BOUNDEDNESS IN FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

ELVAN AKIN-BOHNER AND YOUSSEF N. RAFFOUL

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Using nonnegative definite Lyapunov functionals, we prove general theorems for the boundedness of all solutions of a functional dynamic equation on time scales. We apply our obtained results to linear and nonlinear Volterra integro-dynamic equations on time scales by displaying suitable Lyapunov functionals.

1. Introduction

In this paper, we consider the boundedness of solutions of equations of the form

$$x^{\Delta}(t) = G(t,x(s);\ 0 \le s \le t) := G(t,x(\cdot)) \tag{1.1}$$

on a time scale $\mathbb{T}$ (a nonempty closed subset of real numbers), where $x \in \mathbb{R}^n$ and $G : [0,\infty) \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a given nonlinear continuous function in t and x. For a vector $x \in \mathbb{R}^n$, we take $\|x\|$ to be the Euclidean norm of x. We refer the reader to [8] for the continuous case, that is, $\mathbb{T} = \mathbb{R}$.

In [6], the boundedness of solutions of

$$x^{\Delta}(t) = G(t,x(t)), \quad x(t_0) = x_0, \quad t_0 \ge 0,\ x_0 \in \mathbb{R} \tag{1.2}$$

is considered by using a type I Lyapunov function. Then, in [5], the authors considered nonnegative definite Lyapunov functions and obtained sufficient conditions for the exponential stability of the zero solution. However, the results in either [5] or [6] do not apply to the equations similar to

$$x^{\Delta} = a(t)x + \int_0^t B(t,s) f(x(s))\Delta s, \tag{1.3}$$

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which is the Volterra integro-dynamic equation. In particular, we are interested in applying our results to (1.3) with $f(x) = x^n$, where n is positive and rational. The authors are confident that there is nothing in the literature that deals with the qualitative analysis of Volterra integro-dynamic equations on time scales. Thus, this paper is going to play a major role in any future research that is related to Volterra integro-dynamic equations.

Let $\phi : [0, t_0] \to \mathbb{R}^n$ be continuous, we define $|\phi| = \sup\{\|\phi(t)\| : 0 \le t \le t_0\}$.

We say that solutions of (1.1) are *bounded* if any solution $x(t, t_0, \phi)$ of (1.1) satisfies

$$||x(t, t_0, \phi)|| \le C(|\phi|, t_0), \quad \forall t \ge t_0, \tag{1.4}$$

where C is a constant and depends on t_0. Moreover, solutions of (1.1) are *uniformly bounded* if C is independent of t_0. Throughout this paper, we assume $0 \in \mathbb{T}$ and $[0, \infty) = \{t \in \mathbb{T} : 0 \le t < \infty\}$.

Next, we generalize a "type I Lyapunov function" which is defined by Peterson and Tisdell [6] to Lyapunov functionals. We say $V : [0, \infty) \times \mathbb{R}^n \mapsto [0, \infty)$ is a *type I Lyapunov functional* on $[0, \infty) \times \mathbb{R}^n$ when

$$V(t, x) = \sum_{i=1}^{n} \left(V_i(x_i) + U_i(t)\right), \tag{1.5}$$

where each $V_i : \mathbb{R} \mapsto \mathbb{R}$ and $U_i : [0, \infty) \mapsto \mathbb{R}$ are continuously differentiable. Next, we extend the definition of the derivative of a type I Lyapunov function to type I Lyapunov functionals. If V is a type I Lyapunov functional and x is a solution of (1.1), then (2.11) gives

$$\begin{aligned} [V(t, x)]^{\Delta} &= \sum_{i=1}^{n} \left(V_i(x_i(t)) + U_i(t)\right)^{\Delta} \\ &= \int_0^1 \nabla V[x(t) + h\mu(t)G(t, x(\cdot))] \cdot G(t, x(\cdot))\,dh + \sum_{i=1}^{n} U_i^{\Delta}(t), \end{aligned} \tag{1.6}$$

where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ is the gradient operator. This motivates us to define $\dot{V} : [0, \infty) \times \mathbb{R}^n \mapsto \mathbb{R}$ by

$$\dot{V}(t, x) = [V(t, x)]^{\Delta}. \tag{1.7}$$

Continuing in the spirit of [6], we have

$$\dot{V}(t, x) = \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{V_i(x_i + \mu(t)G_i(t, x(\cdot))) - V_i(x_i)}{\mu(t)} + \sum_{i=1}^{n} U_i^{\Delta}(t), & \text{when } \mu(t) \ne 0, \\ \displaystyle\nabla V(x) \cdot G(t, x(\cdot)) + \sum_{i=1}^{n} U_i^{\Delta}(t), & \text{when } \mu(t) = 0. \end{cases} \tag{1.8}$$

We also use a continuous strictly increasing function $W_i : [0, \infty) \mapsto [0, \infty)$ with $W_i(0) = 0$, $W_i(s) > 0$, if $s > 0$ for each $i \in \mathbb{Z}^+$.

We make use of the above expression in our examples.

Example 1.1. Assume $\phi(t,s)$ is right-dense continuous (rd-continuous) and let

$$V(t,x) = x^2 + \int_0^t \phi(t,s)W(|x(s)|)\Delta s. \tag{1.9}$$

If x is a solution of (1.1), then we have by using (2.10) and Theorem 2.2 that

$$\begin{aligned} \dot{V}(t,x) &= 2x \cdot G(t,x(\cdot)) + \mu(t)G^2(t,x(\cdot)) \\ &\quad + \int_0^t \phi^{\Delta}(t,s)W(|x(s)|)\Delta s + \phi(\sigma(t),t)W(|x(t)|), \end{aligned} \tag{1.10}$$

where $\phi^{\Delta}(t,s)$ denotes the derivative of ϕ with respect to the first variable.

We say that a type I Lyapunov functional $V : [0,\infty) \times \mathbb{R}^n \mapsto [0,\infty)$ is *negative definite* if $V(t,x) > 0$ for $x \neq 0$, $x \in \mathbb{R}^n$, $V(t,x) = 0$ for $x = 0$ and along the solutions of (1.1), we have $\dot{V}(t,x) \leq 0$. If the condition $\dot{V}(t,x) \leq 0$ does not hold for all $(t,x) \in \mathbb{T} \times \mathbb{R}^n$, then the Lyapunov functional is said to be *nonnegative definite*.

