OSCILLATION OF SECOND-ORDER NEUTRAL DELAY AND MIXED-TYPE DYNAMIC EQUATIONS ON TIME SCALES

Y. ŞAHİNER

Received 31 January 2006; Revised 11 May 2006; Accepted 15 May 2006

We consider the equation $(r(t)(y^{\Delta}(t))^{\gamma})^{\Delta} + f(t,x(\delta(t))) = 0$, $t \in \mathbb{T}$, where $y(t) = x(t) + p(t)x(\tau(t))$ and γ is a quotient of positive odd integers. We present some sufficient conditions for neutral delay and mixed-type dynamic equations to be oscillatory, depending on deviating arguments $\tau(t)$ and $\delta(t)$, $t \in \mathbb{T}$.

1. Some preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced by Hilger [6] in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory, see [7] and the monographs by Bohner and Peterson [3, 4], and the references cited therein.

First, we give a short review of the time scales calculus extracted from [3]. For any $t \in \mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{1.1}$$

respectively. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$.

A point $t \in \mathbb{T}$ is said to be right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$. Also, t is said to be right scattered if $\sigma(t) > t$, left scattered if $t > \rho(t)$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left dense points in $\mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, if there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with $|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|$, for all $s \in U$, then f is Δ-differentiable at t, and we call α the derivative of f at t and denote

Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2006, Article ID 65626, Pages 1–9
DOI 10.1155/ADE/2006/65626

it by $f^{\Delta}(t)$,

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \tag{1.2}$$

if t is right scattered. When t is a right dense point, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}, \tag{1.3}$$

provided this limit exists.

If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}$, then f is continuous at t. Furthermore, we assume that $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable. The following formulas are useful:

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t), \qquad (fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t). \tag{1.4}$$

A function F with $F^{\Delta} = f$ is called an antiderivative of f, and then we define

$$\int_a^b f(t)\Delta t = F(b) - F(a), \tag{1.5}$$

where $a, b \in \mathbb{T}$. It is well known that rd-continuous functions possess antiderivatives.

Note that if $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, and

$$\int_a^b f(t)\Delta t = \int_a^b f(t) dt, \tag{1.6}$$

and if $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^{\Delta} = \Delta f$, and

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t). \tag{1.7}$$

If f is rd-continuous, then

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t). \tag{1.8}$$

2. Introduction

In this paper, we are concerned with the oscillatory behavior of the second-order neutral dynamic equation with deviating arguments

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + f(t, x(\delta(t))) = 0, \quad t \in \mathbb{T}, \tag{NE}$$

where $y(t) = x(t) + p(t)x(\tau(t))$, γ is a quotient of positive odd integers, $r, p \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ are positive functions, $\tau, \delta \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \delta(t) = \infty$, and $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $uf(t, u) > 0$ for all $u \ne 0$.

Unless otherwise is stated, throughout the paper, we assume the following conditions:

(H1) $0 \le p(t) < 1$,

(H2) $\int^{\infty} (1/r(t))^{1/\gamma} \Delta t = \infty$,

(H3) there exists a nonnegative function q defined on $\mathbb{T}$ such that $|f(t,u)| \ge q(t)|u|^{\gamma}$.

By a solution of (NE), we mean a nontrivial real-valued function x such that $x(t) + p(t)x(\tau(t))$ and $r(t)[(x(t)+p(t)x(\tau(t)))^{\Delta}]^{\gamma}$ are defined and Δ-differentiable for $t \in \mathbb{T}$, and satisfy (NE) for $t \ge t_0 \in \mathbb{T}$. A solution x has a generalized zero at t in case $x(t) = 0$. We say x has a generalized zero on $[a,b]$ in case $x(t)x(\sigma(t)) < 0$ or $x(t) = 0$ for some $t \in [a,b)$, where $a, b \in \mathbb{T}$ and $a \le b$ (x has a generalized zero at b, in case $x(\rho(b))x(b) < 0$ or $x(b) = 0$). A nontrivial solution of (NE) is said to be oscillatory on $[t_x, \infty)$ if it has infinitely many generalized zeros when $t \ge t_x$; otherwise it is called nonoscillatory. Finally, (NE) is called oscillatory if all its solutions are oscillatory.

