ON THE SYSTEM OF RATIONAL DIFFERENCE EQUATIONS $x_{n+1} = f(x_n, y_{n-k}), y_{n+1} = f(y_n, x_{n-k})$

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We study the global asymptotic behavior of the positive solutions of the system of rational difference equations $x_{n+1} = f(x_n, y_{n-k}), y_{n+1} = f(y_n, x_{n-k}), n = 0, 1, 2, \ldots$, under appropriate assumptions, where $k \in \{1, 2, \ldots\}$ and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, +\infty)$. We give sufficient conditions under which every positive solution of this equation converges to a positive equilibrium. The main theorem in [1] is included in our result.

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1. Introduction

Recently there has been published quite a lot of works concerning the behavior of positive solutions of systems of rational difference equations [2–7]. These results are not only valuable in their own right, but they can provide insight into their differential counterparts.

In [1], Camouzis and Papaschinopoulos studied the global asymptotic behavior of the positive solutions of the system of rational difference equations

$$
\begin{align*}
x_{n+1} &= 1 + \frac{x_n}{y_{n-k}}, \\
y_{n+1} &= 1 + \frac{y_n}{x_{n-k}},
\end{align*}
$$

where $k \in \{1, 2, \ldots\}$ and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, +\infty)$.

To be motivated by the above studies, in this paper, we consider the more general equation

$$
\begin{align*}
x_{n+1} &= f(x_n, y_{n-k}), \\
y_{n+1} &= f(y_n, x_{n-k}),
\end{align*}
$$

where $k \in \{1, 2, \ldots\}$ and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, +\infty)$. We give sufficient conditions under which every positive solution of this equation converges to a positive equilibrium. The main theorem in [1] is included in our result.

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where \( k \in \{1, 2, \ldots \} \), the initial values \( x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, +\infty) \) and \( f \) satisfies the following hypotheses.

1. \( (H_1) \) \( f \in C(E \times E, (0, +\infty)) \) with \( a = \inf_{(u,v) \in E \times E} f(u, v) \in E \), where \( E \in \{(0, +\infty), [0, +\infty)\} \).

2. \( (H_2) \) \( f(u,v) \) is increasing in \( u \) and decreasing in \( v \).

3. \( (H_3) \) There exists a decreasing function \( g \in C((a, +\infty), (a, +\infty)) \) such that
   - (i) For any \( x > a \), \( g(g(x)) = x \) and \( x = f(x, g(x)) \);
   - (ii) \( \lim_{x \to a^+} g(x) = +\infty \) and \( \lim_{x \to +\infty} g(x) = a \).

A pair of sequences of positive real numbers \( \{(x_n, y_n)\}_{n=-k}^{\infty} \) that satisfies (1.2) is a positive solution of (1.2). If a positive solution of (1.2) is a pair of positive constants \((x, y)\), then \((x, y)\) is called a positive equilibrium of (1.2). In this paper, our main result is the following theorem.

**Theorem 1.1.** Assume that \((H_1)\)–\((H_3)\) hold. Then the following statements are true.

(i) Every pair of positive constant \((x, y) \in (a, +\infty) \times (a, +\infty)\) satisfying the equation

\[
y = g(x)
\]

is a positive equilibrium of (1.2).

(ii) Every positive solution of (1.2) converges to a positive equilibrium \((x, y)\) of (1.2) satisfying (1.3) as \( n \to \infty \).

**2. Proof of Theorem 1.1**

In this section we will prove Theorem 1.1. To do this we need the following lemma.

**Lemma 2.1.** Let \( \{(x_n, y_n)\}_{n=-k}^{\infty} \) be a positive solution of (1.2). Then there exists a real number \( L \in (a, +\infty) \) with \( L < g(L) \) such that \( x_n, y_n \in [L, g(L)] \) for all \( n \geq 1 \). Furthermore, if \( \limsup x_n = M \), \( \liminf x_n = m \), \( \limsup y_n = P \), \( \liminf y_n = p \), then \( M = g(p) \) and \( P = g(m) \).