In the case of differential equations or difference equations, it is known that if one can display a negative definite Lyapunov function, or functionals, for (1.1), then boundedness of all solutions follows. In [8], the second author displayed nonnegative Lyapunov functionals and proved boundedness of all solutions of (1.1), in the case $\mathbb{T} = \mathbb{R}$.

2. Calculus on time scales

In this section, we introduce a calculus on time scales including preliminary results. An introduction with applications and advances in dynamic equations are given in [2, 3]. Our aim is not only to unify some results when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ but also to extend them for other time scales such as $h\mathbb{Z}$, where $h > 0$, $q^{\mathbb{N}_0}$, where $q > 1$ and so on. We define the *forward jump operator* σ on $\mathbb{T}$ by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T} \tag{2.1}$$

for all $t \in \mathbb{T}$. In this definition, we put $\inf(\emptyset) = \sup \mathbb{T}$. The *backward jump operator* ρ on $\mathbb{T}$ is defined by

$$\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T} \tag{2.2}$$

for all $t \in \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$, we say t is *left-dense*. The *graininess function* $\mu : \mathbb{T} \mapsto [0,\infty)$ is defined by

$$\mu(t) := \sigma(t) - t. \tag{2.3}$$

$\mathbb{T}$ has left-scattered maximum point m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. Assume $x : \mathbb{T} \mapsto \mathbb{R}^n$. Then we define $x^{\Delta}(t)$ to be the vector (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| [x_i(\sigma(t)) - x_i(s)] - x_i^{\Delta}(t)[\sigma(t) - s] \right| \leq \epsilon \left| \sigma(t) - s \right| \tag{2.4}$$

for all $s \in U$ and for each $i = 1,2,\ldots,n$. We call $x^{\Delta}(t)$ the *delta derivative* of $x(t)$ at t, and it turns out that $x^{\Delta}(t) = x'(t)$ if $\mathbb{T} = \mathbb{R}$ and $x^{\Delta}(t) = x(t+1) - x(t)$ if $\mathbb{T} = \mathbb{Z}$. If $G^{\Delta}(t) = g(t)$, then the Cauchy integral is defined by

$$\int_a^t g(s)\Delta s = G(t) - G(a). \tag{2.5}$$

It can be shown that if $f : \mathbb{T} \mapsto \mathbb{R}^n$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}, \tag{2.6}$$

while if t is right-dense, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}, \tag{2.7}$$

if the limit exists. If $f, g : \mathbb{T} \mapsto \mathbb{R}^n$ are differentiable at $t \in \mathbb{T}$, then the product and quotient rules are as follows:

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t), \tag{2.8}$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)} \quad \text{if } g(t)g^{\sigma}(t) \neq 0. \tag{2.9}$$

If f is differentiable at t, then

$$f^{\sigma}(t) = f(t) + \mu(t)f^{\Delta}(t), \quad \text{where } f^{\sigma} = f \circ \sigma. \tag{2.10}$$

We say $f : \mathbb{T} \mapsto \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense, $\lim_{s \to t^-} f(s)$ exists as a finite number. We say that $p : \mathbb{T} \mapsto \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. We define the set $\mathcal{R}$ of all regressive and rd-continuous functions. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

The following chain rule is due to Poetzsche and the proof can be found in [2, Theorem 1.90].

Theorem 2.1. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{\int_0^1 f'\left(g(t) + h\mu(t)g^{\Delta}(t)\right)dh\right\}g^{\Delta}(t) \tag{2.11}$$

holds.

We use the following result [2, Theorem 1.117] to calculate the derivative of the Lyapunov function in further sections.

Theorem 2.2. *Let $t_0 \in \mathbb{T}^{\kappa}$ and assume $k : \mathbb{T} \times \mathbb{T}^{\kappa} \mapsto \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^{\kappa}$ with $t > t_0$. Also assume that $k(t,\cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose for each $\epsilon > 0$,* *there exists a neighborhood of t, independent U of* $\tau \in [t_0, \sigma(t)]$, *such that*

$$|k(\sigma(t),\tau) - k(s,\tau) - k^{\Delta}(t,\tau)(\sigma(t)-s)| \le \epsilon |\sigma(t)-s| \quad \forall s \in U, \tag{2.12}$$

where k^{Δ} *denotes the derivative of* k *with respect to the first variable. Then*

$$\begin{aligned} g(t) &:= \int_{t_0}^{t} k(t,\tau)\Delta\tau \quad \textit{implies} \quad g^{\Delta}(t) = \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau + k(\sigma(t),t); \\ h(t) &:= \int_{t}^{b} k(t,\tau)\Delta\tau \quad \textit{implies} \quad k^{\Delta}(t) = \int_{t}^{b} k^{\Delta}(t,\tau)\Delta\tau - k(\sigma(t),t). \end{aligned} \tag{2.13}$$

We apply the following Cauchy-Schwarz inequality in [2, Theorem 6.15] to prove Theorem 4.1.

Theorem 2.3. *Let* $a, b \in \mathbb{T}$. *For rd-continuous* $f, g : [a,b] \mapsto \mathbb{R}$,

$$\int_a^b |f(t)g(t)|\Delta t \le \sqrt{\left\{\int_a^b |f(t)|^2 \Delta t\right\}\left\{\int_a^b |g(t)|^2 \Delta t\right\}}. \tag{2.14}$$

If $p : \mathbb{T} \mapsto \mathbb{R}$ is rd-continuous and regressive, then the *exponential function* $e_p(t,t_0)$ is for each fixed $t_0 \in \mathbb{T}$ the unique solution of the initial value problem

$$x^{\Delta} = p(t)x, \qquad x(t_0) = 1 \tag{2.15}$$

on $\mathbb{T}$. Under the addition on $\mathcal{R}$ defined by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t), \quad t \in \mathbb{T}, \tag{2.16}$$

is an Abelian group (see [2]), where the additive inverse of p, denoted by $\ominus p$, is defined by

$$(\ominus p)(t) = \frac{-p(t)}{1+\mu(t)p(t)}, \quad t \in \mathbb{T}. \tag{2.17}$$

We use the following properties of the exponential function $e_p(t,s)$ which are proved in Bohner and Peterson [2].

Theorem 2.4. *If* $p, q \in \mathcal{R}$, *then for* $t, s, r, t_0 \in \mathbb{T}$,

(i) $e_p(t,t) \equiv 1$ *and* $e_0(t,s) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(iii) $1/e_p(t,s) = e_{\ominus p}(t,s) = e_p(s,t)$;
(iv) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$;
(v) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$.

Moreover, the following can be found in [1].

Theorem 2.5. *Let* $t_0 \in \mathbb{T}$.

