

$$\inf\left\{\frac{(y,x)}{\|x\|},\ y \in \alpha(x)\right\} \longrightarrow \infty \quad \text{as } \|x\| \longrightarrow \infty. \tag{3.46}$$

We get the following result.

Theorem 3.3. *If A, α, and β are maximal monotone operators in H satisfying hypotheses (3.1), (3.2), (3.3), (3.45), and (3.46), then for given $a,b \in H$, $g_i \in H$, and $p_i, r_i, k_i > 0, i = \overline{1,N}$, problem (1.3), (1.4) has at least one solution $(u_i)_{i=\overline{1,N}} \in D(A)^N$. The solution is unique up to an additive constant.*

Proof. One uses again the form (2.3) of problem (1.3), (1.4) and approximates it by (3.20). In order to prove the boundedness of u_0^λ and u_{N+1}^λ with respect to λ, consider the auxiliary problem

$$\begin{gathered} v_{i+1}^\lambda - (1+\theta_i)v_i^\lambda + \theta_i v_{i-1}^\lambda = c_i A_\lambda v_i^\lambda + \lambda v_i^\lambda + f_i, \quad i = \overline{1,N}, \\ v_0^\lambda = a, \qquad v_{N+1}^\lambda = b. \end{gathered} \tag{3.47}$$

This problem is a particular case of problem (2.3), where the operator $A_\lambda + \lambda I$ is maximal monotone and strongly monotone and α, β are the subdifferential ∂j of the lower-semicontinuous, convex, and proper function $j : H \to \overline{\mathbb{R}}$ as presented in (1.5). Then Theorem 2.3 implies the existence of a unique solution $(v_i^\lambda)_{i=\overline{1,N}}$ of (3.47).

A multiplication of the difference between (3.20) and (3.47) by $a_i(u_i^\lambda - v_i^\lambda)$ followed by a summation with respect to i leads to

$$\begin{aligned} &\sum_{i=1}^N a_{i-1}\left\|u_i^\lambda - v_i^\lambda - u_{i-1}^\lambda + v_{i-1}^\lambda\right\|^2 \\ &\qquad \le a_N\left(u_{N+1}^\lambda - b - u_N^\lambda + v_N^\lambda, u_N^\lambda - v_N^\lambda\right) - \left(u_1^\lambda - v_1^\lambda - u_0^\lambda + a, u_0^\lambda - a\right) \end{aligned} \tag{3.48}$$

or, equivalently,

$$\begin{aligned} &a_N\left\|u_{N+1}^\lambda - b - u_N^\lambda + v_N^\lambda\right\|^2 + \sum_{i=1}^N a_{i-1}\left\|u_i^\lambda - v_i^\lambda - u_{i-1}^\lambda + v_{i-1}^\lambda\right\|^2 \\ &\qquad \le a_N\left(u_{N+1}^\lambda - b - u_N^\lambda + v_N^\lambda, u_{N+1}^\lambda - b\right) - \left(u_1^\lambda - v_1^\lambda - u_0^\lambda + a, u_0^\lambda - a\right). \end{aligned} \tag{3.49}$$

From this inequality and the boundary conditions in (3.20), we can easily get

$$\begin{aligned} 0 &\le \left(u_1^\lambda - u_0^\lambda, u_0^\lambda - a\right) - a_N\left(u_{N+1}^\lambda - u_N^\lambda, u_{N+1}^\lambda - b\right) \\ &\le \left(v_1^\lambda - a, u_0^\lambda - a\right) + a_N\left(v_N^\lambda - b, u_{N+1}^\lambda - b\right). \end{aligned} \tag{3.50}$$

Since problem (3.47) is a particular case of (3.20), where $D(\beta)$ is bounded, we can use the proof of Theorem 3.2 to deduce the boundedness of v_1^λ and v_N^λ in H with respect to λ. Hence, there exist two constants C_1 and C_2 independent of λ such that

$$0 \le \left(u_1^\lambda - u_0^\lambda, u_0^\lambda - a\right) - a_N\left(u_{N+1}^\lambda - u_N^\lambda, u_{N+1}^\lambda - b\right) \le C_1\left\|u_0^\lambda - a\right\| + C_2\left\|u_{N+1}^\lambda - b\right\|. \tag{3.51}$$

From (3.45), (3.46), and (3.51), we obtain that u_0^λ and u_{N+1}^λ are bounded. Indeed, if not, say $\|u_0^\lambda - a\| \to \infty$ on a subsequence denoted again by λ. By (3.46), it follows that

$$\frac{\left(u_1^\lambda - u_0^\lambda, u_0^\lambda - a\right)}{\left\|u_0^\lambda - a\right\|} \longrightarrow \infty \quad \text{as } \lambda \longrightarrow 0. \tag{3.52}$$

If $R_\lambda = \|u_{N+1}^\lambda - b\|/\|u_0^\lambda - a\|$ is bounded, then we get a contradiction in (3.51). If R_λ is unbounded, then dividing (3.51) by $\|u_{N+1}^\lambda - b\|$ and using condition (3.45) for $x = u_{N+1}^\lambda - b$ and $y = u_{N+1}^\lambda - u_N^\lambda$, we arrive again at a contradiction. This demonstrates the boundedness of u_0^λ and u_{N+1}^λ.

From now on, the proof follows that of Theorem 3.2. □

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