

# ON THIRD-ORDER LINEAR DIFFERENCE EQUATIONS INVOLVING QUASI-DIFFERENCES

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We study the third-order linear difference equation with quasi-differences and its adjoint equation. The main results of the paper describe relationships between the oscillatory and nonoscillatory solutions of both equations.

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## 1. Introduction

Consider the third-order linear difference equation

$$\Delta(p_n \Delta(r_n \Delta x_n)) + q_n x_{n+1} = 0 \quad (\text{E})$$

and its adjoint equation

$$\Delta(r_{n+1} \Delta(p_n \Delta u_n)) - q_{n+1} u_{n+2} = 0, \quad (\text{E}^A)$$

where  $\Delta$  is the forward difference operator defined by  $\Delta x_n = x_{n+1} - x_n$ ,  $(p_n)$ ,  $(r_n)$ , and  $(q_n)$  are sequences of positive real numbers for  $n \in \mathbb{N}$ .

This paper has been motivated by the paper [9], where third-order difference equations

$$\begin{aligned} \Delta^3 v_n - p_{n+1} \Delta v_{n+1} + q_{n+1} v_{n+1} &= 0, \\ \Delta(\Delta^2 u_n - p_{n+1} u_{n+1}) - q_{n+2} u_{n+2} &= 0 \end{aligned} \quad (1.1)$$

had been investigated. As it is noted here, these equations are not adjoint equations and are referred to as *quasi-adjoint* equations.

Equation (E) is a special case of linear  $n$ th-order difference equations with quasi-differences. Such equations have been widely studied in the literature, see, for example, [6, 11] and the references therein. The natural question which arises is to find the adjoint equation to (E) and to examine the connection between solutions of (E) and its adjoint one.

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In the continuous case, it holds (see, e.g., [5, Theorem 8.33]) that

$$\left( \frac{1}{p(t)} \left( \frac{1}{r(t)} x'(t) \right)' \right)' + q(t)x(t) = 0 \quad (1.2)$$

is oscillatory if and only if the adjoint equation

$$\left( \frac{1}{r(t)} \left( \frac{1}{p(t)} x'(t) \right)' \right)' - q(t)x(t) = 0 \quad (1.3)$$

has the same property. In addition, nonoscillatory solutions of these equations satisfy some interesting relationships, see, for example, [2, 5].

The aim of this paper is to investigate oscillatory and asymptotic properties of solutions of (E) and (E<sup>A</sup>). We will prove that (E<sup>A</sup>) is the adjoint equation to (E) and we will give discrete analogues of the above-quoted results for third-order differential equations. Moreover, the oscillation of (E) and (E<sup>A</sup>) is characterized by means of second-order linear difference equations and the problem of the number of oscillatory solutions in a given basis for the solution space of (E) and (E<sup>A</sup>) is investigated. Our results extend and complete results of [7–10] stated for the various forms of third-order difference equations.

A solution  $x$  of (E) is a real sequence  $(x_n)$  defined for all  $n \in \mathbb{N}$  and satisfying (E) for all  $n \in \mathbb{N}$ . A solution of (E) is called *nontrivial* if for any  $n_0 \geq 1$ , there exists  $n > n_0$  such that  $x_n \neq 0$ . Otherwise, the solution is called *trivial*. A nontrivial solution  $x$  of (E) is said to be *oscillatory* if for any  $n_0 \geq 1$ , there exists  $n > n_0$  such that  $x_{n+1}x_n \leq 0$ . Otherwise, the nontrivial solution is said to be *nonoscillatory*. Equation (E) is *oscillatory* if it has an oscillatory solution. The same terminology is used for (E<sup>A</sup>).

Denote quasi-differences  $x^{[i]}$ ,  $i = 0, 1, 2$ , of a solution  $x$  of (E) as follows:

$$x_n^{[0]} = x_n, \quad x_n^{[1]} = r_n \Delta x_n, \quad x_n^{[2]} = p_n \Delta x_n^{[1]}, \quad x_n^{[3]} = \Delta x_n^{[2]}. \quad (1.4)$$

Similarly, denote quasi-differences  $u^{[i]}$ ,  $i = 0, 1, 2$ , of a solution  $u$  of (E<sup>A</sup>) as follows:

$$u_n^{[0]} = u_n, \quad u_n^{[1]} = p_n \Delta u_n, \quad u_n^{[2]} = r_{n+1} \Delta u_n^{[1]}, \quad u_n^{[3]} = \Delta u_n^{[2]}. \quad (1.5)$$

All nonoscillatory solutions  $x$  of (E) can be a priori classified to the following classes:

$$\begin{aligned} N_0 &= \{x : \exists n_x \text{ s.t. } x_n x_n^{[1]} < 0, x_n x_n^{[2]} > 0 \forall n \geq n_x\}, \\ N_1 &= \{x : \exists n_x \text{ s.t. } x_n x_n^{[1]} > 0, x_n x_n^{[2]} < 0 \forall n \geq n_x\}, \\ N_2 &= \{x : \exists n_x \text{ s.t. } x_n x_n^{[1]} > 0, x_n x_n^{[2]} > 0 \forall n \geq n_x\}, \\ N_3 &= \{x : \exists n_x \text{ s.t. } x_n x_n^{[1]} < 0, x_n x_n^{[2]} < 0 \forall n \geq n_x\}, \end{aligned} \quad (1.6)$$

and similarly solutions  $u$  of (E<sup>A</sup>) can be classified to the same classes, whereby quasi-differences  $u^{[i]}$ ,  $i = 1, 2$ , are defined by (1.5), see [4, 3]. Solutions of (E) from the class  $N_0$  are called *Kneser solutions* and solutions of (E<sup>A</sup>) which belong to the class  $N_2$  are called *strongly monotone solutions*.

