

# PERIODIC BOUNDARY VALUE PROBLEMS ON TIME SCALES

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We extend the results concerning periodic boundary value problems from the continuous calculus to time scales. First we use the Schauder fixed point theorem and the concept of lower and upper solutions to prove the existence of the solutions and then we investigate a monotone iterative method which could generate some of them. Since this method does not work on each time scale, a condition containing a Lipschitz constant of right-hand side function and the supremum of the graininess function is introduced.

## 1. Introduction

We recall the basic definitions concerning calculus on time scales. Further details can be found in the survey monography upon this topic [2], its second part [3], or in the paper [6].

*Time scale*  $\mathbb{T}$  is an arbitrary nonempty closed subset of the real numbers  $\mathbb{R}$ . The natural numbers  $\mathbb{N}$ , the integers  $\mathbb{Z}$ , or the union of intervals  $[0, 1] \cup [2, 3]$  are examples of time scales.

For  $t \in \mathbb{T}$  we define the *forward jump operator*  $\sigma : \mathbb{T} \rightarrow \mathbb{T}$  and the *backward jump operator*  $\rho : \mathbb{T} \rightarrow \mathbb{T}$  by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup \{s \in \mathbb{T} : s < t\}, \quad (1.1)$$

where we put  $\inf \emptyset = \sup \mathbb{T}$  and  $\sup \emptyset = \inf \mathbb{T}$ .

We say that a point  $t \in \mathbb{T}$  is *right scattered*, *left scattered*, *right dense*, *left dense* if  $\sigma(t) > t$ ,  $\rho(t) < t$ ,  $\sigma(t) = t$ ,  $\rho(t) = t$ , respectively. A point  $t \in \mathbb{T}$  is *isolated* if it is right scattered and left scattered. A point  $t \in \mathbb{T}$  is *dense* if it is right dense and left dense. Finally, we define the *forward graininess function*  $\mu : \mathbb{T} \rightarrow [0, \infty)$  by

$$\mu(t) := \sigma(t) - t. \quad (1.2)$$

Similarly, we define the *backward graininess function*  $\nu : \mathbb{T} \rightarrow [0, \infty)$  by

$$\nu(t) := t - \rho(t). \quad (1.3)$$

We define the function  $f^\sigma(t) = f(\sigma(t))$  and the modified time scale  $\mathbb{T}^\kappa$  as follows: if  $\mathbb{T}$  has a left-scattered maximum  $m$ , then  $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$ , otherwise  $\mathbb{T}^\kappa = \mathbb{T}$ .

We endow  $\mathbb{T}$  with the topology inherited from  $\mathbb{R}$ . The continuity of the function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is defined in the usual manner. The function  $f$  is said to be *delta differentiable* (the word *delta* is omitted later) at  $t \in \mathbb{T}^\kappa$  if there exists a number (denoted by  $f^\Delta(t)$ ) with the property that given any  $\epsilon > 0$  there is a neighbourhood  $U$  of  $t$  (in the time-scale topology) such that

$$|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s| \quad \forall s \in U. \tag{1.4}$$

If  $f$  is differentiable at every  $t \in \mathbb{T}$ , then  $f$  is said to be *differentiable* on  $\mathbb{T}$ . The second derivative is defined by  $f^{\Delta\Delta}(t) = (f^\Delta(t))^\Delta$ . Similarly, one can define *nabla* derivative  $f^\nabla$  with the aid of the backward jump operator  $\rho$  instead of  $\sigma$ . See [3] for more details.

In the following we often use formula

$$f^\sigma(t) = f(t) + \mu(t)x^\Delta(t). \tag{1.5}$$

We also define the interval with respect to the time scale  $\mathbb{T}$  denoted by

$$[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}. \tag{1.6}$$

We assume further that  $a, b \in \mathbb{T}$ .

A function  $F : \mathbb{T}^\kappa \rightarrow \mathbb{R}$  is called an *antiderivative* of  $f : \mathbb{T} \rightarrow \mathbb{R}$  provided that

$$F^\Delta(t) = f(t) \quad \forall t \in \mathbb{T}^\kappa. \tag{1.7}$$

We then define the *Cauchy integral* by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \forall r, s \in \mathbb{T}. \tag{1.8}$$

We define an *rd-continuous* function as a function that is continuous in all right-dense points and whose left-sided limits exist in left-dense points. The set of all rd-continuous functions will be denoted by  $C_{rd}$ . The set of twice-differentiable functions whose second derivative is rd-continuous will be denoted by  $C_{rd}^2$ . Equivalently  $C_{rd}^2$  could be defined as continuous functions whose second derivatives are rd-continuous (e.g., in [7]). Every rd-continuous function has an antiderivative.

