# ON THE SYSTEM OF RATIONAL DIFFERENCE EQUATIONS $x_{n+1} = f(y_{n-q}, x_{n-s}), y_{n+1} = g(x_{n-t}, y_{n-p})$

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We study the global behavior of positive solutions of the system of rational difference equations $x_{n+1} = f(y_{n-q}, x_{n-s})$, $y_{n+1} = g(x_{n-t}, y_{n-p})$, $n = 0,1,2,\ldots$, where $p,q,s,t \in \{0,1,2,\ldots\}$ with $s \geq t$ and $p \geq q$, the initial values $x_{-s}, x_{-s+1}, \ldots, x_0, y_{-p}, y_{-p+1}, \ldots, y_0 \in (0,+\infty)$. We give sufficient conditions under which every positive solution of this system converges to the unique positive equilibrium.

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## 1. Introduction

In this paper, we study the convergence of positive solutions of a system of rational difference equations. Recently there has been published quite a lot of works concerning the behavior of positive solutions of systems of rational difference equations [1–7, 9, 11]. Not only these results are valuable in their own right, but also they can provide insight into their differential counterparts.

Papaschinopoulos and Schinas [10] studied the oscillatory behavior, the periodicity, and the asymptotic behavior of the positive solutions of systems of rational difference equations

$$x_{n+1} = A + \frac{x_{n-1}}{y_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{x_n}, \quad n = 0,1,\ldots, \tag{1.1}$$

where $A \in (0,+\infty)$ and the initial values $x_{-1}, x_0, y_{-1}, y_0 \in (0,+\infty)$.

Recently, Kulenović and Nurkanović [8] investigated the global asymptotic behavior of solutions of systems of rational difference equations

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \quad y_{n+1} = \frac{d + y_n}{e + x_n}, \quad n = 0,1,\ldots, \tag{1.2}$$

where $a, b, d, e \in (0,+\infty)$ and the initial values $x_0, y_0 \in (0,+\infty)$.

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In this paper, we consider the more general equation

$$x_{n+1} = f(y_{n-q}, x_{n-s}), \qquad y_{n+1} = g(x_{n-t}, y_{n-p}), \tag{1.3}$$

where $p,q,s,t \in \{0,1,2,\ldots\}$ with $s \geq t$ and $p \geq q$, the initial values $x_{-s}, x_{-s+1}, \ldots, x_0, y_{-p}, y_{-p+1}, \ldots, y_0 \in (0,+\infty)$ and $f$ satisfies the following hypotheses.

($H_1$) $f(u,v), g(u,v) \in C(E \times E, (0,+\infty))$ with $a = \inf_{(u,v)\in E\times E} f(u,v) \in E$ and $b = \inf_{(u,v)\in E\times E} g(u,v) \in E$, where $E \in \{(0,+\infty), [0,+\infty)\}$.

($H_2$) $f(u,v)$ and $g(u,v)$ are decreasing in $u$ and increasing in $v$.

($H_3$) Equation

$$x = f(y,x), \qquad y = g(x,y) \tag{1.4}$$

has a unique positive solution $x = \overline{x}$, $y = \overline{y}$.

($H_4$) $f(b,x)$ has only one fixed point in the interval $(a,+\infty)$, denoted by $A$, and $g(a,y)$ has only one fixed point in the interval $(b,+\infty)$, denoted by $B$.

($H_5$) For every $w \in E$, $f(w,x)/x$ and $g(w,x)/x$ are nonincreasing in $x$ in $(0,+\infty)$.