## 2. Relationship between (E) and (E<sup>A</sup>)

Solutions of (E) and (E<sup>A</sup>) are related by the following properties.

**THEOREM 2.1.** (a) *Let  $x, y$  be solutions of (E). Then the sequence  $C = (C_n)$  ( $n \geq 2$ ) such that*

$$C_{n-1} = C(x_{n-1}, y_{n-1}) \equiv \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_{n-1}^{[1]} & y_{n-1}^{[1]} \end{vmatrix} \quad (2.1)$$

*is a solution of (E<sup>A</sup>).*

(b) *Let  $u, v$  be solutions of (E<sup>A</sup>). Then the sequence  $D = (D_n)$  ( $n \geq 2$ ) such that*

$$D_n = D(u_{n-1}, v_{n-1}) \equiv \begin{vmatrix} u_{n-1} & v_{n-1} \\ u_{n-1}^{[1]} & v_{n-1}^{[1]} \end{vmatrix} \quad (2.2)$$

*is a solution of (E).*

*Proof.* Claim (a). For any two solutions  $x, y$  of (E), we have

$$x_n \Delta y_{n-1}^{[2]} - y_n \Delta x_{n-1}^{[2]} = -x_n q_{n-1} y_n + y_n q_{n-1} x_n = 0. \quad (2.3)$$

Therefore,

$$\begin{aligned} \Delta C_{n-1} &= x_n \Delta y_{n-1}^{[1]} + y_{n-1}^{[1]} \Delta x_{n-1} - y_n \Delta x_{n-1}^{[1]} - x_{n-1}^{[1]} \Delta y_{n-1} \\ &= x_n \Delta y_{n-1}^{[1]} + r_{n-1} \Delta y_{n-1} \Delta x_{n-1} - y_n \Delta x_{n-1}^{[1]} - r_{n-1} \Delta x_{n-1} \Delta y_{n-1} \\ &= x_n \Delta y_{n-1}^{[1]} - y_n \Delta x_{n-1}^{[1]}. \end{aligned} \quad (2.4)$$

Using the fact  $x_{n-1}^{[2]} = x_n^{[2]} - \Delta x_{n-1}^{[2]}$  and (2.3), we obtain

$$\begin{aligned} C_{n-1}^{[1]} &= p_{n-1} \Delta C_{n-1} = x_n y_{n-1}^{[2]} - y_n x_{n-1}^{[2]} \\ &= x_n (y_n^{[2]} - \Delta y_{n-1}^{[2]}) - y_n (x_n^{[2]} - \Delta x_{n-1}^{[2]}) = x_n y_n^{[2]} - y_n x_n^{[2]}. \end{aligned} \quad (2.5)$$

By a direct computation in view of (2.3), we get

$$\Delta C_{n-1}^{[1]} = x_{n+1} \Delta y_n^{[2]} + y_n^{[2]} \Delta x_n - y_{n+1} \Delta x_n^{[2]} - x_n^{[2]} \Delta y_n = y_n^{[2]} \Delta x_n - x_n^{[2]} \Delta y_n, \quad (2.6)$$

hence

$$C_{n-1}^{[2]} = r_n \Delta C_{n-1}^{[1]} = x_n^{[1]} y_n^{[2]} - y_n^{[1]} x_n^{[2]}. \quad (2.7)$$

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Finally

$$\begin{aligned}
 \Delta C_{n-1}^{[2]} &= x_{n+1}^{[1]} \Delta y_n^{[2]} + y_n^{[2]} \Delta x_n^{[1]} - y_{n+1}^{[1]} \Delta x_n^{[2]} - x_n^{[2]} \Delta y_n^{[1]} \\
 &= -x_{n+1}^{[1]} q_n y_{n+1} + p_n \Delta y_n^{[1]} \Delta x_n^{[1]} + y_{n+1}^{[1]} q_n x_{n+1} - p_n \Delta x_n^{[1]} \Delta y_n^{[1]} \\
 &= q_n (x_{n+1} y_{n+1}^{[1]} - y_{n+1} x_{n+1}^{[1]}) = q_n C_{n+1},
 \end{aligned} \tag{2.8}$$

that is,  $C_{n-1}$  is a solution of  $(E^A)$ .