We say that a function  $x(t)$  has a *generalized zero* at  $c$  if  $x(c) = 0$  or if  $c$  is left scattered and  $x(\rho(c))x(c) < 0$ .

In this work, we deal with the periodic boundary value problem (BVP)

$$\begin{aligned} x^{\Delta\Delta}(t) &= f(t, x^\sigma(t)) \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) &= x(\sigma^2(b)), \\ x^\Delta(a) &= x^\Delta(\sigma(b)), \end{aligned} \tag{1.9}$$

where  $f$  is a continuous function. Existence theorems are proved for a slightly more general problem:

$$\begin{aligned} x^{\Delta\Delta}(t) &= f(t, x^\sigma(t)) \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) - x(\sigma^2(b)) &= A, \\ x^\Delta(a) - x^\Delta(\sigma(b)) &= B, \end{aligned} \tag{1.10}$$

where  $A, B \in \mathbb{R}$ . A function  $x \in C_{\text{rd}}^2$  is said to be a solution of (1.10) provided that  $x^{\Delta\Delta}(t) = f(t, x^\sigma(t))$  is satisfied for all  $t \in [a, b]_{\mathbb{T}}$  and the boundary conditions hold.

We will use the Schauder fixed point theorem and the concept of lower and upper solutions to prove the existence of solutions of (1.10). First we prove the existence of the solution for the modified problem with bounded right-hand side, then the existence of the solution of (1.10) is proved with the help of lower and upper solutions. Finally, the monotone iterative method for problem (1.9) is extended to time scales. A good source for this method, if  $\mathbb{T} = \mathbb{R}$ , is [5]. However, on time scales the situation is more complicated and additional conditions containing the graininess function have to be introduced.

Similar results concerning BVPs with Dirichlet boundary conditions can be found in [2, Section 6.6] or in Akin's paper [1]. Results for the periodic BVP

$$\begin{aligned} x^{\nabla\Delta}(t) &= f(t, x^\sigma(t)) \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(\rho(a)) &= x(b), \\ x^\Delta(\rho(a)) &= x^\Delta(b) \end{aligned} \tag{1.11}$$

are found in [3, Section 6.3].

These results are due to Topal, and one can find them in [8] too. There, she proves the existence of solutions with the help of the condition  $|\nu(a)| \geq \mu(b)$ , where  $\nu(t) := t - \rho(t)$  and in the proof of the existence of monotone sequences she adds a Lipschitz condition on the right-hand side  $f$ . However, we prove the existence of solutions of (1.10) for all time scales, but for the existence of monotone sequences we use the Lipschitz condition with the constant which depends on the supremum of the graininess function of a particular time scale (see Theorem 4.5).

Interesting and relevant results are also contained in the paper by Cabada, [4]. He works with general  $n$ th order periodic BVPs and the result for

$$\begin{aligned} x^{\Delta\Delta}(t) &= f(t, x(t)) \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) &= x(\sigma(b)), \\ x^\Delta(a) &= x^\Delta(\sigma(b)) \end{aligned} \tag{1.12}$$

is only a special case. In the same way as here, he requires a Lipschitz condition on the function  $f(t, x)$  with a constant  $M$ . But it depends not only on the graininess function but also on the distance  $\sigma(b) - a$ . However, his results are for general  $n$ th order equation whose special second-order case (1.12) differs from problem (1.9) in the right-hand side and in the boundary conditions.

## 2. Bounded right-hand side

We work with the auxiliary BVP

$$\begin{aligned}x^{\Delta\Delta}(t) - x^\sigma(t) &= g(t, x^\sigma(t)) \quad \text{on } [a, b]_{\mathbb{T}}, \\x(a) - x(\sigma^2(b)) &= A, \\x^\Delta(a) - x^\Delta(\sigma(b)) &= B.\end{aligned}\tag{2.1}$$

We prove first that the homogeneous BVP has only a trivial solution.