**Proof.** From \((H_1)\) and \((H_2)\), we have

\[
x_i = f(x_{i-1}, y_{i-1-k}) \geq f(x_{i-1}, y_{i-1-k} + 1) \geq a, \\
y_i = f(y_{i-1}, x_{i-1-k}) \geq f(y_{i-1}, x_{i-1-k} + 1) \geq a,
\]

for every \( 1 \leq i \leq k + 1 \). (2.1)

Since \( \lim_{x \to a^+} g(x) = +\infty \), there exists \( L \in (a, +\infty) \) with \( L < g(L) \) such that

\[
x_i, y_i \in [L, g(L)] \quad \text{for every} \ 1 \leq i \leq k + 1.
\]

(2.2)

It follows from (2.2) and \((H_3)\) that

\[
g(L) = f(g(L), L) \geq x_{k+2} = f(x_{k+1}, y_1) \geq f(L, g(L)) = L, \\
g(L) = f(g(L), L) \geq y_{k+2} = f(y_{k+1}, x_1) \geq f(L, g(L)) = L.
\]

(2.3)

Inductively, we have that \( x_n, y_n \in [L, g(L)] \) for all \( n \geq 1 \).
Let $\limsup x_n = M$, $\liminf x_n = m$, $\limsup y_n = P$, $\liminf y_n = p$, then there exist sequences $l_n \geq 1$ and $s_n \geq 1$ such that
\[
\lim_{n \to \infty} x_{l_n} = M, \quad \lim_{n \to \infty} y_{s_n} = p. \tag{2.4}
\]
Without loss of generality, we may assume (by taking a subsequence) that there exist $A, D \in [m, M]$ and $B, C \in [p, P]$ such that
\[
\lim_{n \to \infty} x_{l_n-1} = A, \quad \lim_{n \to \infty} y_{s_n-1} = B, \\
\lim_{n \to \infty} y_{l_n-k-1} = C, \quad \lim_{n \to \infty} x_{s_n-k-1} = D. \tag{2.5}
\]
Thus, from (1.2), (H2) and (H3), we have
\[
f(M, g(M)) = M = f(A, B) \leq f(M, p), \\
f(p, g(p)) = p = f(C, D) \geq f(p, M), \tag{2.6}
\]
from which it follows that
\[
g(M) \geq p, \quad g(p) \leq M. \tag{2.7}
\]
By (H3), we obtain
\[
p = g(g(p)) \geq g(M). \tag{2.8}
\]
Therefore, $M = g(p)$. By the symmetry, we have also $P = g(m)$. Lemma 2.1 is proven. □

**Proof of Theorem 1.1.**

(i) Is obvious.

(ii) Let $\{(x_n, y_n)\}_{n=-k}^{\infty}$ be a positive solution of (1.2) with the initial conditions $x_0, x_{-1}, \ldots, x_{-k}, y_0, y_{-1}, \ldots, y_{-k} \in (0, +\infty)$. By Lemma 2.1, we have that
\[
a < \liminf x_n = g(P) \leq \limsup x_n = M < +\infty, \\
a < \liminf y_n = g(M) \leq \limsup y_n = P < +\infty. \tag{2.9}
\]
Without loss of generality, we may assume (by taking a subsequence) that there exists a sequence $l_n \geq 4k$ such that
\[
\lim_{n \to \infty} x_{l_n} = M, \\
\lim_{n \to \infty} x_{l_n-j} = M_j \in [g(P), M], \quad \text{for } j \in \{1, 2, \ldots, 3k+1\}, \tag{2.10} \\
\lim_{n \to \infty} y_{l_n-j} = P_j \in [g(M), P], \quad \text{for } j \in \{1, 2, \ldots, 3k+1\}. 
\]
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From (1.2), (2.10) and (H3), we have

\[ f(M, g(M)) = M = f(M_1, P_{k+1}) \leq f(M_1, g(M)) \leq f(M, g(M)), \]  

(2.11)

from which it follows that

\[ M_1 = M, \quad P_{k+1} = g(M). \]  

(2.12)

In a similar fashion, we may obtain that

\[ f(M, g(M)) = M = M_1 = f(M_2, P_{k+2}) \leq f(M_2, g(M)) \leq f(M, g(M)), \]  

(2.13)

from which it follows that

\[ M_2 = M, \quad P_{k+2} = g(M). \]  

(2.14)

Inductively, we have that

\[ M_j = M, \quad P_{k+j} = g(M), \]  

for \( j \in \{1, 2, \ldots, 2k + 1\}, \)  

(2.15)

from which it follows that

\[ \lim_{n \to \infty} x_{l_n-j} = M, \quad \text{for} \quad j \in \{0, 1, \ldots, 2k+1\}, \]  

(2.16)

\[ \lim_{n \to \infty} y_{l_n-j} = g(M), \quad \text{for} \quad j \in \{k+1, \ldots, 3k+1\}. \]

In view (2.16), for any \( 0 < \epsilon < M - a \), there exists some \( l_s \geq 4k \) such that

\[ M - \epsilon < x_{l_n-j} < M + \epsilon, \quad \text{if} \quad j \in \{0, 1, \ldots, 2k+1\}, \]  

\[ g(M + \epsilon) < y_{l_n-j} < g(M - \epsilon), \quad \text{if} \quad j \in \{k+1, \ldots, 2k+1\}. \]  