## 2. Main results

Theorem 2.1. *Assume that ($H_1$)–($H_5$) hold and $\{(x_n, y_n)\}$ is a positive solution of (1.3), then there exists a positive integer $N$ such that*

$$f(B,a) \leq x_n \leq A, \quad g(A,b) \leq y_n \leq B, \quad \text{for } n \geq N. \tag{2.1}$$

*Proof.* Since $a = \inf_{(u,v)\in E\times E} f(u,v) \in E$ and $b = \inf_{(u,v)\in E\times E} g(u,v) \in E$, we have

$$\begin{aligned} \overline{x} &= f(\overline{y},\overline{x}) > f(\overline{y}+1,\overline{x}) \geq a, \\ \overline{y} &= g(\overline{x},\overline{y}) > g(\overline{x}+1,\overline{y}) \geq b. \end{aligned} \tag{2.2}$$

*Claim 1.* $g(A,b) < \overline{y} < B$ and $f(B,a) < \overline{x} < A$.

*Proof of Claim 1.* If $B \leq \overline{y}$, then it follows from ($H_2$), ($H_4$), and ($H_5$) that

$$B = g(a,B) > g(\overline{x},B) = B\frac{g(\overline{x},B)}{B} \geq B\frac{g(\overline{x},\overline{y})}{\overline{y}} = B, \tag{2.3}$$

which is a contradiction. Therefore $\overline{y} < B$. In a similar fashion it is true that $\overline{x} < A$.

Since $\overline{y} < B$ and $\overline{x} < A$, we have that

$$f(B,a) < f(\overline{y},\overline{x}) = \overline{x}, \qquad g(A,b) < g(\overline{x},\overline{y}) = \overline{y}, \tag{2.4}$$

Claim 1 is proven. □

*Claim 2.* (i) For all $n \geq q+1$, $x_{n+1} \leq x_{n-s}$ if $x_{n-s} > A$ and $x_{n+1} \leq A$ if $x_{n-s} \leq A$.

(ii) For all $n \geq t+1$, $y_{n+1} \leq y_{n-p}$ if $y_{n-p} > B$ and $y_{n+1} \leq B$ if $y_{n-p} \leq B$.

*Proof of Claim 2.* We only prove (i) (the proof of (ii) is similar). Obviously

$$x_{n+1} = f(y_{n-q}, x_{n-s}) \leq f(b, x_{n-s}). \tag{2.5}$$

If $x_{n-s} \leq A$, then $x_{n+1} \leq f(b, x_{n-s}) \leq f(b, A) = A$.
If $x_{n-s} > A$, then

$$\frac{f(b, x_{n-s})}{x_{n-s}} \leq \frac{f(b, A)}{A} = 1, \tag{2.6}$$

which implies $x_{n+1} \leq f(b, x_{n-s}) \leq x_{n-s}$. Claim 2 is proven. □

*Claim 3.* (i) There exists a positive integer $N_1$ such that $x_n \leq A$ for all $n \geq N_1$.
(ii) There exists a positive integer $N_2$ such that $y_n \leq B$ for all $n \geq N_2$.

*Proof of Claim 3.* We only prove (i) (the proof of (ii) is similar). Assume on the contrary that Claim 3 does not hold. Then it follows from Claim 2 that there exists a positive integer $R$ such that $x_{n(s+1)+R} \geq x_{(n+1)(s+1)+R} > A$ for every $n \geq 1$. Let $\lim_{n\to\infty} x_{n(s+1)+R} = A_1$, then $A_1 \geq A$.

We know from Claim 2 that $\{x_n\}$ and $\{y_n\}$ are bounded. Let $c = \lim_{n\to\infty} \sup y_{n(s+1)+R-q-1}$, then $c \geq b$ and there exists a sequence $n_k \to \infty$ such that

$$\lim_{k\to\infty} y_{n_k(s+1)+R-q-1} = c. \tag{2.7}$$

By (1.3) we have that

$$x_{n_k(s+1)+R} = f(y_{n_k(s+1)+R-q-1}, x_{(n_k-1)(s+1)+R}), \tag{2.8}$$

from which it follows that

$$A_1 = f(c, A_1) \leq f(b, A_1) = A_1 \frac{f(b, A_1)}{A_1} \leq A_1 \frac{f(b, A)}{A} = A_1. \tag{2.9}$$

This with $(H_2)$ and $(H_4)$ implies $c = b$ and $A_1 = A$. Therefore $\lim_{n\to\infty} y_{n(s+1)+R-q-1} = b$.