LEMMA 2.1. *Let  $\mathbb{T}$  be a time scale. Then the homogeneous BVP*

$$\begin{aligned}x^{\Delta\Delta}(t) - x^\sigma(t) &= 0 \quad \text{on } [a, b]_{\mathbb{T}}, \\x(a) - x(\sigma^2(b)) &= 0, \\x^\Delta(a) - x^\Delta(\sigma(b)) &= 0\end{aligned}\tag{2.2}$$

*has only a trivial solution.*

*Proof.* Equation  $x^{\Delta\Delta}(t) - x^\sigma(t) = 0$  holds on  $[a, b]_{\mathbb{T}}$ . This implies that

$$\int_a^b [x^{\Delta\Delta}(\tau) - x^\sigma(\tau)] \Delta\tau = 0,\tag{2.3}$$

that is,

$$x^\Delta(b) - x^\Delta(a) - \int_a^b x^\sigma(\tau) \Delta\tau = 0.\tag{2.4}$$

Now we take into account that  $x^\Delta(\sigma(b)) = x^\Delta(b) + \mu(b)x^{\Delta\Delta}(b)$  and  $x^\Delta(\sigma(b)) - x^\Delta(a) = 0$  (second boundary condition). Hence,

$$-\mu(b)x(\sigma(b)) - \int_a^b x^\sigma(\tau) \Delta\tau = 0,\tag{2.5}$$

that is,

$$\int_a^b x^\sigma(\tau) \Delta\tau = -\mu(b)x(\sigma(b)).\tag{2.6}$$

Now we distinguish among three cases according to the value of  $x(\sigma(b))$  and  $\mu(b)$ .

(a) We assume that  $x(\sigma(b)) > 0$  and  $\mu(b) > 0$ . From equation (2.6) we get

$$\int_a^b x^\sigma(\tau) \Delta\tau < 0.\tag{2.7}$$

Since  $x(\sigma(b)) > 0$ , there exists a generalized zero  $c$  where the solution  $x(t)$  comes from negative to positive values. At this point the following conditions must hold:

$$x(c) \geq 0, \quad x^\Delta(\rho(c)) \geq 0, \quad x^{\Delta\Delta}(\rho(c)) \geq 0.\tag{2.8}$$

But this implies that  $x(t) \geq 0$ ,  $x^\Delta(t) \geq 0$ , and  $x^{\Delta\Delta}(t) \geq 0$  for all  $t \geq c$ . Similarly we could obtain that if  $d$  is a generalized zero where the function  $x(t)$  comes from positive to negative values, then  $x(t) \leq 0$  for all  $t \geq d$ . Therefore, if BVP (2.2) has a nontrivial solution, it has at most one generalized zero. Since  $x(\sigma(b)) > 0$  and the condition (2.7) holds, there is exactly one generalized zero  $c$  where the function  $x(t)$  comes from negative to positive values. This implies that  $x(\sigma(a)) < 0$ . From the boundary conditions we have that  $x(a) = x(\sigma^2(b)) \geq 0$  and  $x^\Delta(a) = x^\Delta(\sigma(b)) \geq 0$  but this is a contradiction with  $x(\sigma(a)) < 0$  since

$$0 > x(\sigma(a)) = x(a) + \mu(a)x^\Delta(a) \geq 0. \tag{2.9}$$

Therefore this situation could not occur.

(b) We assume that  $x(\sigma(b)) < 0$  and  $\mu(b) > 0$ . This could not happen too, the proof is similar.

(c) We assume that  $x(\sigma(b)) = 0$  or  $\mu(b) = 0$ . This implies that

$$\int_a^b x^\sigma(\tau)\Delta\tau = 0. \tag{2.10}$$

Assume that there exists a nontrivial solution. This implies that there exists a generalized zero  $c$ . Assume that the function changes from negative to positive values at  $c$ . The proof for the other case is similar. We have the conditions (2.8). For  $t > c$  this implies that  $x(t) \geq 0$ ,  $x^\Delta(t) \geq 0$  and  $x^{\Delta\Delta}(t) \geq 0$ . Since there could be only one generalized zero (cf. part (a)), it must be  $x(\sigma(a)) < 0$  and if we consider boundary conditions we come again to the contradiction (2.9).