Claim (b). By the similar argument as in (a), we get

$$\Delta D_n = u_n \Delta v_{n-1}^{[1]} - v_n \Delta u_{n-1}^{[1]}. \tag{2.9}$$

Using the fact  $u_{n-1}^{[2]} = u_{n-2}^{[2]} + \Delta u_{n-2}^{[2]}$ , we obtain

$$\begin{aligned}
 D_n^{[1]} &= r_n \Delta D_n = u_n v_{n-1}^{[2]} - v_n u_{n-1}^{[2]} \\
 &= u_n (v_{n-2}^{[2]} + \Delta v_{n-2}^{[2]}) - v_n (u_{n-2}^{[2]} + \Delta u_{n-2}^{[2]}) \\
 &= u_n (v_{n-2}^{[2]} + q_{n-1} v_n) - v_n (u_{n-2}^{[2]} + q_{n-1} u_n) \\
 &= u_n v_{n-2}^{[2]} - v_n u_{n-2}^{[2]}.
 \end{aligned} \tag{2.10}$$

Using the same argument as before, we get

$$\begin{aligned}
 \Delta D_n^{[1]} &= v_{n-1}^{[2]} \Delta u_n + u_n \Delta v_{n-2}^{[2]} - u_{n-1}^{[2]} \Delta v_n - v_n \Delta u_{n-2}^{[2]} \\
 &= v_{n-1}^{[2]} \Delta u_n - u_{n-1}^{[2]} \Delta v_n.
 \end{aligned} \tag{2.11}$$

Hence

$$D_n^{[2]} = p_n \Delta D_n^{[1]} = u_n^{[1]} v_{n-1}^{[2]} - v_n^{[1]} u_{n-1}^{[2]}. \tag{2.12}$$

Finally

$$\begin{aligned}
 \Delta D_n^{[2]} &= u_n^{[1]} \Delta v_{n-1}^{[2]} + v_n^{[2]} \Delta u_n^{[1]} - v_n^{[1]} \Delta u_{n-1}^{[2]} - u_n^{[2]} \Delta v_n^{[1]} \\
 &= u_n^{[1]} q_n v_{n+1} + r_{n+1} \Delta v_n^{[1]} \Delta u_n^{[1]} - v_n^{[1]} q_n u_{n+1} - r_{n+1} \Delta u_n^{[1]} \Delta v_n^{[1]} \\
 &= -q_n \left[ v_n^{[1]} (\Delta u_n + u_n) - u_n^{[1]} (\Delta v_n + v_n) \right] \\
 &= -q_n (p_n \Delta v_n \Delta u_n + u_n v_n^{[1]} - p_n \Delta u_n \Delta v_n - v_n u_n^{[1]}) = -q_n D_{n+1},
 \end{aligned} \tag{2.13}$$

that is,  $D_n$  is a solution of  $(E)$ . □

Relationship between solutions of  $(E)$  and  $(E^A)$  described in Theorem 2.1 is a discrete analogue of the relationship valid for the differential (1.2) and its adjoint (1.3). For this reason, we call  $(E^A)$  the *adjoint equation* to  $(E)$ . This is in accordance with the definition of the adjoint system to the difference system as the following remark shows.

*Remark 2.2.* According to [1, page 60], if  $X = \{X_n\}$  is a nontrivial solution of the system

$$X_{n+1} = A_n X_n, \quad (2.14)$$

then  $U = \{U_n\}$ , where  $U_n = (X_n^T)^{-1}$  is a solution of the system

$$U_n = A_n^T U_{n+1}. \quad (2.15)$$

System (2.15) is called the *adjoint system* of (2.14).

Equation (E) can be written as a first-order difference system

$$\begin{aligned} \Delta x_n^{[0]} &= \frac{1}{r_n} x_n^{[1]}, \\ \Delta x_n^{[1]} &= \frac{1}{p_n} x_n^{[2]}, \\ \Delta x_n^{[2]} &= -q_n x_{n+1}^{[0]}, \end{aligned} \quad (2.16)$$

for the vector  $X_n = (x_n^{[0]}, x_n^{[1]}, x_n^{[2]})$ . Since  $x_{n+1}^{[0]} = x_n^{[0]} + \Delta x_n^{[0]}$ , we have

$$\Delta x_n^{[2]} = -q_n \left( x_n^{[0]} + \frac{1}{r_n} x_n^{[1]} \right). \quad (2.17)$$

Using the usual convention that no index actually means the index  $n$ , otherwise the index is explicitly specified, we obtain

$$\begin{pmatrix} l x_{n+1}^{[0]} \\ x_{n+1}^{[1]} \\ x_{n+1}^{[2]} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{r} & 0 \\ 0 & 1 & \frac{1}{p} \\ -q & -\frac{q}{r} & 1 \end{pmatrix} \begin{pmatrix} l x^{[0]} \\ x^{[1]} \\ x^{[2]} \end{pmatrix}. \quad (2.18)$$

Hence (E) can be interpreted as the system of the form (2.14). Its adjoint system is

$$\begin{pmatrix} l u^{[0]} \\ u^{[1]} \\ u^{[2]} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -q \\ \frac{1}{r} & 1 & -\frac{q}{r} \\ 0 & \frac{1}{p} & 1 \end{pmatrix} \begin{pmatrix} l u_{n+1}^{[0]} \\ u_{n+1}^{[1]} \\ u_{n+1}^{[2]} \end{pmatrix}. \quad (2.19)$$