Since $\{x_n\}$ and $\{y_n\}$ are bounded, we may assume (by taking a subsequence) that there exist a sequence $l_n \to \infty$ and $\alpha, \beta \in E$ such that

$$\lim_{k\to\infty} x_{l_k(s+1)+R-q-t-2} = \alpha, \qquad \lim_{k\to\infty} y_{l_k(s+1)+R-q-p-2} = \beta. \tag{2.10}$$

By (1.3) we have that

$$y_{l_k(s+1)+R-q-1} = g(x_{l_k(s+1)+R-q-t-2}, y_{l_k(s+1)+R-q-p-2}), \tag{2.11}$$

from which it follows that

$$b = g(\alpha, \beta) > g(\alpha+1, \beta) \geq b. \tag{2.12}$$

This is a contradiction. Claim 3 is proven. □

Let $N = \max\{N_1, N_2\} + 2s + 2p$, then for all $n > N$ we have that

$$\begin{aligned} x_n &\le A, \qquad y_n \le B, \\ x_n &= f(y_{n-q-1}, x_{n-s-1}) \ge f(B, a), \\ y_n &= g(x_{n-t-1}, y_{n-p-1}) \ge g(A, b). \end{aligned} \tag{2.13}$$

Theorem 2.1 is proven. □

Theorem 2.2. *Let $I = [c,d]$ and $J = [\alpha,\beta]$ be intervals of real numbers. Assume that $f \in C(J \times I, I)$ and $g \in C(I \times J, J)$ satisfy the following properties:*

(i) *$f(u,v)$ and $g(u,v)$ are decreasing in $u$ and increasing in $v$;*

(ii) *if $M_1, m_1 \in I$ with $m_1 \le M_1$ and $M_2, m_2 \in J$ with $m_2 \le M_2$ are a solution of the system*

$$\begin{aligned} M_1 &= f(m_2, M_1), \qquad m_1 = f(M_2, m_1), \\ M_2 &= g(m_1, M_2), \qquad m_2 = g(M_1, m_2), \end{aligned} \tag{2.14}$$

*then $M_1 = m_1$ and $M_2 = m_2$.*

*Then the system*

$$x_{n+1} = f(y_{n-q}, x_{n-s}), \quad y_{n+1} = g(x_{n-t}, y_{n-p}), \quad n = 0, 1, \ldots, \tag{2.15}$$

*has a unique equilibrium $(\overline{S}, \overline{T})$ and every solution of (2.15) with the initial values $x_{-s}, x_{-s+1}, \ldots, x_0 \in I$ and $y_{-p}, y_{-p+1}, \ldots, y_0 \in J$ converges to $(\overline{S}, \overline{T})$.*

*Proof.* Let

$$m_1^0 = c, \qquad m_2^0 = \alpha, \qquad M_1^0 = d, \qquad M_2^0 = \beta, \tag{2.16}$$

and for $i = 1, 2, \ldots,$ we define

$$\begin{aligned} M_1^i &= f(m_2^{i-1}, M_1^{i-1}), \qquad m_1^i = f(M_2^{i-1}, m_1^{i-1}), \\ M_2^i &= g(m_1^{i-1}, M_2^{i-1}), \qquad m_2^i = g(M_1^{i-1}, m_2^{i-1}). \end{aligned} \tag{2.17}$$

It is easy to verify that

$$\begin{aligned} m_1^0 \le m_1^1 &= f(M_2^0, m_1^0) \le f(m_2^0, M_1^0) = M_1^1 \le M_1^0, \\ m_2^0 \le m_2^1 &= g(M_1^0, m_2^0) \le g(m_1^0, M_2^0) = M_2^1 \le M_2^0. \end{aligned} \tag{2.18}$$