In recent years, there has been a great deal of work on the oscillatory behavior of solutions of some second-order dynamic equations. To the best of our knowledge, there is very little known about the oscillatory behavior of (NE). Indeed, there are not many results about nonneutral second-order equation in the form of (NE) when $p(t) \equiv 0$. For some oscillation criteria, we refer the reader to the papers [1, 2, 9, 12] and references cited therein.

Subject to our corresponding conditions, Agarwal et al. [2] considered the second-order neutral delay dynamic equation

$$\left(r(t)\left(\left[x(t)+p(t)x(t-\tau)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + f(t, x(t-\delta)) = 0, \tag{2.1}$$

where τ and δ are positive constants. A part of this study contains two main theorems proven by the technique of reduction of order. Previously obtained result about oscillation of first-order delay dynamic equation

$$z^{\Delta}(t) + Q(t)z(h(t)) = 0 \tag{2.2}$$

is used to be compared with (2.1). One of them is the following which is auxiliary for the proof of the first theorem in [2].

Lemma 2.1 [11, Corollary 2]. *Assume $h(t) < t$. Define*

$$\alpha := \limsup_{t\to\infty} \sup_{\lambda \in E_Q} \{\lambda e_{-\lambda Q}(h(t),t)\}, \tag{2.3}$$

where $E_Q = \{\lambda \mid \lambda > 0,\ 1 - \lambda Q(t)\mu(t) > 0,\ t \in \mathbb{T}\}$, and

$$e_{-\lambda Q}(h(t),t) = \exp\int_{h(t)}^{t} \xi_{\mu(s)}(-\lambda Q(s))\Delta s,$$

$$\xi_l(z) = \begin{cases} \dfrac{\log(1+lz)}{l} & \text{if } l \ne 0, \\ z & \text{if } l = 0. \end{cases} \tag{2.4}$$

If $\alpha < 1$, then every solution of (2.2) is oscillatory.

Theorem 2.2 [2, Theorem 3.2]. *Assume that $r^{\Delta}(t) \geq 0$. Then every solution of (2.1) oscillates if*

$$\limsup_{t\to\infty} \sup_{\lambda\in E_A} \{\lambda e_{-\lambda A}(t-\delta,t)\} < 1, \tag{2.5}$$

where

$$A(t) = \frac{q(t)[1-p(t-\delta)]^{\gamma}}{r(t-\delta)} \left(\frac{t-\delta}{2}\right)^{\gamma}. \tag{2.6}$$

Theorem 2.3 [2, Theorem 3.3]. *Assume that $r^{\Delta}(t) \geq 0$. Then every solution of (2.1) oscillates if*

$$\limsup_{t\to\infty} \int_{t-\delta}^{t} A(s)\Delta s > 1. \tag{2.7}$$

Note that the monotonicity condition imposed on r is quite restrictive and therefore Theorem 2.3 applies only to a special class of neutral-type dynamic equations. Also, $\tau(t) = t - \tau$ and $\delta(t) = t - \delta$ being just linear functions cause further restrictions.

The above results are of special importance for us and in fact they motivate our study in this paper. Our purpose here, first of all, is to show that the conclusions of Theorems 2.2 and 2.3 are valid without the monotonicity condition on r and requirements $\tau(t) = t - \tau$ and $\delta(t) = t - \delta$. In the next section, we present some new oscillation criteria under very mild conditions and more general assumptions to extend the above results for the neutral delay and mixed dynamic equations.