(i) *If* $p \in \mathcal{R}^+$, *then* $e_p(t,t_0) > 0$ *for all* $t \in \mathbb{T}$.
(ii) *If* $p \ge 0$, *then* $e_p(t,t_0) \ge 1$ *for all* $t \ge t_0$. *Therefore,* $e_{\ominus p}(t,t_0) \le 1$ *for all* $t \ge t_0$.

3. Boundedness of solutions

In this section, we use a nonnegative definite type I Lyapunov functional and establish sufficient conditions to obtain boundedness of solutions of (1.1).

Theorem 3.1. *Let $D \subset \mathbb{R}^n$. Suppose that there exists a type I Lyapunov functional $V : [0,\infty) \times D \mapsto [0,\infty)$ such that for all $(t,x) \in [0,\infty) \times D$,*

$$\lambda_1 W_1(|x|) \le V(t,x) \le \lambda_2 W_2(|x|) + \lambda_2 \int_0^t \phi_1(t,s) W_3(|x(s)|)\Delta s, \tag{3.1}$$

$$\dot{V}(t,x) \le \frac{-\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \phi_2(t,s) W_5(|x(s)|)\Delta s + L}{1+\mu(t)(\lambda_3/\lambda_2)}, \tag{3.2}$$

where λ_1, λ_2, λ_3, and L are positive constants and $\phi_i(t,s) \ge 0$ is rd-continuous function for $0 \le s \le t < \infty$, $i = 1,2$ such that

$$W_2(|x|) - W_4(|x|) + \int_0^t \big(\phi_1(t,s) W_3(|x(s)|) - \phi_2(t,s) W_5(|x(s)|)\big)\Delta s \le \gamma, \tag{3.3}$$

where $\gamma \ge 0$. If $\int_0^t \phi_1(t,s)\Delta s \le B$ for some $B \ge 0$, then all solutions of (1.1) staying in D are uniformly bounded.

Proof. Let x be a solution of (1.1) with $x(t) = \phi(t)$ for $0 \le t \le t_0$. Set $M = \lambda_3/\lambda_2$. By (2.8) and (2.10) and inequalities (3.1), (3.2), and (3.3) we obtain

$$\begin{aligned}
\big[V(t,x(t))e_M(t,t_0)\big]^\Delta &= \dot{V}(t,x(t))e_M^\sigma(t,t_0) + MV(t,x(t))e_M(t,t_0) \\
&= \big[\dot{V}(t,x(t))(1+\mu(t)M) + MV(t,x(t))\big]e_M(t,t_0) \\
&\le \Big[-\lambda_3 W_4(|x|) - \lambda_3 \int_0^t \phi_2(t,s)W_5(|x(s)|)\Delta s + L\Big]e_M(t,t_0) \\
&\quad + \Big[\lambda_3 W_2(|x|) + \lambda_3 \int_0^t \phi_1(t,s)W_3(|x(s)|)\Delta s\Big]e_M(t,t_0) \\
&\le [\lambda_3\gamma + L]e_M(t,t_0) =: Ke_M(t,t_0),
\end{aligned} \tag{3.4}$$

where we used Theorem 2.5(i). Integrating both sides from t_0 to t, we have

$$\begin{aligned}
V(t,x(t))e_M(t,t_0) &\le V(t_0,\phi) + \frac{K}{M}\int_{t_0}^t e_M^\Delta(\tau,t_0)\Delta\tau \\
&= V(t_0,\phi) + \frac{K}{M}\big(e_M(t,t_0) - 1\big) \le V(t_0,\phi) + \frac{K}{M}e_M(t,t_0).
\end{aligned} \tag{3.5}$$

It follows from Theorem 2.4(iii) that for all $t \ge t_0$,

$$V(t,x(t)) \le V(t_0,\phi)e_{\ominus M}(t,t_0) + \frac{K}{M}. \tag{3.6}$$

From inequality (3.1), we have

$$\begin{aligned} W_1(|x|) &\leq \frac{1}{\lambda_1}\left(V(t_0,\phi)e_{\ominus M}(t,t_0)+\frac{K}{M}\right) \\ &\leq \frac{1}{\lambda_1}\left[\lambda_2 W_2(|\phi|)+\lambda_2 W_3(|\phi|)\int_0^{t_0}\phi_1(t_0,s)\Delta s+\frac{K}{M}\right], \end{aligned} \tag{3.7}$$

where we used the fact Theorem 2.5(ii). Therefore, we obtain

$$|x| \leq W_1^{-1}\left\{\frac{1}{\lambda_1}\left[\lambda_2 W_2(|\phi|)+\lambda_2 W_3(|\phi|)\int_0^{t_0}\phi_1(t_0,s)\Delta s+\frac{K}{M}\right]\right\} \tag{3.8}$$

for all $t \geq t_0$. This concludes the proof. □

In the next theorem, we give sufficient conditions to show that solutions of (1.1) are bounded.

Theorem 3.2. *Let $D \subset \mathbb{R}^n$. Suppose that there exists a type I Lyapunov functional $V : [0,\infty)\times D \mapsto [0,\infty)$ such that for all $(t,x)\in[0,\infty)\times D$,*

$$\begin{aligned} \lambda_1(t)W_1(|x|) \leq V(t,x) &\leq \lambda_2(t)W_2(|x|)+\lambda_2(t)\int_0^t \phi_1(t,s)W_3(|x(s)|)\Delta s, \\ \dot{V}(t,x) &\leq \frac{-\lambda_3(t)W_4(|x|)-\lambda_3(t)\int_0^t\phi_2(t,s)W_5(|x(s)|)\Delta s+L}{1+\mu(t)(\lambda_3(t)/\lambda_2(t))}, \end{aligned} \tag{3.9}$$

where λ_1, λ_2, λ_3 are positive continuous functions, L is a positive constant, λ_1 is nondecreasing, and $\phi_i(t,s)\geq 0$ is rd-continuous for $0\leq s\leq t<\infty$, $i=1,2$, such that

$$W_2(|x|)-W_4(|x|)+\int_0^t\left(\phi_1(t,s)W_3(|x|)-\phi_2(t,s)W_5(|x(s)|)\right)\Delta s \leq \gamma, \tag{3.10}$$

where $\gamma \geq 0$. If $\int_0^t \phi_1(t,s)\Delta s \leq B$ and $\lambda_3(t)\leq N$ for $t\in[0,\infty)$ and some positive constants B and N, then all solutions of (1.1) staying in D are bounded.