Therefore BVP (2.2) has only a trivial solution. □

*Remark 2.2.* It follows from [2, Theorem 4.88] and from Lemma 2.1 that problem

$$\begin{aligned} x^{\Delta\Delta}(t) - x^\sigma(t) &= \tilde{g}(t) \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) - x(\sigma^2(b)) &= A, \\ x^\Delta(a) - x^\Delta(\sigma(b)) &= B, \end{aligned} \tag{2.11}$$

where  $\tilde{g} : \mathbb{T} \rightarrow \mathbb{R}$  is a continuous function, has a unique solution

$$x(t) = z(t) + \int_a^{\sigma(b)} G(t, s)\tilde{g}(s)\Delta s, \tag{2.12}$$

where  $z = z(t)$  is the unique solution of

$$\begin{aligned} x^{\Delta\Delta}(t) - x^\sigma(t) &= 0 \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) - x(\sigma^2(b)) &= A, \\ x^\Delta(a) - x^\Delta(\sigma(b)) &= B, \end{aligned} \tag{2.13}$$

and  $G = G(t, s)$  is Green's function associated with

$$\begin{aligned} x^{\Delta\Delta}(t) - x^\sigma(t) &= \tilde{g}(t) \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) - x(\sigma^2(b)) &= 0, \\ x^\Delta(a) - x^\Delta(\sigma(b)) &= 0. \end{aligned} \tag{2.14}$$

In the following theorem we put  $\Omega = [a, b]_{\mathbb{T}} \times \mathbb{R}$ .

**THEOREM 2.3.** *Let  $\mathbb{T}$  be a time scale,  $a, b \in \mathbb{T}$ . Assume that  $g(t, x)$  is continuous and bounded on  $\Omega$ , then BVP (2.1) has a solution.*

*Proof.* We define the operator

$$Tx(t) := z(t) + \int_a^{\sigma(b)} G(t, s)g(s, x^\sigma(s))\Delta s, \tag{2.15}$$

where  $z$  and  $G$  are as in Remark 2.2. The operator  $T$  maps the Banach space  $C = C([a, \sigma^2(b)]_{\mathbb{T}})$  (equipped with the norm  $\|x\| = \max |x(t)|$ ) into itself. The function  $x = x(t)$  is a solution of (2.1) if and only if  $x$  is a fixed point of  $T$ . In order to apply the Schauder fixed point theorem it is sufficient to prove that  $T$  is a compact operator and maps sufficiently large ball into itself. The proof is similar to the proof of [2, Theorem 6.50] and hence omitted. □

### 3. Lower and upper solutions

We now define lower and upper solutions for the problem (1.10).

**Definition 3.1.** Let  $\mathbb{T}$  be a time scale and  $a, b \in \mathbb{T}$ . Function  $\alpha \in C_{rd}^2$  is a *lower solution* of (1.10) on  $[a, \sigma^2(b)]_{\mathbb{T}}$  if

- (a)  $\alpha^{\Delta\Delta}(t) \geq f(t, \alpha^\sigma(t))$  for all  $t \in [a, b]_{\mathbb{T}}$ ,
- (b)  $\alpha(a) - \alpha(\sigma^2(b)) = A$  and  $\alpha^\Delta(a) - \alpha^\Delta(\sigma(b)) \geq B$ .

Similarly, function  $\beta \in C_{rd}^2$  is an *upper solution* of (1.10) on  $[a, \sigma^2(b)]_{\mathbb{T}}$  if

- (c)  $\beta^{\Delta\Delta}(t) \leq f(t, \beta^\sigma(t))$  for all  $t \in [a, b]_{\mathbb{T}}$ ,
- (d)  $\beta(a) - \beta(\sigma^2(b)) = A$  and  $\beta^\Delta(a) - \beta^\Delta(\sigma(b)) \leq B$ .

The following theorem gives the existence of the solution of the problem (1.10).

**THEOREM 3.2.** *Let  $\mathbb{T}$  be a time scale and  $a, b \in \mathbb{T}$ . Let  $f$  be continuous on  $\Omega$ . Assume that there exist a lower solution  $\alpha$  and an upper solution  $\beta$  of (1.10) such that*

$$\alpha(t) \leq \beta(t) \quad \forall t \in [a, \sigma^2(b)]_{\mathbb{T}}. \tag{3.1}$$

*Then BVP (1.10) has a solution  $x$  with*

$$\alpha(t) \leq x(t) \leq \beta(t) \quad \forall t \in [a, \sigma^2(b)]_{\mathbb{T}}. \tag{3.2}$$