3. Main results

Since we deal with the oscillatory behavior of (NE) on time scales, throughout the paper, we assume that the time scale $\mathbb{T}$ under consideration satisfies $\sup \mathbb{T} = \infty$. We label (NE) as $(\mathrm{NE})_d$ or $(\mathrm{NE})_m$ that refers to neutral delay or mixed dynamic equation if $\delta(t) < t$ or $\delta(t) > t$, respectively.

Theorem 3.1. *Let $E = \{\lambda \mid \lambda > 0,\ 1 - \lambda g(t)\mu(t) > 0\}$. Assume that $\delta(t) < t$. If*

$$\limsup_{t\to\infty} \sup_{\lambda\in E} \{\lambda e_{-\lambda g}(\delta(t),t)\} < 1, \tag{3.1}$$

where $g(t) = [1 - p(\delta(t))]^{\gamma} q(t)$, then $(\mathrm{NE})_d$ is oscillatory.

Proof. Assume, for the sake of contradiction, that $(\mathrm{NE})_d$ has a nonoscillatory solution $x(t)$. We may assume that $x(t)$ is eventually positive, since the proof when $x(t)$ is eventually negative is similar. Because $\delta(t)$, $\tau(t) \to \infty$ as $t \to \infty$, there exists a positive number $t_1 \geq t_0$, such that $x(\delta(t)) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. We also see that $y(t) > 0$ for $t \geq t_1$. We may claim that $y^{\Delta}(t)$ has eventually a fixed sign. If y^{Δ} has a generalized zero on $I = [t_2, \sigma(t_2))$ for some $t_2 > t_1$, then

$$\left(r(t)(y^{\Delta}(t))^{\gamma}\right)^{\Delta}\Big|_I = -f(t, x(\delta(t))) < 0, \tag{3.2}$$

which implies that $y^{\Delta}(t)$ cannot have another generalized zero after it vanishes or changes sign once on the interval I. Suppose that $y^{\Delta}(t) < 0$ for $t \geq t_3 \geq \sigma(t_2)$. It is easy to see from $(\mathrm{NE})_d$ that $r(t)(y^{\Delta}(t))^{\gamma}$ is nonincreasing. So we have

$$r(t)\left(y^{\Delta}(t)\right)^{\gamma} \leq r(t_3)\left(y^{\Delta}(t_3)\right)^{\gamma} = d < 0, \quad t \geq t_3. \tag{3.3}$$

Integration from t_3 to t yields

$$y(t) \leq y(t_3) + d^{1/\gamma} \int_{t_3}^{t} \frac{1}{(r(s))^{1/\gamma}} \Delta s. \tag{3.4}$$

In view of (H2), it follows from (3.4) that the function $y(t)$ takes on negative values for sufficiently large values of t. This contradicts the fact that $y(t)$ is eventually positive, we must have $y^{\Delta}(t) > 0$ for $t \geq t_3$. Using this fact together with $\tau(t) \leq t$ and $x(t) < y(t)$, we see that

$$y(t) = x(t) + p(t)x(\tau(t)) \leq x(t) + p(t)y(\tau(t)) \leq x(t) + p(t)y(t) \tag{3.5}$$

or

$$x(t) \geq [1 - p(t)]y(t), \quad t \geq t_3. \tag{3.6}$$

Because of (H2), we have for sufficiently large $t \geq t_3$,

$$\int_{t_3}^{t} \frac{1}{r^{1/\gamma}(s)} \Delta s > 1. \tag{3.7}$$

By the nonincreasing property of $r^{1/\gamma} y^{\Delta}$,

$$\begin{aligned} y(t) &= y(t_3) + \int_{t_3}^{t} y^{\Delta}(s) \Delta s \\ &\geq \int_{t_3}^{t} \frac{1}{r^{1/\gamma}(s)} \left[r^{1/\gamma}(s) y^{\Delta}(s)\right] \Delta s \geq r^{1/\gamma}(t) y^{\Delta}(t) \int_{t_3}^{t} \frac{1}{r^{1/\gamma}(s)} \end{aligned} \tag{3.8}$$