From here we get

$$\begin{aligned} \Delta u_n^{[0]} &= q_n u_{n+1}^{[2]}, \\ \Delta u_n^{[1]} &= -\frac{1}{r_n} u_{n+1}^{[0]} + \frac{q_n}{r_n} u_{n+1}^{[2]}, \\ \Delta u_n^{[2]} &= -\frac{1}{p_n} u_{n+1}^{[1]}, \end{aligned} \quad (2.20)$$

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and the last equation gives  $\Delta u_{n+1}^{[1]} = -\Delta(p_n \Delta u_n^{[2]})$ . Replacing the shift  $n$  by  $n+1$  and substituting into the second equation, we have

$$-\Delta(p_n \Delta u_n^{[2]}) = -\frac{1}{r_{n+1}} u_{n+2}^{[0]} + \frac{q_{n+1}}{r_{n+1}} u_{n+2}^{[2]}. \quad (2.21)$$

Multiplying this equation by  $-r_{n+1}$  and differentiating it, we obtain

$$\Delta(r_{n+1} \Delta(p_n \Delta u_n^{[2]})) + \Delta(q_{n+1} u_{n+2}^{[2]}) = \Delta u_{n+2}^{[0]}. \quad (2.22)$$

Substituting from the first equation in (2.20), we get

$$\Delta(r_{n+1} \Delta(p_n \Delta u_n^{[2]})) + q_{n+2} u_{n+3}^{[2]} - q_{n+1} u_{n+2}^{[2]} = q_{n+2} u_{n+3}^{[2]}, \quad (2.23)$$

which means that the sequence  $v_n = u_n^{[2]}$  satisfies (E<sup>A</sup>).

*Notation 2.3.* Let  $S$  denote the solution space of (E) and let  $S^*$  denote the solution space of (E<sup>A</sup>). For  $(x, u) \in S \times S^*$ , define  $\mathcal{L} = (\mathcal{L}_n)$ , where

$$\mathcal{L}_n = \mathcal{L}(x_n, u_n) = x_{n+1} u_n^{[2]} - x_{n+1}^{[1]} u_n^{[1]} + x_{n+1}^{[2]} u_{n+1}. \quad (2.24)$$

The functional  $\mathcal{L}$  has the following properties.

**LEMMA 2.4.** *The sequence  $\mathcal{L} : S \times S^* \rightarrow \mathbb{R}$  is a constant which depends only on the choice of solutions  $x$  and  $u$ , and not on  $n$ .*

*Proof.* By a direct computation we get

$$\begin{aligned} \Delta \mathcal{L}_n &= \Delta(x_{n+1} u_n^{[2]} - x_{n+1}^{[1]} u_n^{[1]} + x_{n+1}^{[2]} u_{n+1}) \\ &= x_{n+2} \Delta u_n^{[2]} + u_n^{[2]} \Delta x_{n+1} - x_{n+1}^{[1]} \Delta u_n^{[1]} - u_{n+1}^{[1]} \Delta x_{n+1} \\ &\quad + u_{n+2} \Delta x_{n+1}^{[2]} + x_{n+1}^{[2]} \Delta u_{n+1} \\ &= x_{n+2} q_{n+1} u_{n+2} + r_{n+1} \Delta u_n^{[1]} \Delta x_{n+1} - r_{n+1} \Delta x_{n+1} \Delta u_n^{[1]} \\ &\quad - p_{n+1} \Delta u_{n+1} \Delta x_{n+1}^{[1]} - u_{n+2} q_{n+1} x_{n+2} + p_{n+1} \Delta x_{n+1}^{[1]} \Delta u_{n+1} = 0, \end{aligned} \quad (2.25)$$

which completes the proof.  $\square$

**LEMMA 2.5.** *Let  $x, y, z$  be solutions of (E). Let  $C$  and  $\mathcal{L}$  be defined by (2.1) and (2.24), respectively. Then the sequence  $R = (R_n)$ , where*

$$R_n = \begin{vmatrix} x_n & y_n & z_n \\ x_n^{[1]} & y_n^{[1]} & z_n^{[1]} \\ x_n^{[2]} & y_n^{[2]} & z_n^{[2]} \end{vmatrix} \quad (2.26)$$

satisfies

$$R_n = \mathcal{L}(z_{n-1}, C_{n-1}). \quad (2.27)$$

*Proof.* Expanding  $R_n$  along its third column, we obtain

$$R_n = z_n \begin{vmatrix} x_n^{[1]} & y_n^{[1]} \\ x_n^{[2]} & y_n^{[2]} \end{vmatrix} - z_n^{[1]} \begin{vmatrix} x_n & y_n \\ x_n^{[2]} & y_n^{[2]} \end{vmatrix} + z_n^{[2]} \begin{vmatrix} x_n & y_n \\ x_n^{[1]} & y_n^{[1]} \end{vmatrix}. \quad (2.28)$$

Using (2.24), we have

$$\mathcal{L}(z_{n-1}, C_{n-1}) = z_n C_{n-1}^{[2]} - z_n^{[1]} C_{n-1}^{[1]} + z_n^{[2]} C_n. \quad (2.29)$$

From here, (2.1), (2.5), and (2.7) show that (2.27) holds.  $\square$

### 3. Nonoscillatory solutions of adjoint equations

In this section, we study nonoscillatory solutions. We start with the following auxiliary results.