Proof. Let $M := \inf_{t\geq 0}(\lambda_3(t)/\lambda_2(t)) > 0$ and let x be any solution of (1.1) with $x(t_0)=\phi(t_0)$. Then we obtain

$$\begin{aligned} \left[V(t,x(t))e_M(t,t_0)\right]^{\Delta} &= \dot{V}(t,x(t))e_M^{\sigma}(t,t_0)+MV(t,x(t))e_M(t,t_0) \\ &= \left[\dot{V}(t,x(t))(1+\mu(t)M)+MV(t,x(t))\right]e_M(t,t_0) \\ &\leq \left[-\lambda_3(t)W_4(|x|)-\lambda_3(t)\int_0^t\phi_2(t,s)W_5(|x(s)|)\Delta s+L\right]e_M(t,t_0) \\ &\quad+\left[M\lambda_2(t)W_2(|x|)+M\lambda_2(t)\int_0^t\phi_1(t,s)W_3(|x(s)|)\Delta s\right]e_M(t,t_0) \\ &\leq \left[\lambda_3(t)\gamma+L\right]e_M(t,t_0) \leq (N\gamma+L)e_M(t,t_0) =: Ke_M(t,t_0), \end{aligned} \tag{3.11}$$

because of $M \leq \lambda_3(t)/\lambda_2(t)$, $\lambda_3(t) \leq N$, for $t \in [0,\infty)$ and Theorem 2.5(i). Integrating both sides from t_0 to t, we obtain

$$V(t,x(t))e_M(t,t_0) \leq V(t_0,\phi) + \frac{K}{M}e_M(t,t_0). \tag{3.12}$$

This implies from Theorem 2.4(iii) that for all $t \geq t_0$,

$$V(t,x(t)) \leq V(t_0,\phi)e_{\ominus M}(t,t_0) + \frac{K}{M}. \tag{3.13}$$

From inequality (3.1), we have

$$W_1(|x|) \leq \frac{1}{\lambda_1(t_0)}\left(\lambda_2(t_0)W_2(|\phi|) + \lambda_2(t_0)W_3(|\phi|)\int_0^{t_0}\phi_1(t_0,s)\Delta s + \frac{K}{M}\right) \tag{3.14}$$

for all $t \geq t_0$, where we used the fact Theorem 2.5(ii) and λ_1 is nondecreasing. □

The following theorem is the special case of [8, Theorem 2.6].

Theorem 3.3. *Suppose there exists a continuously differentiable type I Lyapunov functional* $V : [0,\infty) \times \mathbb{R}^n \mapsto [0,\infty)$ *that satisfies*

$$\lambda_1\|x\|^p \leq V(t,x), \quad V(t,x) \neq 0 \quad \text{if } x \neq 0, \tag{3.15}$$

$$[V(t,x)]^\Delta \leq -\lambda_2(t)V(t,x)V^\sigma(t,x) \tag{3.16}$$

for some positive constants λ_1 and p are positive constants, and λ_2 is a positive continuous function such that

$$c_1 = \inf_{0\leq t_0\leq t}\lambda_2(t). \tag{3.17}$$

Then all solutions of (1.1) satisfy

$$\|x\| \leq \frac{1}{\lambda_1^{1/p}}\left[\frac{1}{1/V(t_0,\phi) + c_1(t-t_0)}\right]^{1/p}. \tag{3.18}$$

Proof. For any $t_0 \geq 0$, let x be the solution of (1.1) with $x(t_0) = \phi(t_0)$. By inequalities (3.16) and (3.17), we have

$$[V(t,x)]^\Delta \leq -c_1V(t,x)V^\sigma(t,x). \tag{3.19}$$

Let $u(t) = V(t,x(t))$ so that we have

$$\frac{u^\Delta(t)}{u(t)u^\sigma(t)} \leq -c_1. \tag{3.20}$$

Since $(1/u(t))^{\Delta} = -u^{\Delta}/u(t)u(\sigma(t))$, we obtain

$$\left(\frac{1}{u(t)}\right)^{\Delta} \geq c_1. \tag{3.21}$$

Integrating the above inequality from t_0 to t, we have

$$u(t) \leq \frac{1}{1/u(t_0) + c_1(t-t_0)} \tag{3.22}$$

or

$$V(t,x(t)) \leq \frac{1}{1/V(t_0,\phi) + c_1(t-t_0)}. \tag{3.23}$$

Using (3.15), we obtain

$$\|x\| \leq \frac{1}{\lambda_1^{1/p}}\left[\frac{1}{1/V(t_0,\phi) + c_1(t-t_0)}\right]^{1/p}. \tag{3.24}$$

□

The next theorem is an extension of [7, Theorem 2.6].

Theorem 3.4. *Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov functional $V : [0,\infty) \times D \to [0,\infty)$ such that for all $(t,x) \in [0,\infty) \times D$,*

$$\lambda_1 \|x\|^p \leq V(t,x), \tag{3.25}$$

$$\dot{V}(t,x) \leq \frac{-\lambda_2 V(x) + L}{1+\varepsilon\mu(t)}, \tag{3.26}$$

where $\lambda_1, \lambda_2, p > 0$, $L \geq 0$ are constants and $0 < \varepsilon < \lambda_2$. Then all solutions of (1.1) staying in D are bounded.

Proof. For any $t_0 \geq 0$, let x be the solution of (1.1) with $x(t_0) = \phi$. Since $\varepsilon \in \mathcal{R}^+$, $e_\varepsilon(t,0)$ is well defined and positive. By (3.26), we obtain

$$\begin{aligned}
[V(t,x(t))e_\varepsilon(t,0)]^{\Delta} &= \dot{V}(t,x(t))e_\varepsilon^{\sigma}(t,0) + \varepsilon V(t,x(t))e_\varepsilon(t,0), \\
&\leq (-\lambda_2 V(t,x(t)) + L)e_\varepsilon(t,0) + \varepsilon V(t,x(t))e_\varepsilon(t,0), \\
&= e_\varepsilon(t,0)[\varepsilon V(t,x(t)) - \lambda_2 V(t,x(t)) + L] \leq Le_\varepsilon(t,0).
\end{aligned} \tag{3.27}$$

Integrating both sides from t_0 to t, we obtain

$$V(t,x(t))e_\varepsilon(t,0) \leq V(t_0,\phi) + \frac{L}{\varepsilon}e_\varepsilon(t,0). \tag{3.28}$$

Dividing both sides of the above inequality by $e_\varepsilon(t,0)$ and then using (3.25) and Theorem 2.5, we obtain

$$\|x\| \le \left\{\frac{1}{\lambda_1}\right\}^{1/p}\left[V(t_0,\phi)+\frac{L}{\varepsilon}\right]^{1/p} \quad \text{for all } t \ge t_0. \tag{3.29}$$

This completes the proof. □

Remark 3.5. In Theorem 3.4, if $V(t_0,\phi)$ is uniformly bounded, then one concludes that all solutions of (1.1) that stay in D are uniformly bounded.