and using (3.7), we get

$$y(t) \geq r^{1/\gamma}(t) y^{\Delta}(t), \quad t \geq t_3. \tag{3.9}$$

There exists a number $t_* = \delta(t_3) < t_3 \leq t$ such that the following holds from inequalities (3.6) and (3.9):

$$x(\delta(t)) \geq [1 - p(\delta(t))] r^{1/\gamma}(\delta(t)) y^{\Delta}(\delta(t)), \quad t \geq t_*. \tag{3.10}$$

In view of $(\mathrm{NE})_d$ and (H3), we have

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q(t) x^{\gamma}(\delta(t)) \leq 0. \tag{3.11}$$

Substituting (3.10) into the last inequality, we obtain for $t \geq t_*$,

$$z^{\Delta}(t) + [1 - p(\delta(t))]^{\gamma} q(t) z(\delta(t)) \leq 0, \tag{3.12}$$

where $z(t) = r(t)(y^{\Delta}(t))^{\gamma}$ is an eventually positive solution. This contradicts condition (3.1), the proof is complete. □

Remark 3.2. In case that $\mathbb{T} = \mathbb{N}$, (2.2) reduces to the first-order delay difference equation

$$z_{n+1} - z_n + Q_n z_{n-h} = 0, \tag{3.13}$$

where $h_n = n - h$, $h \in \mathbb{N}$ and $n > h \geq 1$. Erbe and Zhang [5] proved that (3.13) is oscillatory provided that

$$\limsup_{n\to\infty} \sum_{i=n-h}^{n} Q_i > 1. \tag{3.14}$$

In the proof of Theorem 2.3, first (2.1) is reduced to a first-order delay dynamic equation in the form of (2.2) and then, by similar steps of the proof of well-known oscillation criterion given by Ladas et al. [8] for (2.2) when $\mathbb{T} = \mathbb{R}$, a contradiction is obtained in view of condition (2.7). But when $\mathbb{T} = \mathbb{N}$, considering definition (1.7), condition (2.7) is derived as

$$\limsup_{n\to\infty} \sum_{i=n-h}^{n-1} Q_i > 1 \tag{3.15}$$

which is not the same as condition (3.14).

To overcome this difficulty, we intend to use the following sufficient condition established by Şahiner and Stavroulakis [10] for (2.2) to be oscillatory on any time scale $\mathbb{T}$.

Lemma 3.3 [9, Theorem 2.4]. *Assume that $h(t) < t$. If*

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} Q(s)\Delta s > 1, \tag{3.16}$$

then (2.2) is oscillatory.

Theorem 3.4. *Assume that $\delta(t) < t$. If*

$$\limsup_{t\to\infty} \int_{\delta(t)}^{\sigma(t)} [1 - p(\delta(s))]^{\gamma} q(s)\Delta s > 1, \tag{3.17}$$

then $(NE)_d$ is oscillatory.

Proof. Suppose the contrary that x is a nonoscillatory solution of $(\mathrm{NE})_d$ and following the same steps as in Theorem 3.1, we obtain (3.12). The rest of the proof is exactly the same as that of Lemma 3.3, see [10]. The proof is complete. □

Remark 3.5. The above theorems are applicable even if r is not monotone and deviating arguments $\tau(t)$ and $\delta(t)$ are variable functions of t. Moreover, in case $r(t) > (t/2)^\gamma$ for sufficiently large t, Theorems 3.1 and 3.4 are stronger than Theorems 2.2 and 2.3.