*Proof.* We consider the modified BVP

$$\begin{aligned} x^{\Delta\Delta}(t) - x^\sigma(t) &= F(t, x^\sigma(t)) \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) - x(\sigma^2(b)) &= A, \quad x^\Delta(a) - x^\Delta(\sigma(b)) = B, \end{aligned} \quad (3.3)$$

where  $F(t, x)$  is defined by

$$F(t, x) = \begin{cases} f(t, \beta^\sigma(t)) - \beta^\sigma(t) & \text{if } x \geq \beta^\sigma(t), \\ f(t, x) - x & \text{if } \alpha^\sigma(t) \leq x \leq \beta^\sigma(t), \\ f(t, \alpha^\sigma(t)) - \alpha^\sigma(t) & \text{if } x \leq \alpha^\sigma(t). \end{cases} \quad (3.4)$$

The function  $F$  is bounded and continuous on  $\Omega$ . By Theorem 2.3 the problem (3.3) has a solution  $x(t)$ . To complete the proof it suffices to show that  $\alpha(t) \leq x(t) \leq \beta(t)$  for all  $t \in [a, \sigma^2(b)]_{\mathbb{T}}$  (since  $F(t, x) = f(t, x) - x$  whenever  $\alpha^\sigma(t) \leq x \leq \beta^\sigma(t)$ ).

We show that  $\alpha(t) \leq x(t)$  for all  $t \in [a, \sigma^2(b)]_{\mathbb{T}}$ . Assume by contradiction that this is not the case. Then the function  $z := \alpha - x$  must attain positive maximum in  $[a, \sigma^2(b)]_{\mathbb{T}}$ . Set

$$z(c) = \max_{t \in [a, \sigma^2(b)]_{\mathbb{T}}} \{z(t)\} > 0. \quad (3.5)$$

We now distinguish between two cases, according to the position of the point  $c$  in  $[a, \sigma^2(b)]_{\mathbb{T}}$ .

(a) Let  $c \in (a, \sigma^2(b))_{\mathbb{T}}$ . The proof is similar to the second part of the proof of [2, Theorem 6.54]. By analysis of four different types of points on time scales we come to the conditions  $z(c) > 0$ ,  $z^\Delta(c) \leq 0$ ,  $z^{\Delta\Delta}(\rho(c)) \leq 0$ . But  $z^{\Delta\Delta}(\rho(c)) \leq 0$  implies the following contradiction:

$$\begin{aligned} 0 &\geq \alpha^{\Delta\Delta}(\rho(c)) - x^{\Delta\Delta}(\rho(c)) \\ &= \alpha^{\Delta\Delta}(\rho(c)) - x^\sigma(\rho(c)) - f(\rho(c), \alpha^\sigma(\rho(c))) + \alpha^\sigma(\rho(c)) \\ &> 0, \end{aligned} \quad (3.6)$$

where the last inequality follows the definition of the lower solution and the fact that  $z^\sigma(\rho(c)) = z(c)$ .

(b) Let  $c \in \{a, \sigma^2(b)\}$ . Using the first boundary condition from (3.3) and, Definition 3.1(b), we have

$$\begin{aligned} \alpha(a) - \alpha(\sigma^2(b)) &= A = x(a) - x(\sigma^2(b)), \\ \alpha(a) - x(a) &= \alpha(\sigma^2(b)) - x(\sigma^2(b)), \\ z(a) &= z(\sigma^2(b)). \end{aligned} \quad (3.7)$$

Then at both points  $a$  and  $\sigma^2(b)$  the maximum of  $z$  must be attained. Hence delta derivatives satisfy

$$z^\Delta(a) \leq 0 \leq z^\Delta(\sigma(b)). \quad (3.8)$$

But from the definition of the lower solution we have

$$\begin{aligned} \alpha^\Delta(a) - \alpha^\Delta(\sigma(b)) &\geq B = x^\Delta(a) - x^\Delta(\sigma(b)), \\ \alpha^\Delta(a) - x^\Delta(a) &\geq \alpha^\Delta(\sigma(b)) - x^\Delta(\sigma(b)), \\ z^\Delta(a) &\geq z^\Delta(\sigma(b)). \end{aligned} \tag{3.9}$$

Comparing (3.8) and (3.9) we have

$$z^\Delta(a) = 0 = z^\Delta(\sigma(b)). \tag{3.10}$$

Now we must distinguish between two possibilities.