4. Applications to Volterra integro-dynamic equations

In this section, we apply our theorems from the previous section and obtain sufficient conditions that insure the boundedness and uniform boundedness of solutions of Volterra integro-dynamic equations. We begin with the following theorem.

THEOREM 4.1. *Suppose $B(t,s)$ is rd-continuous and consider the scalar nonlinear Volterra integro-dynamic equation*

$$x^\Delta = a(t)x(t)+\int_0^t B(t,s)x^{2/3}(s)\Delta s, \quad t\ge 0,\ x(t)=\phi(t) \text{ for } 0\le t\le t_0, \tag{4.1}$$

where ϕ is a given bounded continuous initial function on $[0,\infty)$, and a is a continuous function on $[0,\infty)$. Suppose there are positive constants ν, β_1, β_2, with $\nu \in (0,1)$, and $\lambda_3 = \min\{\beta_1,\beta_2\}$ such that

$$\begin{aligned}&\Big[2a(t)+\mu(t)a^2(t)+\mu(t)|a(t)|\int_0^t |B(t,s)|\Delta s+\int_0^t |B(t,s)|\Delta s\\ &\quad +\nu\int_{\sigma(t)}^\infty |B(u,t)|\Delta u\Big](1+\mu(t)\lambda_3) \le -\beta_1,\end{aligned} \tag{4.2}$$

$$\left\{\frac{2}{3}\left[1+\mu(t)|a(t)|+\mu(t)\int_0^t |B(t,s)|\Delta s\right]-\nu\right\}(1+\mu(t)\lambda_3)\le -\beta_2, \tag{4.3}$$

$$\begin{aligned}&\int_0^t\int_t^\infty |B(u,s)|\Delta u\Delta s<\infty, \qquad \int_0^t |B(t,s)|\Delta s<\infty,\\ &|B(t,s)|\ge \nu\int_t^\infty |B(u,s)|\Delta u,\end{aligned} \tag{4.4}$$

then all solutions of (4.1) are uniformly bounded.

Proof. Let

$$V(t,x)=x^2(t)+\nu\int_0^t\int_t^\infty |B(u,s)|\Delta u x^2(s)\Delta s. \tag{4.5}$$

Using Theorem 2.2, we have along the solutions of (4.1) that

$$\begin{aligned}
\dot{V}(t,x) &= 2x(t)\Big(a(t)x(t)+\int_0^t B(t,s)x^{2/3}(s)\Delta s\Big) \\
&\quad +\mu(t)\Big(a(t)x(t)+\int_0^t B(t,s)x^{2/3}(s)\Delta s\Big)^2 \\
&\quad -\nu\int_0^t |B(t,s)|x^2(s)\Delta s+\nu\int_{\sigma(t)}^{\infty}|B(u,t)|x^2(t)\Delta u \\
&\le 2a(t)x^2(t)+2\int_0^t |B(t,s)|\,|x(t)|x^{2/3}(s)\Delta s \\
&\quad +\mu(t)a^2(t)x^2(t)+2\mu(t)|a(t)|\int_0^t |B(t,s)|\,|x(t)|x^{2/3}(s)\Delta s \\
&\quad +\mu(t)\Big(\int_0^t B(t,s)x^{2/3}(s)\Delta s\Big)^2 \\
&\quad +\nu\int_{\sigma(t)}^{\infty}|B(u,t)|x^2(t)\Delta u-\nu\int_0^t |B(t,s)|x^2(s)\Delta s.
\end{aligned} \tag{4.6}$$

Using the fact that $ab \le a^2/2+b^2/2$ for any real numbers a and b, we have

$$2\int_0^t |B(t,s)|\,|x(t)|x^{2/3}(s)\Delta s \le \int_0^t |B(t,s)|\big(x^2(t)+x^{4/3}(s)\big)\Delta s. \tag{4.7}$$

Also, using Theorem 2.3, one obtains

$$\begin{aligned}
\Big(\int_0^t |B(t,s)|x^{2/3}(s)\Delta s\Big)^2 &= \Big(\int_0^t |B(t,s)|^{1/2}|B(t,s)|^{1/2}x^{2/3}(s)\Delta s\Big)^2 \\
&\le \int_0^t |B(t,s)|\Delta s\int_0^t |B(t,s)|x^{4/3}(s)\Delta s.
\end{aligned} \tag{4.8}$$

A substitution of the above two inequalities into (4.6) yields

$$\begin{aligned}
\dot{V}(t,x) &\le \Big[2a(t)+\mu(t)a^2(t)+\mu(t)|a(t)|\int_0^t |B(t,s)|\Delta s \\
&\quad +\int_0^t |B(t,s)|\Delta s+\nu\int_{\sigma(t)}^{\infty}|B(u,t)|\Delta u\Big]x^2(t) \\
&\quad +\Big[1+\mu(t)|a(t)|+\mu(t)\int_0^t |B(t,s)|\Delta s\Big]\int_0^t |B(t,s)|x^{4/3}(s)\Delta s \\
&\quad -\nu\int_0^t |B(t,s)|x^2(s)\Delta s.
\end{aligned} \tag{4.9}$$

To further simplify (4.9), we make use of Young's inequality, which says that for any two nonnegative real numbers w and z, we have

$$wz \le \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } \frac{1}{e} + \frac{1}{f} = 1. \tag{4.10}$$

Thus, for $e = 3/2$ and $f = 3$, we get

$$\begin{aligned}\int_0^t |B(t,s)| x^{4/3}(s)\Delta s &= \int_0^t |B(t,s)|^{1/3} |B(t,s)|^{2/3} x^{4/3}(s)\Delta s \\ &\le \int_0^t \left(\frac{|B(t,s)|}{3} + \frac{2}{3}|B(t,s)| x^2(s)\right)\Delta s.\end{aligned} \tag{4.11}$$