Example 3.6. Consider the following neutral delay dynamic equation:

$$\left(\frac{1}{t}\left(\left[x(t)+p(t)x\left(\frac{t}{2}\right)\right]^{\Delta}\right)^{3}\right)^{\Delta}+q(t)x^{3}(\sqrt{t})=0. \tag{3.18}$$

$r(t)$ satisfies (H2) but it is not increasing. Moreover, delay terms $\tau(t) = t/2$ and $\delta(t) = \sqrt{t}$ are not in the form of $t-\tau$ and $t-\delta$ for any constants $\tau, \delta > 0$, respectively. Therefore, Theorems 2.2 and 2.3 cannot be applied to (3.18). On the other hand, if

$$\limsup_{t\to\infty}\sup_{\lambda\in E}\left\{\lambda e_{-\lambda g}(\sqrt{t},t)\right\}<1, \tag{3.19}$$

or

$$\limsup_{t\to\infty}\int_{\sqrt{t}}^{\sigma(t)}\left[1-p(\sqrt{t})\right]^{3}q(s)\Delta s>1 \tag{3.20}$$

is satisfied, then by Theorem 3.1 or 3.4, respectively, (3.18) is oscillatory.

Remember that (NE) is a mixed-type neutral dynamic equation when $\delta(t) > t$, because of that the equation contains both delay and advanced arguments. Now, we state some sufficient conditions for mixed-type neutral dynamic equations $(\mathrm{NE})_m$ to be oscillatory. We just give an outline for the proof of next theorem.

Theorem 3.7. *Assume that $\delta(t) > t$ and $\tau(\delta(t)) < t$. If*

$$\limsup_{t\to\infty}\sup_{\lambda\in E}\left\{\lambda e_{-\lambda g}(\tau(\delta(t)),t)\right\}<1, \tag{3.21}$$

where $g(t)$ and E are as defined in Theorem 3.1, then $(\mathrm{NE})_m$ is oscillatory.

Proof. Assume that $(\mathrm{NE})_m$ has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t)$ is eventually positive. Proceeding as in the proof of Theorem 3.1, it is known that $x(t) < y(t)$ and $y^{\Delta}(t) > 0$. Therefore, for sufficiently large t_4, we obtain instead of (3.6),

$$y(\tau(t)) \le y(t) = x(t)+p(t)x(\tau(t)) \le x(t)+p(t)y(\tau(t)) \tag{3.22}$$

or

$$x(t) \ge [1-p(t)]y(\tau(t)), \quad t \ge t_4. \tag{3.23}$$

Using this with inequality (3.9), we get

$$x(\delta(t)) \ge [1-p(\delta(t))]r^{1/\gamma}(\tau(\delta(t)))y^{\Delta}(\tau(\delta(t))). \tag{3.24}$$

At the end, we obtain

$$z^{\Delta}(t)+\left[1-p(\delta(t))\right]^{\gamma}q(t)z(\tau(\delta(t)))\leq 0, \tag{3.25}$$

where $z(t)=r(t)(y^{\Delta}(t))^{\gamma}$ is an eventually positive solution. This contradicts condition (3.29), the proof is complete. □

Theorem 3.8. *Assume that $\delta(t)>t$ and $\tau(\delta(t))<t$. If*

$$\limsup_{t\to\infty}\int_{\tau(\delta(t))}^{\sigma(t)}\left[1-p(\delta(s))\right]^{\gamma}q(s)\Delta s>1, \tag{3.26}$$

then (NE)$_d$ is oscillatory.

Example 3.9. Consider the following mixed-type neutral dynamic equation:

$$\left(\frac{1}{t}\left(\left[x(t)+\left(1-\frac{1}{t}\right)x(\sqrt{t})\right]^{\Delta}\right)^{1/3}\right)^{\Delta}+\frac{t}{\sigma(t)t^{1/3}}x^{1/3}\left(\frac{t^2}{64}\right)=0,\quad t\geq 9. \tag{3.27}$$

$r(t)$ satisfies (H2). Assumptions of Theorem 3.8 which are $\delta(t)=t^2/64>t$ and $\tau(\delta(t))=t/8<t$ hold for $t\geq 9$. Since

$$\int_{t/8}^{\sigma(t)}\left(1-\left(1-\frac{64}{s^2}\right)\right)^{1/3}\frac{s}{\sigma(s)s^{1/3}}\Delta s\geq\frac{t}{8}\int_{t/8}^{\sigma(t)}\frac{4}{s\sigma(s)}\Delta s=\frac{1}{2}\left(8-\frac{t}{\sigma(t)}\right)\geq\frac{7}{2}, \tag{3.28}$$

condition (3.26) is satisfied. Therefore (3.27) is oscillatory.