(i) Let  $a$  be right scattered. Then  $z^\sigma(a) = z(a) + \mu(a)z^\Delta(a) = z(a)$  and at  $\sigma(a)$  the function  $z$  attains also its maximum. But  $\sigma(a) \in (a, \sigma^2(b))_{\mathbb{T}}$ , a contradiction (see (3.6)).

(ii) Let  $a$  be right-dense. Then

$$\begin{aligned} 0 &\geq \alpha^{\Delta\Delta}(a) - x^{\Delta\Delta}(a) = \alpha^{\Delta\Delta}(a) - x(a) - f(t, \alpha(a)) + \alpha(a) \\ &= \alpha^{\Delta\Delta}(a) - f(t, \alpha(a)) + \alpha(a) - x(a) > 0, \end{aligned} \tag{3.11}$$

a contradiction. In the first equality we considered BVP (3.3) and the definition of the function  $F(t, x)$ , then we rearranged the terms and finally we used the definition of the lower solution ( $\alpha^{\Delta\Delta}(a) - f(t, \alpha^\sigma(a)) \geq 0$ ) and the assumption that  $z(a) > 0$ .

The situation with  $\sigma^2(b)$  is similar. Therefore this part of the proof is omitted.

Hence we proved that  $\alpha(t) \leq x(t)$  for all  $t \in [a, \sigma^2(b)]_{\mathbb{T}}$ . One can show by the same way that  $x(t) \leq \beta(t)$ . Therefore (3.2) holds and so  $x(t)$  solves BVP (1.10).  $\square$

#### 4. Monotone iterative methods

We extend here the results from the continuous case (cf. [5, Section 2.5]) to the problem on time scales (1.9). First, we define the sector in the Banach space

$$[u, v]_2 := \{w \in C_{rd}^2 : u \leq w \leq v\} \tag{4.1}$$

and the maximal and minimal solution of (1.9).

*Definition 4.1.* It is said that  $\bar{u}$  is a *maximal solution* ( $\underline{u}$  a *minimal solution*) of (1.9) in  $[\alpha, \beta]_2$  if it is a solution with  $\alpha \leq \bar{u} \leq \beta$  on  $[a, \sigma^2(b)]_{\mathbb{T}}$  ( $\alpha \leq \underline{u} \leq \beta$ ) and every solution  $v$  of (1.9) is such that  $v \leq \bar{u}$  on  $[a, \sigma^2(b)]_{\mathbb{T}}$  ( $\underline{u} \leq v$ , resp.).

This method consists in generating sequences that converge monotonically to maximal and minimal solutions. We denote

$$\bar{\mu} = \sup_{t \in \mathbb{T}} \{\mu(t)\}. \tag{4.2}$$

The theory of this method is based on the following *maximum principle*.

LEMMA 4.2. Let  $M > 0$  be such that if  $\bar{\mu} > 0$  then

$$M \leq \frac{1}{\bar{\mu}^2} \tag{4.3}$$



and  $x \in C_{\text{id}}^2([a, \sigma^2(b)]_{\mathbb{T}})$  be such that

$$\begin{aligned} x^{\Delta\Delta}(t) - Mx(t) &\leq 0 \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) &= x(\sigma^2(b)), \\ x^{\Delta}(a) &\leq x^{\Delta}(\sigma(b)). \end{aligned} \tag{4.4}$$

Then  $x(t) \geq 0$  on  $[a, \sigma^2(b)]_{\mathbb{T}}$ .

*Proof.* We assume by contradiction that there exists a point where function  $x(t)$  is negative. Denote  $m \in [a, \sigma^2(b)]_{\mathbb{T}}$ , where  $\min_{t \in [a, \sigma^2(b)]_{\mathbb{T}}} \{x(t)\} = x(m) < 0$ . For simplicity assume that  $m = \min\{t \in [a, \sigma^2(b)]_{\mathbb{T}} : x(t) = x(m)\}$ . We distinguish between two cases.