From (i) and (2.18) we obtain

$$\begin{aligned} m_1^1 &= f(M_2^0, m_1^0) \le f(M_2^1, m_1^1) = m_1^2, \\ m_1^2 &= f(M_2^1, m_1^1) \le f(m_2^1, M_1^1) = M_1^2, \\ M_1^2 &= f(m_2^1, M_1^1) \le f(m_2^0, M_1^0) = M_1^1, \\ m_2^1 &= g(M_1^0, m_2^0) \le g(M_1^1, m_2^1) = m_2^2, \\ m_2^2 &= g(M_1^1, m_2^1) \le g(m_1^1, M_2^1) = M_2^2, \\ M_2^2 &= g(m_1^1, M_2^1) \le g(m_1^0, M_2^0) = M_2^1. \end{aligned} \tag{2.19}$$

By induction it follows that for $i = 0,1,\dots,$

$$\begin{aligned} m_1^i &\le m_1^{i+1} \le \cdots \le M_1^{i+1} \le M_1^i, \\ m_2^i &\le m_2^{i+1} \le \cdots \le M_2^{i+1} \le M_2^i. \end{aligned} \tag{2.20}$$

On the other hand, we have $x_n \in [m_1^0, M_1^0]$ for any $n \ge -s$ and $y_n \in [m_2^0, M_2^0]$ for any $n \ge -p$ since $x_{-s}, x_{-s+1}, \dots, x_0 \in [m_1^0, M_1^0]$ and $y_{-p}, y_{-p+1}, \dots, y_0 \in [m_2^0, M_2^0]$. For any $n \ge 0$, we obtain

$$\begin{aligned} m_1^1 = f(M_2^0, m_1^0) \le x_{n+1} = f(y_{n-q}, x_{n-s}) \le f(m_2^0, M_1^0) = M_1^1, \\ m_2^1 = g(M_1^0, m_2^0) \le y_{n+1} = g(x_{n-t}, y_{n-p}) \le g(m_1^0, M_2^0) = M_2^1. \end{aligned} \tag{2.21}$$

Let $k = \max\{s+1, p+1\}$. It follows that for any $n \ge k$,

$$\begin{aligned} m_1^2 = f(M_2^1, m_1^1) \le x_{n+1} = f(y_{n-q}, x_{n-s}) \le f(m_2^1, M_1^1) = M_1^2, \\ m_2^2 = g(M_1^1, m_2^1) \le y_{n+1} = g(x_{n-t}, y_{n-p}) \le g(m_1^1, M_2^1) = M_2^2. \end{aligned} \tag{2.22}$$

By induction, for $l = 0,1,\dots,$ we obtain that for any $n \ge lk$,

$$m_1^{l+1} \le x_{n+1} \le M_1^{l+1}, \qquad m_2^{l+1} \le y_{n+1} \le M_2^{l+1}. \tag{2.23}$$

Let

$$\begin{aligned} \lim_{n\to\infty} m_1^n = m_1, \qquad \lim_{n\to\infty} m_2^n = m_2, \\ \lim_{n\to\infty} M_1^n = M_1, \qquad \lim_{n\to\infty} M_2^n = M_2. \end{aligned} \tag{2.24}$$

By the continuity of $f$ and $g$, we have from (2.17) that

$$\begin{aligned} M_1 = f(m_2, M_1), \qquad M_2 = g(m_1, M_2), \\ m_2 = g(M_1, m_2), \qquad m_1 = f(M_2, m_1). \end{aligned} \tag{2.25}$$

Using assumption (ii), it follows from (2.23) that

$$\lim_{n\to\infty} x_n = m_1 = M_1 = \overline{S}, \qquad \lim_{n\to\infty} y_n = m_2 = M_2 = \overline{T}. \tag{2.26}$$