Remark 3.10. Theorems 3.7 and 3.8 are also valid for (NE)$_d$. If we assume $\tau(t)<t$ instead of $\tau(t)\leq t$, assumption $\tau(\delta(t))<t$ is already satisfied when $\delta(t)<t$ and the proofs do not change. Assumption $\tau(t)<t$ implies the immediate result $\tau(\delta(t))<\delta(t)$. Therefore, we conclude the following which are stronger conditions for neutral delay dynamic equation (NE)$_d$.

Corollary 3.11. *Assume that $\tau(t)<t$ and $\delta(t)<t$. If*

$$\limsup_{t\to\infty}\sup_{\lambda\in E}\left\{\lambda e_{-\lambda g}(\tau(\delta(t)),t)\right\}<1, \tag{3.29}$$

where $g(t)$ and E are as defined in Theorem 3.1, then (NE)$_d$ is oscillatory.

Corollary 3.12. *Assume that $\tau(t)<t$ and $\delta(t)<t$. If*

$$\limsup_{t\to\infty}\int_{\tau(\delta(t))}^{\sigma(t)}\left[1-p(\delta(s))\right]^{\gamma}q(s)\Delta s>1, \tag{3.30}$$

then (NE)$_d$ is oscillatory.

We note that obtained results in this section generalize and extend some sufficient conditions about oscillation previously established to neutral and nonneutral differential difference and dynamic equations.

References

[1] R. P. Agarwal, M. Bohner, and S. H. Saker, *Oscillation of second order delay dynamic equations*, to appear in The Canadian Applied Mathematics Quarterly.

[2] R. P. Agarwal, D. O'Regan, and S. H. Saker, *Oscillation criteria for second-order nonlinear neutral delay dynamic equations*, Journal of Mathematical Analysis and Applications **300** (2004), no. 1, 203–217.

[3] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser Boston, Massachusetts, 2001.

[4] M. Bohner and A. Peterson (eds.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Massachusetts, 2003.

[5] L. H. Erbe and B. G. Zhang, *Oscillation of discrete analogues of delay equations*, Differential and Integral Equations **2** (1989), no. 3, 300–309.

[6] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, Würzburg, 1988.

[7] ______, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results in Mathematics **18** (1990), no. 1-2, 18–56.

[8] G. Ladas, Ch. G. Philos, and Y. G. Sficas, *Sharp conditions for the oscillation of delay difference equations*, Journal of Applied Mathematics and Simulation **2** (1989), no. 2, 101–111.

[9] Y. Şahiner, *Oscillation of second-order delay differential equations on time scales*, Nonlinear Analysis **63** (2005), no. 5–7, e1073–e1080.

[10] Y. Şahiner and I. P. Stavroulakis, *Oscillations of first order delay dynamic equations*, to appear in Dynamic Systems and Applications.

[11] B. G. Zhang and X. Deng, *Oscillation of delay differential equations on time scales*, Mathematical and Computer Modelling **36** (2002), no. 11–13, 1307–1318.

[12] B. G. Zhang and Z. Shanliang, *Oscillation of second-order nonlinear delay dynamic equations on time scales*, Computers & Mathematics with Applications **49** (2005), no. 4, 599–609.

Y. Şahiner: Department of Mathematics , Atilim University, 06836 Incek-Ankara, Turkey
E-mail address: ysahiner@atilim.edu.tr