LEMMA 3.1. *There always exists nonoscillatory solution  $u$  of  $(E^A)$  with the property*

$$u_n > 0, \quad u_n^{[1]} > 0, \quad u_n^{[2]} > 0 \quad \text{for } n \in \mathbb{N}, \quad (3.1)$$

*that is,  $(E^A)$  has a strongly monotone solution.*

For the proof, see [4, Theorem 3.2].

LEMMA 3.2. *If a solution  $y$  of  $(E)$  satisfies for some integer  $m > 1$  that*

$$y_m \geq 0, \quad y_m^{[1]} \leq 0, \quad y_m^{[2]} > 0, \quad (3.2)$$

*then*

$$y_k > 0, \quad y_k^{[1]} < 0, \quad y_k^{[2]} > 0 \quad (3.3)$$

*for each  $k \in \mathbb{N}$  such that  $1 \leq k < m$ .*

The proof follows from the proof of [3, Proposition 2].

The existence of Kneser solutions of  $(E)$  is ensured by the following result.

THEOREM 3.3. *There always exists nonoscillatory solution  $x$  of  $(E)$  with the property*

$$x_n > 0, \quad x_n^{[1]} < 0, \quad x_n^{[2]} > 0 \quad \text{for } n \in \mathbb{N}, \quad (3.4)$$

*that is,  $(E)$  has a Kneser solution.*

*Proof.* Let  $x = (x(n))$ ,  $y = (y(n))$ ,  $z = (z(n))$  be a basis of the solution space  $S$  of  $(E)$ . For  $k \in \mathbb{N}$ , define

$$\omega_k(n) = a_k x(n) + b_k y(n) + c_k z(n), \quad (3.5)$$

where  $a_k, b_k, c_k$  are chosen such that

$$\omega_k(k) = 0, \quad \omega_k(k+1) = 0, \quad a_k^2 + b_k^2 + c_k^2 = 1. \quad (3.6)$$

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Then  $\omega_k^{[1]}(k) = 0$ . By [3, Lemma 1],  $\omega_k(k+2) \neq 0$ . Without loss of generality, assume that  $\omega_k(k+2) > 0$ . Then

$$\omega_k^{[1]}(k+1) = r_{k+1}\Delta\omega_k(k+1) = r_{k+1}(\omega_k(k+2) - \omega_k(k+1)) > 0, \quad (3.7)$$

hence

$$\omega_k^{[2]}(k) = p_k\Delta\omega_k^{[1]}(k) = p_k(\omega_k^{[1]}(k+1) - \omega_k^{[1]}(k)) > 0. \quad (3.8)$$

Since

$$\omega_k(k) = 0, \quad \omega_k^{[1]}(k) = 0, \quad \omega_k^{[2]}(k) > 0, \quad (3.9)$$

by Lemma 3.2

$$\omega_k(n) > 0, \quad \omega_k^{[1]}(n) < 0, \quad \omega_k^{[2]}(n) > 0, \quad \text{for } 1 \leq n < k. \quad (3.10)$$

Put  $A_k = (a_k, b_k, c_k)$ . Then  $\|A_k\| = 1$  for each  $k$ . The unit ball is compact in  $\mathbb{R}^3$ , so  $(A_k)$  has a convergent subsequence  $(A_{k_i})$ . Denote

$$A = \lim_{i \rightarrow \infty} A_{k_i} = (a, b, c). \quad (3.11)$$

Then  $a^2 + b^2 + c^2 = 1$  and

$$\omega(n) = \lim_{i \rightarrow \infty} \omega_{k_i} = \lim (a_{k_i}x(n) + b_{k_i}y(n) + c_{k_i}z(n)) \quad (3.12)$$

is a nontrivial solution of (E). Then in view of (3.10) and the fact that  $k$  is arbitrary integer, we get

$$\omega(n) \geq 0, \quad \omega^{[1]}(n) \leq 0, \quad \omega^{[2]}(n) \geq 0 \quad \text{for } n \geq 1. \quad (3.13)$$

If  $\omega(n_0) = 0$  for some  $n = n_0$ , then  $\omega(n) = 0$  for all  $n \geq n_0$  which is a contradiction with the fact that  $\omega$  is a nontrivial solution. Thus  $\omega(n) > 0$  for every  $n \geq 1$ , and so

$$\Delta(\omega^{[2]}(n)) = -q(n)\omega(n+1) < 0 \quad \text{for } n \geq 1. \quad (3.14)$$

Hence,  $\omega^{[2]}$  is decreasing and so  $\omega^{[2]}(n) > 0$  for  $n \in \mathbb{N}$ . From here  $\Delta(\omega^{[1]}(n)) > 0$  for  $n \in \mathbb{N}$ , which implies that  $\omega^{[1]}$  is increasing and  $\omega^{[1]}(n) < 0$  for  $n \in \mathbb{N}$ .  $\square$