By substituting the above inequality into (4.9), we arrive at

$$\begin{aligned}\dot V(t,x) \le{}& \Big[2a(t) + \mu(t)a^2(t) + \mu(t)|a(t)| \int_0^t |B(t,s)|\Delta s \\ &+ \int_0^t |B(t,s)|\Delta s + \nu \int_{\sigma(t)}^\infty |B(u,t)|\Delta u \Big] x^2(t) \\ &+ \left[-\nu + \frac{2}{3}\left(1 + \mu(t)|a(t)| + \mu(t)\int_0^t |B(t,s)|\Delta s\right)\right] \int_0^t |B(t,s)| x^2(s)\Delta s \\ &+ \frac{1}{3}\left(1 + \mu(t)|a(t)| + \mu(t)\int_0^t |B(t,s)|\Delta s\right)\int_0^t |B(t,s)|\Delta s.\end{aligned} \tag{4.12}$$

Multiplying and dividing the above inequality by $1 + \mu(t)\lambda_3$, and then applying conditions (4.2) and (4.3), $\dot V(t,x)$ reduces to

$$\dot V(t,x) \le \frac{-\beta_1 x^2(t) - \beta_2 \int_0^t |B(t,s)| x^2(s)\Delta s + L}{1 + \mu(t)\lambda_3}, \tag{4.13}$$

where $L = 1/3(1 + \mu(t)|a(t)| + \mu(t)\int_0^t |B(t,s)|\Delta s)\int_0^t |B(t,s)|\Delta s(1 + \mu(t)\lambda_3)$. By taking $W_1 = W_2 = W_4 = x^2(t)$, $W_3 = W_5 = x^2(s)$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \min\{\beta_1, \beta_2\}$, $\phi_1(t,s) = \nu\int_t^\infty |B(u,s)|\Delta u$, and $\phi_2(t,s) = |B(t,s)|$, we see that conditions (3.1) and (3.2) of Theorem 3.1 are satisfied. Next we make sure that condition (3.3) holds. Use (4.4) to obtain

$$\begin{aligned}&W_2(|x|) - W_4(|x|) + \int_0^t (\phi_1(t,s)W_3(|x(s)|) - \phi_2(t,s)W_5(|x(s)|))\Delta s \\ &\quad = x^2(t) - x^2(t) + \int_0^t \left(\nu\int_t^\infty |B(u,s)|\Delta u - |B(t,s)|\right)x^2(s)\Delta s \le 0.\end{aligned} \tag{4.14}$$

Thus condition (3.3) is satisfied with $\gamma = 0$. An application of Theorem 3.1 yields the results. □

Remark 4.2. In the case $\mathbb{T} = \mathbb{R}$, the second author in [8] took $\nu = 1$ in the displayed Lyapunov functional. On the other hand, in our theorem, we had to incorporate such ν

in the Lyapunov functional, otherwise, condition (4.5) may only hold if $B(t,s) = 0$ for all $t \in \mathbb{T}$ with $0 \le s \le t < \infty$ for a particular time scale. For example, if we take $\mathbb{T} = \mathbb{Z}$, then condition (4.5) reduces to $|B(t,s)| \ge \nu \sum_{u=t}^{\infty} |B(u,s)|$, which can only hold if $B(t,s) = 0$ for $\nu = 1$.

Remark 4.3. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ for all t and hence Theorem 4.1 reduces to [8, Example 2.3].

Remark 4.4. We assert that Theorem 4.1 can be easily generalized to handle scalar nonlinear Volterra integro-dynamic equations of the form

$$x^{\Delta} = a(t)x(t) + \int_0^t B(t,s)f(s,x(s))\Delta s, \tag{4.15}$$

where $|f(t,x(t))| \le x^{2/3}(t) + M$ for some positive constant M.

For the next theorem, we consider the scalar Volterra integro-dynamic equation

$$x^{\Delta}(t) = a(t)x(t) + \int_0^t B(t,s)f(s,x(s))\Delta s + g(t,x(t)), \tag{4.16}$$

where $t \ge 0$, $x(t) = \phi(t)$ for $0 \le t \le t_0$, ϕ is a given bounded continuous initial function, $a(t)$ is continuous for $t \ge 0$, and $B(t,s)$ is right-dense continuous for $0 \le s \le t < \infty$. We assume $f(t,x)$ and $g(t,x)$ are continuous in x and t and satisfy

$$|g(t,x)| \le \gamma_1(t) + \gamma_2(t)|x(t)|, \qquad |f(t,x)| \le \gamma(t)|x(t)|, \tag{4.17}$$

where γ and γ_2 are positive and bounded, and γ_1 is nonnegative and bounded.

For the next theorem, we need the identity

$$|x(t)|^{\Delta} = \frac{x(t) + x^{\sigma}(t)}{|x(t)| + |x^{\sigma}(t)|} x^{\Delta}(t). \tag{4.18}$$

Its proof can be found in [4].

Theorem 4.5. *Suppose there exist constants $k > 1$ and ε, α with $0 < \varepsilon < \alpha$ such that*

$$\left[a(t) + \gamma_2(t) + k\int_{\sigma(t)}^{\infty} |B(u,t)|\Delta u\gamma(t)\right](1 + \varepsilon\mu(t)) \le -\alpha < 0, \tag{4.19}$$

where $k = 1 + \zeta$ for some $\zeta > 0$. Suppose

$$(1 + \mu(t)\varepsilon)|B(t,s)| \ge \lambda \int_t^{\infty} |B(u,s)|\Delta u, \tag{4.20}$$

where $\lambda \ge k\alpha/\zeta$, $0 \le s < t \le u < \infty$,

$$\int_0^{t_0}\int_{t_0}^{\infty} |B(u,s)|\Delta u\gamma(s)\Delta s \le \rho < \infty \quad \forall t_0 \ge 0, \tag{4.21}$$

and for some positive constant L,

$$\gamma_1(t)(1+\varepsilon\mu(t)) \le L. \tag{4.22}$$

Then all solutions of (4.16) are uniformly bounded.