Theorem 2.2 is proven. □

Theorem 2.3. *If $(H_1)$–$(H_5)$ hold and the system*

$$\begin{aligned} M_1 = f(m_2, M_1), \qquad M_2 = g(m_1, M_2), \\ m_2 = g(M_1, m_2), \qquad m_1 = f(M_2, m_1), \end{aligned} \tag{2.27}$$

*with* $f(B,a) \le m_1 \le M_1 \le A$ *and* $g(A,b) \le m_2 \le M_2 \le B$ *has the unique solution* $m_1 = M_1 = \overline{x}$ *and* $m_2 = M_2 = \overline{y}$*, then every solution of (1.3) converges to the unique positive equilibrium* $(\overline{x},\overline{y})$.

*Proof.* Let $\{(x_n, y_n)\}$ is a positive solution of (1.3). By Theorem 2.1, there exists a positive integer $N$ such that $f(B,a) \le x_n = f(y_{n-q},x_{n-s}) \le A$ and $g(A,b) \le y_n = g(x_{n-t}, y_{n-p}) \le B$ for all $n \ge N$. Since $f$, $g$ satisfy the conditions (i) and (ii) of Theorem 2.2 in $I = [f(B,a),A]$ and $J = [(A,b),B]$, it follows that $\{(x_n, y_n)\}$ converges to the unique positive equilibrium $(\overline{x},\overline{y})$. □

## 3. Examples

In this section, we will give two applications of the above results.

*Example 3.1.* Consider equation

$$x_{n+1} = \frac{c+x_{n-s}}{a+y_{n-q}}, \qquad y_{n+1} = \frac{d+y_{n-p}}{b+x_{n-t}}, \tag{3.1}$$

where $p,q,s,t \in \{0,1,2,\dots\}$ with $s \ge t$ and $p \ge q$, the initial values $x_{-s},x_{-s+1},\dots,x_0$, $y_{-p},y_{-p+1},\dots,y_0 \in (0,+\infty)$ and $a,b,c,d \in (0,+\infty)$. If $a > 1$ and $b > 1$, then every positive solution of (3.1) converges to the unique positive equilibrium.

*Proof.* Let $E = [0,+\infty)$, it is easy to verify that $(H_1)$–$(H_5)$ hold for (3.1). In addition, if

$$\begin{aligned} M_1 &= \frac{c+M_1}{a+m_2}, & M_2 &= \frac{d+M_2}{b+m_1}, \\ m_2 &= \frac{d+m_2}{b+M_1}, & m_1 &= \frac{c+m_1}{a+M_2}, \end{aligned} \tag{3.2}$$

with $0 \le m_1 \le M_1$ and $0 \le m_2 \le M_2$, then we have

$$\begin{aligned} (M_1 - m_1)(a-1) &= m_1M_2 - M_1m_2, \\ (M_2 - m_2)(b-1) &= M_1m_2 - m_1M_2, \end{aligned} \tag{3.3}$$

from which it follows that $M_1 = m_1$ and $M_2 = m_2$. Moreover, it is easy to verify that (3.2) have the unique solution

$$\begin{aligned} M_1 = m_1 = \overline{x} &= \frac{-(a-1)(b-1)+c-d+\sqrt{[(a-1)(b-1)+d-c]^2+4c(a-1)(b-1)}}{2(a-1)}, \\ M_2 = m_2 = \overline{y} &= \frac{-(a-1)(b-1)+d-c+\sqrt{[(a-1)(b-1)+c-d]^2+4d(a-1)(b-1)}}{2(b-1)}. \end{aligned} \tag{3.4}$$

It follows from Theorems 2.1 and 2.3 that every positive solution of (3.1) converges to the unique positive equilibrium $(\overline{x},\overline{y})$. □

*Example 3.2.* Consider equation

$$x_{n+1} = a + \frac{x_{n-s}}{y_{n-q}}, \qquad y_{n+1} = b + \frac{y_{n-p}}{x_{n-t}}, \tag{3.5}$$

where $p,q,s,t \in \{0,1,2,\dots\}$ with $s \geq t$ and $p \geq q$, the initial values $x_{-s}, x_{-s+1}, \dots, x_0$, $y_{-p}, y_{-p+1}, \dots, y_0 \in (0,+\infty)$ and $a,b \in (0,+\infty)$. If $a > 1$ and $b > 1$, then every positive solution of (3.5) converges to the unique positive equilibrium.