**THEOREM 3.4.** *Every nonoscillatory solution of  $(E^A)$  is strongly monotone if and only if every nonoscillatory solution of (E) is a Kneser solution.*

*Proof.* Let every nonoscillatory solution of  $(E^A)$  be strongly monotone. Assume by contradiction that there exists solution  $y$  of (E) which belongs to the class  $N_i$ , where  $i \in \{1, 2, 3\}$ . Let  $x$  be a Kneser solution of (E). Without loss of generality, we may suppose that  $x_n > 0$  and  $y_n > 0$  for large  $n$ . Then the sequence  $C$  defined by (2.1) is according to

Theorem 2.1 solution of  $(E^A)$  and in view of (2.5) and (2.7) it satisfies, for large  $n$ ,

$$\begin{aligned} C_{n-1} > 0, & \quad C_{n-1}^{[1]} < 0 \quad (\text{if } i = 1) \\ C_{n-1} > 0, & \quad C_{n-1}^{[2]} < 0 \quad (\text{if } i = 2) \\ C_{n-1}^{[1]} < 0, & \quad C_{n-1}^{[2]} > 0 \quad (\text{if } i = 3). \end{aligned} \quad (3.15)$$

This is a contradiction with the fact that  $C$  is strongly monotone solution.

Now suppose that every solution of  $(E)$  is a Kneser solution. Assume by contradiction that there exists solution  $v$  of  $(E^A)$  which belongs to the class  $N_i$ , where  $i \in \{0, 1, 3\}$ . Let  $u$  be a strongly monotone solution of  $(E^A)$ . Without loss of generality, we may suppose that  $u_n > 0$  and  $v_n > 0$  for large  $n$ . Then the sequence  $D$  defined by (2.2) is according to Theorem 2.1 solution of  $(E)$  and it satisfies, for large  $n$ ,

$$\begin{aligned} D_n < 0, & \quad D_n^{[2]} > 0 \quad (\text{if } i = 0) \\ D_n^{[1]} < 0, & \quad D_n^{[2]} < 0 \quad (\text{if } i = 1) \\ D_n < 0, & \quad D_n^{[1]} < 0 \quad (\text{if } i = 3). \end{aligned} \quad (3.16)$$

This is a contradiction with the fact that  $D$  is a Kneser solution.  $\square$

#### 4. Oscillatory properties of adjoint equations

LEMMA 4.1. *Let  $u$  be a strongly monotone solution and  $v$  an oscillatory solution of  $(E^A)$ . Then their Casoratian  $D$  defined by (2.2) is an oscillatory solution of  $(E)$ .*

*Proof.* By Theorem 2.1,  $D$  is a solution of  $(E)$ . We will show that  $D$  is an oscillatory solution. Without loss of generality, we may suppose that  $u$  satisfies (3.1). Since  $v$  is an oscillatory solution, there exist increasing sequences of positive integers  $(i_n)$  and  $(j_n)$ , with properties

$$\begin{aligned} v_{i_n} &\leq 0, & v_{i_n}^{[1]} &> 0 & \text{for } n \in \mathbb{N}, \\ v_{j_n} &\geq 0, & v_{j_n}^{[1]} &< 0 & \text{for } n \in \mathbb{N}. \end{aligned} \quad (4.1)$$

From the above inequalities, (2.2), and (3.1), we have

$$D_{i_n+1} = u_{i_n} v_{i_n}^{[1]} - v_{i_n} u_{i_n}^{[1]} > 0 \quad \text{for } n \in \mathbb{N}, \quad (4.2)$$

and similarly  $D_{j_n+1} < 0$  for  $n \in \mathbb{N}$ . Hence the sequence  $D$  is an oscillatory solution of  $(E)$ .  $\square$

LEMMA 4.2. *Let  $x$  be a Kneser solution and  $y$  an oscillatory solution of  $(E)$ . Then their Casoratian  $C$  defined by (2.1) is an oscillatory solution of  $(E^A)$ .*

*Proof.* By Theorem 2.1,  $C$  is a solution of  $(E^A)$ . We will show that  $C$  is an oscillatory solution. Without loss of generality, we may suppose that  $x$  satisfies (3.4). Because  $y$  is an oscillatory solution, there exist increasing sequences of positive integers  $(i_n)_1^\infty$  and  $(j_n)_1^\infty$

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with properties

$$\begin{aligned} i_1 > M, \quad y_{i_n} \leq 0, \quad y_{i_n}^{[1]} > 0 \quad \text{for } n \in \mathbb{N}, \\ j_1 > M, \quad y_{j_n} \geq 0, \quad y_{j_n}^{[1]} < 0 \quad \text{for } n \in \mathbb{N}, \end{aligned} \quad (4.3)$$

where  $M = \min\{n \in \mathbb{N} : y_n y_{n+1} \leq 0\}$ .