Proof. Define

$$V(t,x(\cdot)) = |x(t)| + k\int_0^t\int_t^\infty |B(u,s)|\Delta u|f(s,x(s))|\Delta s. \tag{4.23}$$

Along the solutions of (4.16), we have

$$\begin{aligned}
\dot V(t,x) = {} & \frac{x(t)+x^\sigma(t)}{|x(t)|+|x^\sigma(t)|}x^\Delta(t) + k\int_{\sigma(t)}^\infty |B(u,t)|\Delta u|f(t,x(t))| \\
& - k\int_0^t |B(t,s)||f(s,x(s))|\Delta s \le a(t)|x(t)| + \int_0^t |B(t,s)||f(s,x(s))|\Delta s \\
& + |g(t,x(t))| + k\int_{\sigma(t)}^\infty |B(u,t)|\Delta u|f(t,x(t))| - k\int_0^t |B(t,s)||f(s,x(s))|\Delta s \\
\le {} & \left[a(t)+\gamma_2(t)+k\int_{\sigma(t)}^\infty |B(u,t)|\Delta u\gamma(t)\right]|x(t)| \\
& + (1-k)\int_0^t |B(t,s)||f(s,x(s))|\Delta s + \gamma_1(t) \\
= {} & \left[a(t)+\gamma_2(t)+k\int_{\sigma(t)}^\infty |B(u,t)|\Delta u\gamma(t)\right]|x(t)|\frac{1+\mu(t)\varepsilon}{1+\mu(t)\varepsilon} \\
& - \zeta(1+\mu(t)\varepsilon)\int_0^t |B(t,s)||f(s,x(s))|\Delta s\frac{1}{1+\mu(t)\varepsilon} + (1+\mu(t)\varepsilon)\gamma_1(t)\frac{1}{1+\mu(t)\varepsilon} \\
\le {} & -\alpha|x(t)|\frac{1}{1+\mu(t)\varepsilon} - \zeta\lambda\int_0^t\int_t^\infty |B(u,s)|\Delta u|f(s,x(s))|\Delta s\frac{1}{1+\mu(t)\varepsilon} + \frac{L}{1+\mu(t)\varepsilon} \\
= {} & -\alpha\left[|x(t)| + k\int_0^t\int_t^\infty |B(u,s)|\Delta u|f(s,x(s))|\Delta s\right]\frac{1}{1+\mu(t)\varepsilon} + \frac{L}{1+\mu(t)\varepsilon} \\
= {} & \frac{-\alpha V(t,x)+L}{1+\mu(t)\varepsilon}.
\end{aligned} \tag{4.24}$$

The results follow form Theorem 3.4 and Remark 3.5. □

In the next theorem, we establish sufficient conditions that guarantee the boundedness of all solutions of the vector Volterra integro-dynamic equation

$$x^\Delta = Ax(t) + \int_0^t C(t,s)x(s)\Delta s + g(t), \tag{4.25}$$

where $t \geq 0$, $x(t) = \phi(t)$ for $0 \leq t \leq t_0$, ϕ is a given bounded continuous initial $k \times 1$ vector function. Also, A and $C(t,s)$ are $k \times k$ matrix with $C(t,s)$ being continuous on $\mathbb{T} \times \mathbb{T}$, g, x are $k \times 1$ vector functions that are continuous for $t \in \mathbb{T}$. If D is a matrix, then $|D|$ means the sum of the absolute values of the elements.

Theorem 4.6. *Suppose $C^T(t,s) = C(t,s)$. Let I be the $k \times k$ identity matrix. Assume there exist positive constants L, ν, ξ, β_1, β_2, λ_3, and $k \times k$ positive definite constant symmetric matrix B such that*

$$[A^T B + BA + \mu(t) A^T BA] \leq -\xi I, \tag{4.26}$$

$$\begin{aligned} \Big[-\xi + |A^T Bg| + |Bg| + \int_0^t |B|\,|C(t,s)|\Delta s + \mu(t) \int_0^t |A^T B|\,|C(t,s)|\Delta s \\ + \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u \Big] (1 + \mu(t)\lambda_3) \leq -\beta_1, \end{aligned} \tag{4.27}$$

$$\Big[|B| - \nu + \mu(t) \Big((g^T B)^2 + 1 + |A^T B| + \int_0^t |C(t,s)|\Delta s \Big) \Big] (1 + \mu(t)\lambda_3) \leq -\beta_2, \tag{4.28}$$

$$\big(\mu(t)|g^T g| + |Bg|\big)(1 + \mu(t)\lambda_3) + \mu(t)|A^T Bg| = L, \tag{4.29}$$

$$|C(t,s)| \geq \nu \int_{\sigma(t)}^{\infty} |C(u,s)|\Delta u, \tag{4.30}$$

$$\int_0^t \int_t^{\infty} |C(u,s)|\Delta u \Delta s < \infty, \qquad \int_0^t |C(t,s)|\Delta s < \infty. \tag{4.31}$$

Then there exists an $r_1 \in (0,1]$ such that

$$r_1 x^T x \leq x^T Bx \leq x^T x. \tag{4.32}$$

Proof. Let the matrix B be defined by (4.26) and define

$$V(t,x) = x^T Bx + \nu \int_0^t \int_t^{\infty} |C(u,s)|\Delta u x^2(s) \Delta s. \tag{4.33}$$

Here $x^T x = x^2 = (x_1^2 + x_2^2 + \cdots + x_k^2)$. Using the product rule given in (2.8), we have along the solutions of (4.25) that

$$\begin{aligned} \dot{V}(t,x) &= (x^{\Delta})^T Bx + (x^{\sigma})^T Bx^{\Delta} - \nu \int_0^t |C(t,s)| x^2(s)\Delta s + \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2 \\ &= (x^{\Delta})^T Bx + (x + \mu(t)x^{\Delta})^T Bx^{\Delta} - \nu \int_0^t |C(t,s)| x^2(s)\Delta s + \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2 \\ &= (x^{\Delta})^T Bx + x^T Bx^{\Delta} + \mu(t)(x^{\Delta})^T Bx^{\Delta} - \nu \int_0^t |C(t,s)| x^2(s)\Delta s + \nu \int_{\sigma(t)}^{\infty} |C(u,t)|\Delta u x^2. \end{aligned} \tag{4.34}$$

Substituting the right-hand side of (4.25) for x^Δ into (4.34) and making use of (4.26), we obtain

$$\begin{aligned}\dot{V}(t,x) = {} & \left[Ax+\int_0^t C(t,s)x(s)\Delta s+g\right]^T Bx+x^T B\left[Ax+\int_0^t C(t,s)x(s)\Delta s+g\right] \\ & +\mu(t)\left[Ax+\int_0^t C(t,s)x(s)\Delta s+g\right]^T B\left[Ax+\int_0^t C(t,s)x(s)\Delta s+g\right] \\ & -\nu\int_0^t |C(t,s)|x^2(s)\Delta s+\nu\int_{\sigma(t)}^\infty |C(u,t)|\Delta u x^2.\end{aligned} \tag{4.35}$$