(2.17)

From (1.2) and (2.17), we have

\[ y_{l_n-k} = f(y_{l_n-k-1}, x_{l_n-2k-1}) < f(g(M - \epsilon), M - \epsilon) = g(M - \epsilon). \]  

(2.18)

Also (1.2), (2.17) and (2.18) implies

\[ x_{l_n+1} = f(x_{l_n}, y_{l_n-k}) > f(M - \epsilon, g(M - \epsilon)) = M - \epsilon. \]  

(2.19)

Inductively, it follows that

\[ y_{l_n+n-k} < g(M - \epsilon) \quad \forall n \geq 0, \]  

\[ x_{l_n+n} > M - \epsilon \quad \forall n \geq 0. \]  

(2.20)
Since $\limsup x_n = M$ and $\liminf y_n = g(M)$, we have
\[
\lim_{n \to \infty} x_n = M, \quad \lim_{n \to \infty} y_n = g(M).
\] (2.21)
Thus $\lim_{n \to \infty} (x_n, y_n) = (M, P)$ with $P = g(M)$. Theorem 1.1 is proven. □

3. Examples
To illustrate the applicability of Theorem 1.1, we present the following examples.

Example 3.1. Consider equation
\[
\begin{align*}
  x_{n+1} &= \frac{p + x_n}{1 + y_{n-k}}, \quad n = 0, 1, \ldots, \\
y_{n+1} &= \frac{p + y_n}{1 + x_{n-k}},
\end{align*}
\] (3.1)
where $k \in \{1, 2, \ldots \}$, the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, +\infty)$ and $p \in (0, +\infty)$. Let $E = (0, +\infty)$ and
\[
f(x, y) = \frac{p + x}{1 + y} \quad (x \geq 0, y \geq 0), \quad g(x) = \frac{p}{x} \quad (x > 0).
\] (3.2)
It is easy to verify that (H 1)–(H3) hold for (3.1). It follows from Theorem 1.1 that
(i) every pair of positive constant $(x, y) \in (0, +\infty) \times (0, +\infty)$ satisfying the equation
\[
xy = p
\] (3.3)
is a positive equilibrium of (3.1).
(ii) every positive solution of (3.1) converges to a positive equilibrium $(x, y)$ of (3.1) satisfying (3.3) as $n \to \infty$.

Example 3.2. Consider equation
\[
\begin{align*}
  x_{n+1} &= 1 + \frac{x_n}{y_{n-k}}, \quad n = 0, 1, \ldots, \\
y_{n+1} &= 1 + \frac{y_n}{x_{n-k}},
\end{align*}
\] (3.4)
where $k \in \{1, 2, \ldots \}$ and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, +\infty)$. Let $E = (0, +\infty)$ and
\[
f(x, y) = 1 + \frac{x}{y} \quad (x > 0, y > 0), \quad g(x) = \frac{x}{x - 1} \quad (x > 1).
\] (3.5)
It is easy to verify that (H 1)–(H3) hold for (3.4). It follows from Theorem 1.1 that
(i) every pair of positive constant $(x, y) \in (1, +\infty) \times (1, +\infty)$ satisfying the equation
\[
xy = x + y
\] (3.6)
is a positive equilibrium of (3.4);
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(ii) every positive solution of (3.4) converges to a positive equilibrium \((x,y)\) of (3.4) satisfying (3.6) as \(n \to \infty\).

Example 3.3. Consider equation

\[
\begin{align*}
x_{n+1} &= p + \frac{A + x_n}{q + y_{n-k}}, \\
y_{n+1} &= p + \frac{A + y_n}{q + x_{n-k}},
\end{align*}
\]

where \(k \in \{1,2,\ldots\}\), the initial conditions \(x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0, +\infty)\), \(A \in (0, +\infty)\) and \(p, q \in [0,1]\) with \(p + q = 1\). Let \(E = (0, +\infty)\) if \(p > 0\) and \(E = [0, +\infty)\) if \(p = 0\) and

\[f(x, y) = p + \frac{A + x}{q + y},\]

then \(a = \inf_{(x,y) \in E \times E} f(x, y) = p\). Let \(g(x) = (pq + px + A)/(x - p)\) \((x > p)\). It is easy to verify that (H1)–(H3) hold for (3.7). It follows from Theorem 1.1 that

(i) every pair of positive constant \((x, y) \in (p, +\infty) \times (p, +\infty)\) satisfying the equation

\[xy = pq + px + py + A\]

is a positive equilibrium of (3.7);

(ii) every positive solution of (3.7) converges to a positive equilibrium \((x,y)\) of (3.7) satisfying (3.9) as \(n \to \infty\).

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