Assume that  $y_{j_n}^{[2]} > 0$  for some  $n \in \mathbb{N}$ . Then by Lemma 3.2, we get

$$y_k > 0, \quad y_k^{[1]} < 0 \quad \text{for } 1 \leq k < j_n, \quad (4.4)$$

which is a contradiction with  $j_1 > M$ . Hence  $y_{j_n}^{[2]} \leq 0$  for  $n \in \mathbb{N}$ . From here and using (2.7) follows

$$C_{j_n-1}^{[2]} = x_{j_n}^{[1]} y_{j_n}^{[2]} - y_{j_n}^{[1]} x_{j_n}^{[2]} > 0 \quad \text{for } n \in \mathbb{N}. \quad (4.5)$$

By similar argument as before, we obtain  $y_{i_n}^{[2]} \geq 0$  for  $n \in \mathbb{N}$ , which implies that

$$C_{i_n-1}^{[2]} = x_{i_n}^{[1]} y_{i_n}^{[2]} - y_{i_n}^{[1]} x_{i_n}^{[2]} < 0 \quad \text{for } n \in \mathbb{N}. \quad (4.6)$$

By [3, Lemma 2], it follows from inequalities (4.5) and (4.6) that  $C$  is an oscillatory solution of  $(E^A)$ . The proof is now complete.  $\square$

Our next result characterizes the existence of oscillatory solutions of the adjoint equations.

**THEOREM 4.3.** *Equation  $(E^A)$  is oscillatory if and only if  $(E)$  is oscillatory.*

The proof follows from Theorem 3.3 and Lemmas 3.1, 4.1, and 4.2.

In the sequel, we study the existence of an oscillatory solution in terms of second-order equations.

**THEOREM 4.4.** (a) *If  $u$  is a nonoscillatory solution of  $(E^A)$ , then two linearly independent solutions of  $(E)$  satisfy the second-order difference equation*

$$p_{n+1} \Delta \left( \frac{r_{n+1} \Delta x_{n+1}}{u_{n+1}} \right) + \frac{u_n^{[2]}}{u_{n+1} u_{n+2}} x_{n+2} = 0. \quad (4.7)$$

(b) *If  $x$  is a nonoscillatory solution of  $(E)$ , then two linearly independent solutions of  $(E^A)$  satisfy the second-order difference equation*

$$r_{n+1} \Delta \left( \frac{p_n \Delta u_n}{x_{n+1}} \right) + \frac{x_{n+1}^{[2]}}{x_{n+1} x_{n+2}} u_{n+1} = 0. \quad (4.8)$$

*Proof.* Claim (a). Let  $u$  be a fixed nonoscillatory solution of  $(E^A)$  such that  $u_n > 0$  for  $n \geq N$ . Let  $L : S \rightarrow \mathbb{R}$  be the functional on  $S$  defined by  $L(x) = \mathcal{L}(x_n, u_n)$ . The set

$$K = \{x \in S : L(x) = 0\} \quad (4.9)$$

is the kernel of linear functional  $L$  defined on  $S$ . Then  $x \in K$  satisfies

$$u_{n+1}x_{n+1}^{[2]} - u_n^{[1]}x_{n+1}^{[1]} + u_n^{[2]}x_{n+1} = 0. \quad (4.10)$$

Multiplying the last equation by  $(u_{n+1}u_{n+2})^{-1}$ , we get

$$\frac{u_{n+1}x_{n+1}^{[2]} - u_n^{[1]}x_{n+1}^{[1]}}{u_{n+1}u_{n+2}} + \frac{u_{n+1}^{[1]}x_{n+1}^{[1]} - u_n^{[1]}x_{n+1}^{[1]}}{u_{n+1}u_{n+2}} + \frac{u_n^{[2]}x_{n+1}}{u_{n+1}u_{n+2}} = 0. \quad (4.11)$$

From here and using

$$\begin{aligned} p_{n+1}\Delta\left(\frac{x_{n+1}^{[1]}}{u_{n+1}}\right) &= \frac{u_{n+1}p_{n+1}\Delta x_{n+1}^{[1]} - x_{n+1}^{[1]}p_{n+1}\Delta u_{n+1}}{u_{n+1}u_{n+2}} \\ &= \frac{u_{n+1}x_{n+1}^{[2]} - u_n^{[1]}x_{n+1}^{[1]}}{u_{n+1}u_{n+2}}, \end{aligned} \quad (4.12)$$

we obtain

$$p_{n+1}\Delta\left(\frac{x_{n+1}^{[1]}}{u_{n+1}}\right) + \frac{x_{n+1}^{[1]}\Delta u_n^{[1]}}{u_{n+1}u_{n+2}} + \frac{u_n^{[2]}x_{n+1}}{u_{n+1}u_{n+2}} = 0. \quad (4.13)$$

In view of the identity

$$x_{n+1}^{[1]}\Delta u_n^{[1]} + u_n^{[2]}x_{n+1} = u_n^{[2]}\Delta x_{n+1} + u_n^{[2]}x_{n+1} = u_n^{[2]}x_{n+2}, \quad (4.14)$$

(4.13) can be rewritten in the form (4.7). Since  $\dim K = \dim S - 1 = 2$ , we get the conclusion.