*Proof.* Let $E = (0,+\infty)$, it is easy to verify that $(H_1)$–$(H_5)$ hold for (3.5). In addition, if

$$\begin{aligned} M_1 &= a + \frac{M_1}{m_2}, \qquad & M_2 &= b + \frac{M_2}{m_1}, \\ m_2 &= b + \frac{m_2}{M_1}, \qquad & m_1 &= a + \frac{m_1}{M_2}, \end{aligned} \tag{3.6}$$

with $0 \leq m_1 \leq M_1$ and $0 \leq m_2 \leq M_2$, then (3.6) have the unique solution

$$\begin{aligned} M_1 &= m_1 = \overline{x} = \frac{ab-1}{b-1}, \\ M_2 &= m_2 = \overline{y} = \frac{ab-1}{a-1}. \end{aligned} \tag{3.7}$$

It follows from Theorems 2.1 and 2.3 that every positive solution of (3.5) converges to the unique positive equilibrium $(\overline{x}, \overline{y}) = ((ab-1)/(b-1), (ab-1)/(a-1))$. □

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## References

[1] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Eigenvalue characterization of a system of difference equations*, Nonlinear Oscillations **7** (2004), no. 1, 3–47.

[2] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, Mathematics and Its Applications, vol. 404, Kluwer Academic, Dordrecht, 1997.

[3] E. Camouzis and G. Papaschinopoulos, *Global asymptotic behavior of positive solutions on the system of rational difference equations* $x_{n+1} = 1 + x_n/y_{n-m}$, $y_{n+1} = 1 + y_n/x_{n-m}$, Applied Mathematics Letters **17** (2004), no. 6, 733–737.

[4] C. Çinar, *On the positive solutions of the difference equation system* $x_{n+1} = 1/y_n$, $y_{n+1} = y_n/x_{n-1}y_{n-1}$, Applied Mathematics and Computation **158** (2004), no. 2, 303–305.

[5] D. Clark and M. R. S. Kulenović, *A coupled system of rational difference equations*, Computers & Mathematics with Applications **43** (2002), no. 6-7, 849–867.

[6] D. Clark, M. R. S. Kulenović, and J. F. Selgrade, *Global asymptotic behavior of a two-dimensional difference equation modelling competition*, Nonlinear Analysis **52** (2003), no. 7, 1765–1776.

[7] E. A. Grove, G. Ladas, L. C. McGrath, and C. T. Teixeira, *Existence and behavior of solutions of a rational system*, Communications on Applied Nonlinear Analysis **8** (2001), no. 1, 1–25.

[8] M. R. S. Kulenović and M. Nurkanović, *Asymptotic behavior of a system of linear fractional difference equations*, Journal of Inequalities and Applications **2005** (2005), no. 2, 127–143.

[9] G. Papaschinopoulos and C. J. Schinas, *On a system of two nonlinear difference equations*, Journal of Mathematical Analysis and Applications **219** (1998), no. 2, 415–426.

[10] ———, *On the system of two nonlinear difference equations* $x_{n+1} = A + x_{n-1}/y_n$, $y_{n+1} = A + y_{n-1}/x_n$, International Journal of Mathematics and Mathematical Sciences **23** (2000), no. 12, 839–848.

[11] X. Yang, *On the system of rational difference equations* $x_n = A + y_{n-1}/x_{n-p}y_{n-q}$, $y_n = A + x_{n-1}/x_{n-r}y_{n-s}$, Journal of Mathematical Analysis and Applications **307** (2005), no. 1, 305–311.

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