(i) Assume that  $m$  is left-dense. Since at  $m$  the function achieves its minimum we have that  $x^{\Delta}(m) = 0$  and  $x^{\Delta\Delta}(m) > 0$ , but this leads to the contradiction

$$0 < x^{\Delta\Delta}(m) \leq Mx(m) < 0. \tag{4.5}$$

(ii) Assume that  $m$  is left scattered. From the minimality of  $x(t)$  at  $m$  we obtain that

$$x^{\Delta}(m) \geq 0, \tag{4.6}$$

$$x^{\Delta}(\rho(m)) < 0. \tag{4.7}$$

Using (4.6), (4.7), the positivity of graininess function, and the fact that  $x^{\Delta}(m) = x^{\Delta}(\rho(m)) + \mu(\rho(m))x^{\Delta\Delta}(\rho(m))$  we have that

$$x^{\Delta\Delta}(\rho(m)) > 0. \tag{4.8}$$

Inequalities (4.8) and (4.4) (note that  $M > 0$ ) imply that

$$x(\rho(m)) > 0. \tag{4.9}$$

We recall again that  $x(m) < 0$ . This implies that  $x(\rho(m)) + \mu(\rho(m))x^{\Delta}(\rho(m)) < 0$  and after rearranging and taking inequality (4.7) into account we obtain that

$$\mu(\rho(m)) > -\frac{x(\rho(m))}{x^{\Delta}(\rho(m))}. \tag{4.10}$$

Using the inequalities (4.4) and (4.6), we have that

$$x^{\Delta}(\rho(m)) + \mu(\rho(m))Mx(\rho(m)) \geq x^{\Delta}(\rho(m)) + \mu(\rho(m))x^{\Delta\Delta}(\rho(m)) = x^{\Delta}(m) \geq 0 \tag{4.11}$$

and we rearrange it to the following form:

$$M \geq -\frac{x^{\Delta}(\rho(m))}{\mu(\rho(m))x(\rho(m))}. \tag{4.12}$$

Table 4.1

$t$	0	1	2	3	4	5
$v(t)$	2	3	-1	1	1	2
$v^\Delta(t)$	1	-4	2	0	1	—
$v^{\Delta\Delta}(t)$	-5	6	-2	1	—	—

Table 4.2

$t$	-6	-5	-4	-3	-2	-1	$[-1,0]$	0	$\frac{1}{2}$	$\frac{3}{2}$
$v(t)$	2	3	-1	1	1	$a$	$\cosh(\sqrt{2}t)$	1	1	2
$v^\Delta(t)$	1	-4	2	0	$a-1$	$\sqrt{2}b$	$\sqrt{2}\sinh(\sqrt{2}t)$	0	1	—
$v^{\Delta\Delta}(t)$	-5	6	-2	$a-1$	$1-a+\sqrt{2}b$	$2a$	$2\cosh(\sqrt{2}t)$	2	—	—

By modifying inequality (4.12) with the help of (4.10) and the fact that  $\bar{\mu} \geq \mu(t)$  for all  $t \in \mathbb{T}$  we obtain contradiction with the assumption (4.3):

$$M \geq -\frac{x^\Delta(\rho(m))}{\mu(\rho(m))x(\rho(m))} = \frac{1}{\mu(\rho(m))} \frac{1}{-x(\rho(m))/x^\Delta(\rho(m))} > \frac{1}{\mu^2(\rho(m))} \geq \frac{1}{\bar{\mu}^2}. \quad (4.13)$$

□

Note that in the case that the condition (4.3) is not satisfied, the assertion of Lemma 4.2 does not have to hold. We illustrate this in the following two examples.

*Example 4.3.* We suppose the time scale  $\mathbb{T} = \mathbb{Z}$  and the following problem:

$$\begin{aligned} x^{\Delta\Delta}(t) - 2x(t) &\leq 0 \quad \text{on } [0,3]_{\mathbb{T}}, \\ x(0) &= x(5), \\ x^\Delta(0) &\leq x^\Delta(4). \end{aligned} \quad (4.14)$$

We define a function  $v(t)$  in Table 4.1.

The first inequality in (4.14) is satisfied in each point of the interval  $[0,3]_{\mathbb{T}}$  and the boundary conditions are satisfied too. Thus  $v(t)$  is a solution of (4.14) although  $v(2) = -1 < 0$ .

One should introduce an example with mixed time scale too.