Claim (b). Let  $x$  be a fixed nonoscillatory solution of (E) such that  $x_n > 0$  for  $n \geq N$ . Let  $L^* : S^* \rightarrow \mathbb{R}$  be the functional on  $S$  defined by  $L^*(u) = \mathcal{L}(x_n, u_n)$ . The set

$$K^* = \{u \in S^* : L^*(u) = 0\} \quad (4.15)$$

is the kernel of linear functional defined on  $S^*$ . Then  $u \in K^*$  satisfies

$$x_{n+1}u_n^{[2]} - x_{n+1}^{[1]}u_n^{[1]} + x_{n+1}^{[2]}u_{n+1} = 0. \quad (4.16)$$

Multiplying the last equation by  $(x_{n+1}x_{n+2})^{-1}$ , we get

$$\frac{x_{n+1}u_n^{[2]} - x_{n+1}^{[1]}u_n^{[1]}}{x_{n+1}x_{n+2}} + \frac{x_{n+1}^{[2]}u_{n+1}}{x_{n+1}x_{n+2}} = 0. \quad (4.17)$$

From here using

$$\begin{aligned} r_{n+1}\Delta\left(\frac{u_n^{[1]}}{x_{n+1}}\right) &= \frac{x_{n+1}r_{n+1}\Delta u_n^{[1]} - u_n^{[1]}r_{n+1}\Delta x_{n+1}}{x_{n+1}x_{n+2}} \\ &= \frac{x_{n+1}u_n^{[2]} - x_{n+1}^{[1]}u_n^{[1]}}{x_{n+1}x_{n+2}}, \end{aligned} \quad (4.18)$$

we get (4.8). Since  $\dim K^* = \dim S^* - 1 = 2$ , we get the conclusion.  $\square$

COROLLARY 4.5. (a) *If (E) is oscillatory, then there exists a basis for S consisting of one nonoscillatory solution and two oscillatory solutions.*

(b) *If (E<sup>A</sup>) is oscillatory, then there exists a basis for S\* consisting of one nonoscillatory solution and two oscillatory solutions.*

*Proof.* By Theorem 3.3 and Lemma 3.1, there exist Kneser solution of (E) and strongly monotone solution of (E<sup>A</sup>). Assume that (E) and (E<sup>A</sup>) are oscillatory. In view of Theorem 4.4 and its proof, (E) has two independent solutions which must be oscillatory. Similarly, there exist two independent oscillatory solutions of (E<sup>A</sup>). Because  $\dim S = \dim S^* = 3$ , the proof is complete.  $\square$

THEOREM 4.6. (a) *If (4.7), where u is a nonoscillatory solution of (E<sup>A</sup>), is oscillatory, then (E) and (E<sup>A</sup>) are oscillatory.*

(b) *If (4.8), where x is a nonoscillatory solution of (E), is oscillatory, then (E) and (E<sup>A</sup>) are oscillatory.*

*Proof.* Assume that (4.7) is oscillatory, that is, there exists an oscillatory solution x of (4.7). Using the same argument as in the proof of Theorem 4.4, we obtain that  $x \in K$ , where K is defined by (4.9). Hence x is an oscillatory solution of (E). By Theorem 4.3, (E<sup>A</sup>) is oscillatory, too.

Similarly, if u is an oscillatory solution of (4.8), then  $u \in K^*$ . Hence u is oscillatory solution of (E<sup>A</sup>) and this implies that (E) is oscillatory too.  $\square$

THEOREM 4.7. *Equation (E) is oscillatory if and only if (4.8) is oscillatory.*

*Proof.* Assume that (E) is oscillatory, that is, there exists an oscillatory solution y of (E). By Theorem 3.3, there exists a Kneser solution x of (E). According to (2.27),

$$0 = \begin{vmatrix} x_n & y_n & x_n \\ x_n^{[1]} & y_n^{[1]} & x_n^{[1]} \\ x_n^{[2]} & y_n^{[2]} & x_n^{[2]} \end{vmatrix} = \mathcal{L}(x_{n-1}, C_{n-1}), \quad (4.19)$$

where C is defined by (2.1). By Theorem 2.1, C is solution of (E<sup>A</sup>). From Lemma 4.2 follows that C is an oscillatory solution of (E<sup>A</sup>). In view of Lemma 2.4,

$$L^*(C) = \mathcal{L}(x_n, C_n) = 0, \quad (4.20)$$

hence  $C \in K^*$ , where  $K^*$  is defined by (4.15). From here and the proof of Theorem 4.4, we get the fact that C is an oscillatory solution of (4.8). The opposite statement follows from Theorem 4.6.  $\square$

*Open problems.*

- (1) It is an open problem whether the existence of an oscillatory solution of (E<sup>A</sup>) implies the oscillation of (4.7). To solve this problem, it would be useful to find the functional  $\mathcal{L}^*$  defined on  $S^* \times S$  with similar properties to those of  $\mathcal{L}$  described in Lemmas 2.4 and 2.5.
- (2) To generalize results of this paper to the linear nth-order difference equations involving quasi-differences.

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