By noting that the right side of (4.35) is scalar and by recalling that B is a symmetric matrix, expression (4.35) simplifies to

$$\begin{aligned}\dot{V}(t,x) = {} & x^T(A^TB+BA+\mu(t)A^TBA)x+2x^TBg+2\int_0^t x^TBC(t,s)x(s)\Delta s \\ & +\mu(t)\Big[2x^TA^TBg+2g^TB\int_0^t C(t,s)x(s)\Delta s+2x^TA^TB\int_0^t C(t,s)x(s)\Delta s \\ & \qquad +\int_0^t x^T(s)C(t,s)\Delta sB\int_0^t C(t,s)x(s)\Delta s+g^TBg\Big] \\ & -\nu\int_0^t |C(t,s)|x^2(s)\Delta s+\nu\int_{\sigma(t)}^\infty |C(u,t)|\Delta u x^2 \\ \le {} & -\xi x^2+2|x^T||Bg|+2\int_0^t |x^T||B||C(t,s)||x(s)|\Delta s \\ & +\mu(t)\Big[\int_0^t |C(t,s)|2|g^TB||x(s)|\Delta s+2\int_0^t |x^T||A^TB||C(t,s)||x(s)|\Delta s \\ & \qquad +\int_0^t x^T(s)C(t,s)B\Delta s\int_0^t C(t,s)x(s)\Delta s+|g^Tg|+2|x^T||A^TBg|\Big] \\ & -\nu\int_0^t |C(t,s)|x^2(s)\Delta s+\nu\int_{\sigma(t)}^\infty |C(u,t)|\Delta u x^2.\end{aligned} \tag{4.36}$$

Next, we perform some calculations to simplify inequality (4.36),

$$\begin{aligned} 2|x^T||Bg| = 2|x^T||Bg|^{1/2}|Bg|^{1/2} &\le x^2|Bg|+|Bg|, \\ 2|x^T||A^TBg|2 = |x^T||A^TBg|^{1/2}|A^TBg|^{1/2} &\le x^2|A^TBg|+|A^TBg|, \\ 2\int_0^t |x^T||B||C(t,s)||x(s)|\Delta s &\le \int_0^t |B||C(t,s)|(x^2+x^2(s))\Delta s, \\ \int_0^t |C(t,s)|2|g^TB||x(s)|\Delta s &\le \int_0^t |C(t,s)|(|g^TB|^2+x^2(s))\Delta s, \\ 2\int_0^t |x^T||A^TB||C(t,s)||x(s)|\Delta s &\le \int_0^t |A^TB||C(t,s)|(x^2+x^2(s))\Delta s.\end{aligned} \tag{4.37}$$

Finally,

$$\begin{aligned}
&\int_0^t x^T(s)C(t,s)\Delta sB\int_0^t C(t,s)x(s)\Delta s\\
&\quad\le |B|\left|\int_0^t x^T(s)C(t,s)\Delta s\right|\left|\int_0^t C(t,s)x(s)\Delta s\right|\\
&\quad\le \frac{|B|\big(\int_0^t x^T(s)C(t,s)\Delta s\big)^2}{2}+\frac{|B|\big(\int_0^t C(t,s)x(s)\Delta s\big)^2}{2}\\
&\quad= |B|\left(\int_0^t C(t,s)x(s)\Delta s\right)^2\\
&\quad= |B|\left(\int_0^t |C(t,s)|^{1/2}|C(t,s)|^{1/2}|x(s)|\Delta s\right)^2\\
&\quad\le |B|\int_0^t |C(t,s)|\Delta s\int_0^t |C(t,s)|x^2(s)\Delta s.
\end{aligned}\tag{4.38}$$

A substitution of the above inequalities into (4.36) yields

$$\begin{aligned}
\dot V(t,x)\le{}&\bigg[-\xi+\mu(t)|A^TBg|+|Bg|+\int_0^t|B||C(t,s)|\Delta s\\
&\quad+\mu(t)\int_0^t|A^TB||C(t,s)|\Delta s+\nu\int_{\sigma(t)}^{\infty}|C(u,t)|\Delta u\bigg]x^2\\
&+\bigg[|B|-\nu+\mu(t)\Big((g^TB)^2+1+|A^TB|\\
&\qquad+|B|\int_0^t|C(t,s)|\Delta s\Big)\bigg]\int_0^t|C(t,s)|x^2(s)\Delta s\\
&+\mu(t)(|A^TBg|+|g^TBg|)+|Bg|.
\end{aligned}\tag{4.39}$$

Multiplying and dividing the above inequality by $1+\mu(t)\lambda_3$, and then applying conditions (4.30) and (4.31) $\dot V(t,x)$ reduces to

$$\dot V(t,x)\le\frac{-\beta_1x^2-\beta_2\int_0^t|C(t,s)|x^2(s)\Delta s+L}{1+\mu(t)\lambda_3},\tag{4.40}$$

where $L=(\mu(t)(|A^TBg|+|g^TBg|)+|Bg|)(1+\mu(t)\lambda_3)$. By taking $W_1=r_1x^Tx$, $W_2=x^TBx$, $W_4=x^Tx$, $W_3=W_5=x^2(s)$, $\lambda_1=\lambda_2=1$ and $\lambda_3=\min\{\beta_1,\beta_2\}$, $\phi_1(t,s)=\nu\int_t^\infty|C(u,s)|\Delta u$, and $\phi_2(t,s)=|C(t,s)|$, we see that conditions (3.1) and (3.2) of Theorem 3.1 are satisfied. Next we make sure that condition (3.3) holds. Using (4.29) and (4.32), we obtain

$$\begin{aligned}
&W_2(|x|)-W_4(|x|)+\int_0^t(\phi_1(t,s)W_3(|x(s)|)-\phi_2(t,s)W_5(|x(s)|))\Delta s\\
&\quad=x^TBx-x^Tx+\int_0^t\left(\nu\int_t^\infty|C(u,s)|\Delta u-|C(t,s)|\right)x^2(s)\Delta s\le 0.
\end{aligned}\tag{4.41}$$

Thus condition (3.3) is satisfied with $\gamma = 0$. An application of Theorem 3.1 yields the results. □

Remark 4.7. It is worth mentioning that Theorem 4.6 is new when $\mathbb{T} = \mathbb{R}$.

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Elvan Akin-Bohner: Department of Mathematics and Statistics, University of Missouri-Rolla, 310 Rolla Building, Rolla, MO 65409-0020, USA
E-mail address: akine@umr.edu

Youssef N. Raffoul: Department of Mathematics, University of Dayton, Dayton, OH 45469-2316, USA
E-mail address: youssef.raffoul@notes.udayton.edu