*Example 4.4.* We suppose the time scale  $\mathbb{T} = \{-6; -5; -4; -3; -2\} \cup [-1,0] \cup \{1/2; 3/2\}$  and a similar problem:

$$\begin{aligned} x^{\Delta\Delta}(t) - 2x(t) &\leq 0 \quad \text{on } [-6,0]_{\mathbb{T}}, \\ x(-6) &= x\left(\frac{3}{2}\right), \\ x^\Delta(-6) &\leq x^\Delta\left(\frac{1}{2}\right). \end{aligned} \quad (4.15)$$

We define a function  $v(t)$  in Table 4.2.

where  $a = \cosh(-\sqrt{2})$  and  $b = \sinh(-\sqrt{2})$ . We see again that the first inequality in (4.15) is satisfied in each point of the interval  $[-6, 0]_{\mathbb{T}}$  and also both boundary conditions hold but  $v(-4) = -1 < 0$ .

From the properties of Green’s function one could obtain that if  $G(t, s)$  is Green’s function for the problem

$$\begin{aligned} x^{\Delta\Delta}(t) - Mx(t) + h(t) &= 0 \quad \text{on } [a, b]_{\mathbb{T}}, \\ x(a) &= x(\sigma^2(b)), \\ x^{\Delta}(a) &= x^{\Delta}(\sigma(b)), \end{aligned} \tag{4.16}$$

then  $x(t)$  is a solution of (1.9) if and only if

$$u = Tu := \int_a^{\sigma(b)} G(t, s)[f(s, x^{\sigma}(s)) + Mx(s)]\Delta s. \tag{4.17}$$

Now we prove the monotone iterative method theorem.

**THEOREM 4.5.** *Let  $f : [a, b]_{\mathbb{T}} \times \mathbb{R} \rightarrow \mathbb{R}$  be continuous and let  $\alpha(t)$  and  $\beta(t)$  be lower and upper solutions of (1.9) with  $\alpha \leq \beta$ . Assume that there exists  $M > 0$  such that (4.3) holds, and for all  $t \in [a, b]_{\mathbb{T}}$  and any  $u, v \in \mathbb{R}$  with  $\alpha(t) < u < v < \beta(t)$ , we have*

$$f(t, v^{\sigma}) - f(t, u^{\sigma}) \geq -M(v - u). \tag{4.18}$$

Then the sequences  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  defined by

$$\begin{aligned} \alpha_0 &= \alpha, & \alpha_{n+1} &= T\alpha_n, \\ \beta_0 &= \beta, & \beta_{n+1} &= T\beta_n, \end{aligned} \tag{4.19}$$

converge uniformly to the minimal and the maximal solutions of (1.9).

*Proof.* Let the functions  $u$  and  $v$  be such that  $\alpha \leq u \leq v \leq \beta$ . Denote  $u_1 = Tu$  and  $v_1 = Tv$ . Taking the definition of the operator  $T$  (4.17) and the assumption (4.18) into account we obtain that

$$\begin{aligned} (v_1^{\Delta\Delta} - u_1^{\Delta\Delta}) - M(v_1 - u_1) &= -f(t, v^{\sigma}) + f(t, u^{\sigma}) - M(v - u) \leq 0, \\ (v_1 - u_1)(a) &= (v_1 - u_1)(\sigma^2(b)), & (v_1^{\Delta} - u_1^{\Delta})(a) &= (v_1 - u_1)(\sigma(b)). \end{aligned} \tag{4.20}$$

Now we use Lemma 4.2 to obtain that  $Tv - Tu = v_1 - u_1 \geq 0$  and thus operator  $T$  is monotone on interval  $[\alpha, \beta]$ .

Similarly it can be shown that

$$\alpha = \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \leq \dots \leq \beta_n \leq \dots \leq \beta_1 \leq \beta_0 = \beta, \tag{4.21}$$

in other words, sequences  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  are bounded and monotone.

Using Arzelá-Ascoli theorem, we can show that  $T$  is completely continuous. This, together with boundness of the defining sequences, implies that there exist some subsequences such that

$$\alpha_{n_k} \longrightarrow \underline{u}, \quad \beta_{n_k} \longrightarrow \bar{u} \tag{4.22}$$

uniformly on  $[a, \sigma^2(b)]$ . But the monotonicity of  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  implies that

$$\alpha_n \longrightarrow \underline{u}, \quad \beta_n \longrightarrow \bar{u}. \quad (4.23)$$

Going to the limit in (4.19),  $\underline{u}$  and  $\bar{u}$  are solutions of (1.9). Each solution  $u \in [\alpha, \beta]$  satisfies  $u \in [\underline{u}, \bar{u}]$ , we prove this using monotonicity of operator  $T$ .  $\square$

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